Discrete mathematics johnsonbaugh 7th edition pdf online student

[Berlekamp, 2001, 2003] contains a general theory of games as well as analyses of many specific games. = (1.12)n (1000). Then, by symmetry, we need only consider 7 × 7 boards with squares (i, j) removed where i ≤ j ≤ 4. Rules of inference, brief, valid arguments, are used within a larger argument. k 34. How many ways can the Counting Methods and the Pigeonhole Principle officers be chosen if either Greg is secretary or he is not an officer? Suppose that we have a set S whose elements can be ordered. k=1 (g) Find a formula for sn,n−2 , n ≥ 3, and prove it. Operations research studies the very broad category of problems of optimizing the performance of a system. a d e1 e2 e3 b 29. Now n − 6 ≤ 6i, so by the inductive assumption, the greedy algorithm is optimal for n − 6. Example 5.2 In the binary tree of Figure 5.1, vertex b is the left child of vertex a and vertex c is the right child of vertex a. A family has n children. 83. That is, for each y ∈ R+ , f −1 (y) is the logarithm to the base 2 of y. (c) v is a left child of w in T1 if and only if f(v) is a left child of f(w) in T2 . A left child of vertex v is a child designated as “left.” 21. 95. Show how Algorithm 3.2 executes in case key = ‘C’. Proof Suppose that E = {x1 , . 389 Recurrence Relations If we solve equation (1.10) for qn+1 and substitute into the demand equation for time n + 1, pn+1 = a − bqn+1 , we obtain the recurrence relation b pn+1 = a − pn k for the price. At line 2, since i > j (2 > 2) is false, we proceed to line 4, where we set k to 2. What is the Fundamental Theorem of Arithmetic? e f i h g b 29. (The set D2 is defined before Exercise 1.) 17. Since we always have R ⊆ τ(R), it follows that R = τ(R). Our method of deriving a theta notation for the sequence defined by (3.1) and (3.2) illustrates a general method of handling such recurrence relations. A player who cannot make a move loses. 0111100010 21. Example 1.36 If f is a bijection from a finite set X to a finite set Y , then |X| = |Y |, that is, the sets have the same cardinality and are the same size. If C is not a simple cycle, then vi = vj , for some i < j < n. Example 2.13 The sequence 2, 4, 8, 16, . 2 = (−1)k+1 from Exercise 33. We could find such an i: We could check whether s1 ≥ A and if true, stop. We have proved that A ∪ B = B. , n} containing an even number of elements is 2n−1 . One way to “eliminate” this expression is to take a = b = 1, in which case the Binomial Theorem becomes 2n = (1 + 1)n = n k=0 C(n, k)1n−k 1k = n C(n, k). Write an algorithm to generate an n-vertex random binary tree. Then at each iteration, it adds to the current tree a minimum-weight edge that does not complete a cycle. (The initial call sets ab val to ∞ if PT is a box vertex or to −∞ if PT is a circle vertex.) Output: The game tree with PT evaluated } } } else { contents(PT) = ∞ for each child C of PT { alpha beta prune(C, box, PT level + 1, n, E, content(PT)) c val = contents(C) if (c val ≤ ab val) { contents(PT) = ab val return } if (c val < contents(PT)) contents(PT) = c val } } 23. q = −1, r = 2 1 1 1 1 1 5 = + = + + 22. If we succeed, this part of the example is complete. How is the height of a binary tree related to the number of its terminal vertices? Let x ∈ X ∪ (Y − X). If P ends at v3 [see Figure 2.9(b)], then v is in the same component as v3 in G , which is in the same component as v1 in G (since edge e in G is incident on v1 and v3). 5 ➜ Introduction to Discrete Probability† Probability was developed in the seventeenth century to analyze games and, in this earliest form, directly involved counting. Similarly, if b = 0, b ∈ X. 239 Algorithms Next, we write a recursive algorithm that computes factorials. p1 = np − q 0 5, and 3 divides n2 − 1. Let the domain of discourse be X × Y . ∴ ¬p ∴ q 8. Show that G with e removed is still connected. By definition, the proposition ∀x P(x) is true when P(x) is true for all x in the domain of discourse. f(m, n) = m 18. By the second form of the Pigeonhole Principle, at least two of these values coincide. Theorem 6.9 now gives P(E1 ∪ E2) = P(E1) + P(E2) − P(E1 ∩ E2) = P(E1) + P(E2). R is a symmetric relation if and only if the matrix of R is symmetric about the main diagonal. We then set x0 = s. Someone is a professional athlete and does not play soccer. Write a program that generates Pascal’s triangle to level n, for arbitrary n. For example, ab is in L, for if we take α = λ, then α ∈ L and the first rule states that ab = aαb ∈ L. The puzzle of Exercise 29 has four solutions. Theorem 3.2 If a is a nonnegative integer, b is a positive integer, and r = a mod b, then gcd(a, b) = gcd(b, r). } 30. Define sibling in a rooted tree. Ackermann’s function A(m, n) is defined by the recurrence relations A(m, 0) = A(m − 1, 1), m≥1 A(m, n) = A(m − 1, A(m, n − 1)), m ≥ 1, n ≥ 1 and initial conditions A(0, n) = n + 1, 4. We track the partial product in Result; so Result is set to x a a2 a4 a8 a16 Current Value of n n mod 2 Result Quotient When n Divided by 2 29 14 7 3 1 1 0 1 1 1 a Unchanged a · a4 = a5 a5 · a8 = a13 13 a · a16 = a29 14 7 3 1 0 Figure 2.4 Computing a29 using repeated squaring. A palindrome over X is a string α for which α = αR (i.e., a string that reads the same forward and backward). Figure 1.4 shows the way the tree of Figure 1.2 would be drawn (with v1 as root). This fact can be proved by simply exhibiting one (see Exercise 21). ln(1 + i) Since ln 2 = 0.6931472 . 4. Which are onto? Among the axioms are ■ For all real numbers x and y, xy = yx. But looking at the moves the knight makes, we see that vertices vi where i is odd correspond to white squares, and vertices vi where i is even correspond to black squares. (The sets X and Y may or may not be the same.) The function defined by (1.1) assigns to each nonnegative real number t the value 55t. How many selections are there in which neither Ben nor Francisco is an officer? The number of solutions in nonnegative integers of 1 4 10 C(n + 1, k)an+1−k bk k=0 1 6 n + C(n + 1, n + 1)a0 bn+1 = 1 [C(n, k) + C(n, k − 1)]an+1−k bk k=1 1 1 n + C(n + 1, n + 1)a0 bn+1 3. For each element k ∈ X − {m}, let ⌈ ⌉ ⌈k 26. The complete graph on five vertices K5 6. Show a configuration in which the maximum number of communication links are broken with communication among all offices still possible. a = cs mod z = 2891 mod 221 = 63 Counting Methods and the Pigeonhole Principle 1 2 3 4 † 5 † 6 7 8 Basic Principles Problem-Solving Corner: Counting Permutations and Combinations Problem-Solving Corner: Combinations Generalized Permutations and Combinations Algorithms for Generating Permutations and Combinations Introduction to Discrete Probability Discrete Probability Theory Binomial Coefficients and Combinatorial Identities The Pigeonhole Principle Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises There’s only so many hands in a deck o’ cards. At price p1 , the manufacturer will be willing to supply the quantity q2 at time n = 2, as we can see by moving horizontally to the supply curve. Is the relation 18. If x ∈ B, then x is one of 1, 2, 3. , n + 1} that contain an even number of elements. f is onto by definition. Prove that for all sets X, Y , and Z, if X ⊆ Y and Y ⊂ Z, then X ⊂ Z. Since n is odd, 2m < n < 2m+1 . Since limn→∞ f(n)/g(n) = 0, taking ε = 1, there exists N such that f(n) g(n) < 1, for all n ≥ N. Prove that if P(X) ⊆ P(Y), then X ⊆ Y for all sets X and Y . Theorem 2.3 Let P be a path from a to z in a network G satisfying the following conditions: (a) For each properly oriented edge (i, j) in P, Fij < Cij . Determine the truth value of the given statement and its negation. When we draw a graph, the only information of importance is which vertices are connected by which edges. If n is even, there exists k such that n = 2k. Let A = {1, 2, 3}, B = {2, 3, 4}, C = {3, 4, 5} so that there is an element that is in C, but not in A. Since all months and dates are equally likely and we are ignoring February 29 birthdays, the size of the sample space is 365n . Which do you think are true? If we repeat an experiment n times and observe the values x1 , . (b) Find a formula for bn . 6 36 36 18 Events E1 and E2 are mutually exclusive if E1 ∩ E2 = ∅. Given the arrow diagram for a one-to-one, onto function f from X to Y , we can obtain the arrow diagram for f −1 by reversing the direction of each arrow. Each of the five pairs (P1 , P2), (P1 , P3), (P1 , P4), (P1 , P5), (P1 , P6), has the value “similar” or “dissimilar.” By the third form of the Pigeonhole Principle, there are at least 5/2 = 3 pairs with the same value; that is, there are three pairs (P1 , Pi), (P1 , Pj), (P1 , Pk) all similar or all dissimilar. Define a function f from A to the set of equivalence classes of A by the rule f(x) = [x]. Basis Step b2 = 2b1 + c1,1 ≥ 2b1 ≥ b1 Inductive Step Assume that the statement holds for k < n. p ∧ q 40. Consider the first statement in which we assume that CX∪Y (x) = CX (x)+CY (x) for all x ∈ U and prove that X ∩ Y = ∅. , xn } 7. e1 v1 v4 27. Now suppose that we have n items containing duplicates—specifically, ni identical objects of type i, for i = 1, . √ We now argue √ by cases. [In fact, any polynomial in n of degree k is (nk), even if some of its coefficients are negative. Give an example (different from those of Example 1.1) of an axiom in Euclidean geometry. Suppose that n > 1 and consider a convex (n+2)-sided polygon (see the following figure). Example: If the sequence is 6.2 8.9 4.2 8.9, the algorithm returns the value 4. Trace Algorithm 2.4 for the input 34 34 12. Fibonacci signed much of his work “Leonardo Bigollo.” Bigollo translates as “traveler” or “blockhead.” There is some evidence that Fibonacci enjoyed having his contemporaries consider him a blockhead for advocating the new number system. X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z) (Conclusion) We should be able to use the definitions of intersection (∩) and set difference (−) to complete the proof. Now γ = aabb ∈ L and the first rule states that aaabbb = aγb ∈ L. [Photo by the author. 5/36 21 · 276 C(7, 5)C(31 − 7, 2) = = 0.002204158 19. At this point, we have derived a contradiction: x + y ≥ 2 and x + y < 2. First we explicitly solve (3.2) in case n is a power of 2. w1 v1 v6 w3 w2 v5 T2 In Exercises 7–9, determine whether each pair of rooted trees is isomorphic. ∀x∀y(x2 < y + 1) 43. The argument must conclude by proving that P(x) is true. By (5), T is a tree. ¬(∀x P(x)) and ∃x ¬P(x) have the same truth values. Therefore, all mathematicians are algebraists. 443 Graph Theory 7. Define sequence. Therefore, P contains a vertex v of degree 1 (see Figure 2.4). Algorithm 1.1 Finding the Maximum of Three Numbers This algorithm finds the largest of the numbers a, b, and c. At every vertex with one child, we add an additional child. Figure 1.5 A set that is not a function. Thus we conclude that for all real numbers x and y, if x + y ≥ 2, then either x ≥ 1 or y ≥ 1. If a person invests in a tax-sheltered annuity, the money invested, as well as the interest earned, is not subject to taxation until withdrawn from the account. Exercises In Exercises 1–3, a path from the source a to the sink z in a network is given. We show that if n = 2i + 1 is odd, bn = Ci , where Ci denotes the ith Catalan number. It may, however, help us to understand the statement. Show that 2 2 − fn+1 = fn fn+3 fn+2 for all n ≥ 1. Write a program that generates all permutations of the elements {1, . What is 9. i=0 It is sometimes useful to change not only the name of the index, but to change its limits as well. Define a function f from {1, 2, 3, 4} to {a, b, c, d, e} as f = {(1, a), (2, c), (3, b), (4, d)}. The equivalence is immediate from a truth table: †A starred 84 exercise indicates a problem of above-average difficulty. G1 = ({v1 }, ∅) 2 3 9. mi + 1, (m − 1)i + 1 Section 5 Review 1. Example 3.2 v1 v5 v4 v3 Figure 3.5 A graph with no Hamiltonian cycle. Therefore, our solution is optimal. Then fn+2 = fn+1 + fn √ n−1 √ n−2 1+ 5 1+ 5 ≥ + 2 2 √ n−2 √ 1+ 5 1+ 5 = +1 2 2 √ n−2 √ 2 1+ 5 1+ 5 = 2 2 √ n 1+ 5 = , 2 and the Inductive Step is complete. Let Y be the set of integer upper bounds for X. ¬(p ∧ q) ∨ (r ∧ ¬p) 54. Give an example of a tree T that does not satisfy the following property: If v and w are vertices in T, there is a unique path from v to w. Now the k-element subsets of Y can be divided into two disjoint classes: 1. r1 k vertices r2 k + 1 vertices Figure 8.15 Two binary trees that give worst-case run time 3n + 1 for Algorithm 8.13 when n = 2k + 1 is odd. We cannot select any edges incident on B or G since B and G must each have degree 2. (6, 9), (3, 6), (2, 3), (2, 5), (1, 2), (1, 4), (4, 7), (7, 8) 16. To prove if x ∈ X ∩ (Y − Z), then x ∈ (X ∩ Y) − (X ∩ Z), we begin by assuming that (the arbitrary element) x is in X ∩ (Y − Z). Explain the rule of thumb: To estimate the time to double the investment, divide 70 by r. Of course, such an algorithm might not terminate (because of bad choices at the intersections). X is a subset of Y if every element of X is an element of Y. It is not the case that: You heard the “Flying Pigs” rock concert or you heard the “Y2K” rock concert or you do not have sore eardrums. Show that for any events E1 and E2 , P(E1 ∩ E2) ≥ P(E1) + P(E2) − 1. The probability of E, the complement of E, satisfies P(E) + P(E) = 1. Find a theta notation for this algorithm. The function f is not onto. n Therefore, there exists i such that si ≥ A. If j = rand(3, 5) = 3, the sequence does not change. What is the ceiling of x? We have three possibilities for selecting the additional edge (shown in color in Figure 8.8). This function is onto because each element in Y has at least one arrow pointing to it. The first permutation follows the last permutation. There are Sn−1 such (n − 1)-bit strings. The netid is given by the next 14 bits, which identifies the network. Thus, Case ←: If X∩Y = ∅, then CX∪Y (x) = CX (x)+CY (x) for all x ∈ U. Graph Theory In Exercises 7–9, prove that the graphs G1 and G2 are not isomorphic. Since (n/2)n becomes arbitrarily large as n becomes large, we cannot have nn ≤ C2n for some constant C and for all sufficiently large n. , Xk denote the distinct equivalence classes. If n is odd, the algorithm partitions the array into subclasses of sizes (n − 1)/2 and (n + 1)/2. We just observed that m is even. Define antisymmetric relation. We use induction on n to show that the greedy solution gives an optimal solution for n ≥ 1. Therefore, the number of eight-bit strings with no two 0’s in a row is 1 + 8 + C(7, 2) + C(6, 3) + C(5, 4). ... 4 5 G2 7 8 6 10 5 2 9 4 3 6. Example 4.3 Let α = 13246 and β = 1342 be strings over {1, 2, 3, 4, 5, 6}. Somebody loves somebody. What we have just computed are the members of [10110100]. Jill can sing. Consider an optimal solution S for n. Is q nonincreasing? (a) Show that a2 = 1, a3 ≤ 3, a4 ≤ 4. Tournament Sort. Let D = {(x, x) | x ∈ X}. 11 Chapter Self-Test 6 4 5 5 1. dn = 33. Thus edges (b, d) and (b, f) are not in H. How many strings contain the letter A if repetitions are not allowed? The name of the index can be chosen in any convenient way. What is the value of the hexadecimal number hn hn−1 . (4.1) Assuming that these statements are true, it is reasonable to conclude The bug is in module 17. Since the relation is reflexive, the main diagonal must consist of 1’s. In particular, a is an upper bound for X so x≤a for all x ∈ X. Thus Cn divides (n + 2)Cn+1 . Therefore, z ≤ ta for every z ∈ tX and the proof of part (a) is complete. Counting Methods and the Pigeonhole Principle ■ Counting the number of members of a set in two different ways leads to an equation. Inductive Step {b} {c} {b, c} Figure 4.3 Subsets of X = {a, b, c} divided into two classes: those that contain a and those that do not contain a. Let (v0 , . The second form of the Pigeonhole Principle can be reduced to the first form by letting X be the set of pigeons and Y be the set of pigeonholes. How many numbers are there? Example: If the sequence is AMY BRUNO ELIE DAN ZEKE, the algorithm returns the value 2. Draw the subtree rooted at Aphrodite. The netid must not consist of all 1’s. 2 4 R 4 3 1 4 492 G G W B W W 2 3 R G G B W G 4 (c) Find all subgraphs of your graph of part (a) satisfying properties (8.1) and (8.2). Looking for specified text in a document when running a word processor is yet another example of a searching problem. At line 4, i is reset to 8 and we return to line 2. The Inductive Step is complete, and we have proved that the algorithm is correct. Basis Step (n = 1) Inductive Step f12 = 12 = 1 = 1 · 1 = f1 f2 n+1 fk2 = k=1 16. The chapter concludes with a discussion of the Pigeonhole Principle, which often allows us to prove the existence of an object with certain properties. f 14 g In a precedence graph, the vertices model certain actions. How is this formula derived? Again, by the inductive assumption, these boards can be tiled. Therefore, 3 2 is irrational. Since there are C(4, 2) ways to select two suits, the number of hands containing cards of exactly two suits is 5! − (2 · 4! − 3!). p ∧ q ∴p 12. How many ways can (integer) points be assigned to the problems if the total of the points is 100 and each problem is worth at least five points? If today is Monday, then it is raining. (a) p ∧ q → r (b) p ∨ q → ¬r (c) p ∧ (q → r) (d) p → (q → r) (a) We first evaluate p ∧ q because → is evaluated last. The problem had been posed in the mid-1800s and for years no one had succeeded in giving a proof. If m < n − m + 1, the best case is that p is at index 1 in t. Now suppose that x ∈ A ∪ (B ∪ C). ∅ 8. , sn and a value x. Write an algorithm that finds the second-smallest element among a, b, and c. Prove that for every n ∈ Z, 44. Write an algorithm that reverses the sequence s1 , . Determine whether each function in Exercises 23–28 is one-to-one, onto, or both. To prove that f is onto, let m be an integer. This latter formula is true because one of {min(x, y), max(x, y)} equals x and the other equals y. Choose a ∈ / X. 488 F E H G 19. Encrypt 584 using the public key z, n. Reversing each side, we find that α = β. Suppose that {aij } is a sequence indexed over pairs of positive integers. I ate my hat. How many selections are there in which Dolph is an officer and Francisco is not an officer? 2! 9! 4! 5! 4! 1! 2! 4! 4! 1! The solution to Example 3.1 assumes a nice form. FE 38. ¬r From 3 and 5 Now 4 and 6 give a contradiction. 1 1 · 3 · 5 · · · (2n + 1) − 2 2 · 4 · 6 · · · (2n + 2) 1 1 1 + + ··· + 22 − 1 32 − 1 (n + 1)2 − 1 = 1 1 3 − − 4 2(n + 1) 2(n + 2) 10. Therefore, it suffices to count the number of routes from the lower-left corner to the upper-right corner of a 2 × 1 grid. Since pj → q is false, (2.2) is also false. Divide the routes into classes based on when the route crosses the slanted line shown. If A is the matrix of R (relative to some ordering), we compute A2 . Since there are nm vertices, this cycle has nm edges; therefore, the Hamiltonian cycle contains an odd number of edges. Let k be a fixed integer, 0 ≤ k ≤ n − 1. From the argument in the preceding paragraph, we conclude that |P(Y)| = |P(X)| . The latter point is considered in more detail in Examples 3.5 and 6.3. We conclude this introductory section by defining some special graphs that appear frequently in graph theory. In how many ways can 10 distinct books be divided among three students if the first student gets five books, the second three books, and the third two books? This notation suggests that we are interpreting the relation as an ordering of the elements in X. If the pair a, b, a > b, requires n ≥ 1 modulus operations when input to the Euclidean algorithm, how are a and b related to the Fibonacci sequence? Answer Exercises 26–29 to give an argument that shows that in a group of 10 persons there are at least two such that either the difference or sum of their ages is divisible by 16. Ten heads were not obtained. This leaves order 1 for person 2. Each vertex in T becomes an internal vertex in T ∗ . For the graph G = (V, E), find V, E, all parallel edges, all loops, and all isolated vertices, and state whether G is a simple graph. 1 ➜ Introduction Consider the directed graph in Figure 1.1, which represents an oil pipeline network. , and nt identical objects of type t. But what if (x, y) ∈ R and (y, z) ∈ S? ⎪ ⎪⎝0 2 0 2 1⎠ 1 1 1 1 2 By multiplying, ⎛ 2 ⎪0 ⎪ A4 = A2 A2 = ⎪ ⎪2 ⎝0 1 0 3 1 2 1 2 1 3 0 1 0 2 0 2 1 ⎠⎛ 1 2 ⎪0 1 ⎪ ⎪⎪ ⎪ 1 ⎪ ⎪2 1⎠ ⎝0 2 1 0 3 1 2 1 2 1 3 0 1 0 2 0 2 1 ⎠ 1 1 ⎪ ⎪1 ⎪ ⎪, 1⎠ 2 we find that a b A4 = c d e b c d ⎛ a 9 3 11 1 ⎪ 3 15 7 11 ⎪ ⎪ 11 7 15 3 ⎪ ⎝ 1 11 3 9 6 8 8 6 e⎠ 6 8 ⎪ ⎪8 ⎪ ⎪. For n = 0, (4.12) becomes a= a(r 1 − 1) , r−1 which is true. if (sm−1 > sn) 16. Trace Algorithm 4.4 (the tromino tiling algorithm) when n = 8 and the missing square is four from the left and two from the top. Reflexive, symmetric, transitive 10. (x, y) ∈ R if 3 divides x + 2y. However, unlike the Euclidean algorithm, this proof does not lead to a technique to compute s and t. Therefore there are 3(n −n)/2 reflexive and antisymmetric relations on an n-element set. G1 has vertices of degree 3, but G2 has no vertices of degree 3. For the graph of Exercise 39, find a path with no repeated edges from d to e containing all the edges. Theorem 1.17 Let m and n be integers, m > 1, n > 1, with prime factorizations m = pa11 pa22 · · · pakk and n = pb11 pb22 · · · pbkk . The most general technique is called alpha-beta pruning. The height of T is 3. (b, b) 2. Strassen’s algorithm to multiply two n × n matrices runs in time (nlg 7). x is set to x ∗ x mod z = 25 mod 21 = 4, and n is set to 5. By Definition 5.4, P(x) is true for every x in D, the domain of discourse. An axiom is a proposition that is assumed to be true. Find r1 . (If it is both connected and acyclic, it would be a tree and thus have five edges.)Aconnected graph with n vertices and more than n−1 edges must contain a cycle. By using first De Morgan’s laws (3.3) and then (3.4), we can obtain equivalent hypotheses, each of which is a clause. The converse of p → q is q → p. What is the probability of getting a sum of 6 or doubles given that at least one die shows 2? We use induction on i, where n = 2i . k=0 Now multiply by n to obtain n(1 + x)n−1 = n k=0 C(n, k)an+1−k bk n−1 n−1 C(n − 1, k)xk k=0 =n n C(n − 1, k − 1)xk−1 k=1 379 Counting Methods and the Pigeonhole Principle = n k=1 = = n k=1 n n(n − 1)! xk−1 (n − k)! (k − 1)! 1 = 2 n! kxk−1 (n − k)! k! = C(n, k)kxk−1 . The work was coordinated on the Internet. Thus ∪{S | S ∈ S} = X. Therefore X ⊂ Z. (e) Finds the siblings of v. Then A (B ∪ C) = {4} and (A B) ∪ (A C) = {1, 4} ∪ {3, 4} = {1, 3, 4}. First, we choose positions for A, C, and E [C(6, 3) ways]. Assume that G has a flow F . How many comparisons does tournament sort require to find the largest element? e a h c 6. Suppose that a graph has an adjacency matrix of the form e A= 4. 36 Therefore, 1 P(E ∩ F) 1 36 P(E | F) = = . 32. Allowing equality yields what we call a “nondecreasing sequence.” That is, the sequence a is nondecreasing if an ≤ an+1 for all n. An edge in G connects two vertices if the corresponding faces in G are separated by a boundary. Directly under a folder f are the folders and files contained in f . Since edge e is in G , there is a path in G from v to v1 . Exercise 9, Section 1 12. We assume that a > c. What is the probability that the wheel lands on a black number twice in a row? 345 Counting Methods and the Pigeonhole Principle Examples of events that might occur when the previous experiments are performed are ■ ■ Obtaining a 4 when rolling a six-sided die. At line 8, we place one tromino in the center. 1 k Proof Let g = gcd(m, n). If b ≤ c, then 3. An integer n is even if there exists an integer k such that n = 2k. How do we denote x is not an element of the set X? b e g i h d f 32. P(1/2) 15. P ≡ Q 69. We put (1, 2) in R and (2, 3) in S and ensure that (1, 3) is not in R ∪ S. The addition operator + is a binary operator on the set of integers. The disjunction of p and q, denoted p ∨ q, is the proposition p or q. (The argument is the same if a < c.) We may then cancel 2c from both sides of 2a 3b = 2c 3d to obtain 2a−c 3b = 3d . We assume that the demand is given by the equation pn = a − bqn ; that is, at time n, the quantity qn of the product will be sold at price pn . Definition 6.13 Let E and F be events, and assume that P(F) > 0. In a tree representation of an expression, the internal vertices represent operators, and the operators operate on the subtrees. X ∪ (Y − Z) = (X ∪ Y) − (X ∪ Z) for all sets X, Y , and Z. In particular, if d is in D, then P(d) is true. At lines 15 and 16, vertex 2 is added to the minimal spanning tree and edge (1, 2) is added to E. In each case, which meaning do you think is intended? The recurrence relation now follows. We have shown that the price fluctuates between two values when k = b. We have proved that for every n ∈ Z, if n2 is even, then n is even. The discussion following Example 1.5 shows that n has a divisor d satisfying 2 ≤ d < n. For example, if we have the situation shown in Figure 9.3, the first player (box) should move to the position represented by vertex B, since this position is the most valuable. The three nonisomorphic trees are shown in Figures 8.1 and 8.3. Theorem 8.3 There are three nonisomorphic trees with five vertices. Show that each such string represents a binary tree. Modify Algorithm 3.6 so that it tracks the parent p of a vertex c (p is the parent of c if c was visited from p). How is the union of X and Y denoted? Logic is the beginning of wisdom, Valeris, not the end. To put it another way, X = Y if the following two conditions hold: ■ For every x, if x ∈ X, then x ∈ Y , and ■ For every x, if x ∈ Y , then x ∈ X. We will give a direct proof of the following statement. But now v6 is in both V1 and V2 , which is a contradiction since V1 and V2 are disjoint. (The front, back, left, and right are of different colors.) 14. f ◦ g = {(x, z) | (x, y) ∈ g and (y, z) ∈ f for some y ∈ Y }. Another useful proposition is p if and only if q, which is considered to be true precisely when p and q have the same truth values (i.e., p and q are both true or p and q are both false). It is possible to extend the definition of recurrence relation to include functions indexed over n-tuples of positive integers. Here s or {sn } denotes the entire sequence s1 , s2 , s3 , ... The entry in row v and column e is 1 if e is incident on v and 0 otherwise. (b) How many possibilities are there for the months of the birthdays of five people? 285 Introduction to Number Theory Example 3.4 We show how Algorithm 3.3 finds gcd(504, 396). Find v4 . Proof Let X be a set with n elements. Develop a graph model for Erd˝os numbers. , π Figure 1.1 Sets of numbers. , sn max max sum3(s, n) { // max is the maximum sum seen so far. Example 7.9 Use equation (7.4) to find the sum 1 + 2 + · · · + n. , sm // q is the position in the sequence sm+1 , . Since f(n) ≥ 0 for all n ∈ Z, f(n) = −1 for all n ∈ Z. 37. fn+1 f1 fn+1 fki . (b) 00000000, 01000000, 10000000, 11000000, 00010000, 01010000, 10010000, 11010000, 00100000, 00110000, 01100000, 01110000, 10100000, 10110000, 11100000, 11110000 lists one member of each equivalence class. Noting that the expansion of 29 in powers of 2, that is the binary expansion, is 29 = 1 + 4 + 8 + 16, we see that we can compute a29 as a29 = a1 · a4 · a8 · a16 , which uses 3 additional multiplications for a total of 7 multiplications. ABCD+∗/E− 14. For example, the string bbaaac may be written b2 a3 c. By the analog of Theorem 5.6 for “trinary” trees, log3 (n − 1) ≤ h = f(n). 10!/(5! · 3! · 2!) 15. Then f(m − 1) = (m − 1) + 1 = m. The spanning tree to be used is drawn in black. Since i = 1 and power = 2, ci ∗ power = 0 ∗ 2 = 0. We call a board with one square missing a deficient board. No 10. [Requires calculus] The definition of lim f(x) = L x→a is: For every ε > 0, there exists δ > 0 such that for all x if 0 < |x − a| < δ, then |f(x) − L| < ε. We first show that if (a, b) = (c, d), then a = c and b = d. This property is equivalent to the two forms of induction (see Exercises 31–33). Suppose first that si+1 > large. In general, |X1 × X2 × · · · × Xn | = |X1 | · |X2 | · · · |Xn |. The graph of a single-elimination tournament is a full binary tree (see Figure 5.2). To find the maximum sum of consecutive values, we must at least look at each element in the sequence, which takes time (n). Now 0 ≤ s < φ. Rules of Inference for Quantified Statements We conclude this section by introducing some rules of inference for quantified statements and showing how they can be used with rules of inference for propositions (see Section 4). Then there exists a such that (x, a) ∈ S and (a, y) ∈ R, and there exists b such that (y, b) ∈ S and (b, z) ∈ R. (b) A designated vertex, the sink, has no outgoing edges. , bn) precisely when a1 = b1 , a2 = b2 , . Another characterization of the components of a graph G = (V, E) is obtained by defining a relation R on the set of vertices V by the rule v1 Rv2 if there is a path from v1 to v2 . b 4 (b) Find another self-complementary graph. When we divide a by b, the possible remainders are 0, 1, . For example, if n = 4, we might insert the parentheses as shown: (a1 a2)(a3 a4). Thus the right side of equation (7.3) counts the number of subsets of X. We call (vi , . 12 4 9 3 6 3 7 3 6 8 10 6 2 9 14. We call the function f an isomorphism. i=1 ■ Directly enumerate some of the items to be counted. Since the integers increase without bound, there exists n ∈ Z such that 1/(b − a) < n. Eighty proposes the following version of binary search: binary search2(s, i, j, key) { if (i > j) return 0 k = (i + j)/2 if (key == sk) return k k1 = binary search2(s, i, k − 1, key) k2 = binary search2(s, k + 1, j, key) return k1 + k2 } (a) Show that binary search2 is correct; that is, if key is present, the algorithm returns its index, but if key is not present, it returns 0. If n = 6, binary insertion sort sorts five items (eight comparisons—worst case) and then inserts the sixth item in the sorted five-item list (three comparisons—worst case) for a total of eleven comparisons in the worst case. In the Inductive Step of the Strong Form of Mathematical Induction, if you assume that case n−p is true, there will be p Basis Steps: n = n0 , n = n0 +1, . Then, there exists an edge x in T1 that is not in T2 . Amin, Amin co-authored a paper with Peter J. Therefore, the pair of propositions in part (a) always has the same truth values. The programs that perform the search are called search engines. Suppose that x ∈ A ∩ B. , cj p=i q=m+1 r=i // copy smaller of sp and sq while (p ≤ m ∧ q ≤ j) { if (sp < sq) { cr = sp p=p+1 } else { cr = sq q=q+1 } r =r+1 } // copy remainder of first sequence while (p ≤ m) { cr = sp p=p+1 r =r+1 } // copy remainder of second sequence while (q ≤ j) { cr = sq q=q+1 r =r+1 } } 415 Recurrence Relations Example 3.6 Figure 3.1 shows how Algorithm 3.5 merges the sequences 1, 3, 4, 8 and p 134 q 2456 2456 r r 1 2 2, 4, 5, 6. For example, Bill is taking Computer Science and Art, and Mary is taking Mathematics. 1(1!) + 2(2!) + · · · + n(n!) = (n + 1)! − 1 4. 0! = 1! = 1, Example 4.2 Example 4.3 3! = 3 · 2 · 1 = 6, 6! = 6 · 5 · 4 · 3 · 2 · 1 = 720 Use induction to show that n! ≥ 2n−1 for all n ≥ 1. In this case, if the remaining (n − 1)-bit string does not contain 000, neither will the n-bit string. You did not hear the “Flying Pigs” rock concert and you did not hear the “Y2K” rock concert, but you have sore eardrums. Find the matrix A1 of the relation R1 relative to the orderings 1, 2, 3; x, y. The Fibonacci sequence is defined by the linear homogeneous, second-order recurrence relation fn − fn−1 − fn−2 = 0, n ≥ 3, and initial conditions f1 = 1, f2 = 1. where NX (respectively, NO) is the number of rows, columns, or diagonals containing an X (respectively, O) that X (respectively, O) might complete. A tape is to be released that can hold C seconds. Since G is connected, there is a path P in G from v to v1 . a d e 6. If we then move nDn−1 to the left side of the equation (to obtain an expression equal to Cn), we obtain Dn − nDn−1 = −Dn−1 + (n − 1)Dn−2 . ∀x∀y¬P(x, y) 10. Therefore there exists an integer k ≥ 1 satisfying Cn+1 = kCn . Input: A, B, n Output: C, the product of A and B matrix product(A, B, n) { for i = 1 to n for j = 1 to n { Cij = 0 for k = 1 to n Cij = Cij + Aik ∗ Bkj } return C } The constants that are suppressed in the theta notation may be important. Then C is a proper subset of A since C is a subset of A but C does not equal A. Therefore q2 is even, and Example 2.1 tells us the q is even. How many strings can be formed by ordering the letters ILLINOIS if some I appears before some L? The algorithm then returns a [= gcd(a, 0)]. Thus R is an equivalence relation on X. n n n Therefore x = (m + 1)/n is a rational number satisfying a < x < b. Then k − 1 < x ≤ k and 2x ≤ 2k. Since n = 3 > 0, the body of the while loop executes. The asymptotic time of this algorithm cannot be further improved. Program Number of Program Lines Number of return Statements Number of Function Calls 1 2 3 4 5 66 41 68 90 75 20 10 5 34 12 1 2 8 5 14 A similarity graph G is constructed as follows. This construction is due to Ira Gessel and was forwarded to the author by Arthur Benjamin. We set to ∞ [= val(a)] at line 16. f(n) = (1), 14. Let g be a function from X to Y and let f be a function from Y to Z. The equation n! = n · (n − 1)!, which happens to be true even when n = 1, shows how to decompose the original problem (compute n!) into increasingly simpler subproblems [compute (n − 1)!, compute (n − 2)!, . Resolution, the topic of Section 3, is a special proof technique that can be automated. , zk occurs twice. After each word is read, it is inserted into the tree. Now n ≥ am−1 + am ≥ ai + am , so n − ai ≥ am . That k1 is not necessarily equal to k2 is the reason that we must denote the two integers with different symbols.) The missing part of our proof is the argument to show that m + n is odd. 56. Write a program that determines whether a graph is a bipartite graph. The contestant wins a modest amount ($40) if exactly five numbers, in any order, match those among the seven distinct numbers randomly drawn by a lottery representative. Show that b1 = 0 and b2 = 1. If we now input n = 13 to Algorithm 1.8, the algorithm returns 0 because 13 is prime. A recurrence relation relates the nth element of a sequence to its predecessors. 1.2i · 100(n − i) + 1.2n d0 kn (kn + 1) ≤ n, 2 √ we can deduce kn ≤ 2n. We want to build the lowest-cost road system that will connect the six cities. Since A ⊆ B, x ∈ B. Experiment 30. Path Simple Path? Since the hypothesis q is true and the conclusion p is false, the converse is false. Since logb n = O(loga n) and logb n = (loga n), we conclude that logb n = (loga n). In other words, the (n − 1) disk problem must have been solved, which required at least dn−1 moves. In Algorithm 4.2, if n = 0, the function does not invoke itself. R. Therefore, CX (x) + CY (x) = 0. In proving statement n + 1, we were permitted to make use of statement n; indeed, the trick in constructing a proof using mathematical induction is to relate statement n to statement n + 1. c 4 In Exercises 13–18, find a maximal flow in each network starting with the flow given. v and w are siblings if they have the same parent. Show that Algorithm 4.3 examines (n3) edges in the worst case. Therefore, we cannot simultaneously have f(n) = O(g(n)) and g(n) = O(f(n)). For s R s to be true, the first four bits of s and s must coincide. d k 34. Compound interest is interest on interest. Alpha-beta pruning works as follows. Draw all nonisomorphic binary trees having two vertices. For example, ASCII (American Standard Code for Information Interchange) represents each character by a string of seven bits. factorial(n) { if (n == 0) return 1 return n ∗ factorial(n − 1) } 7. SALAD 6. Another possible way a message could be intercepted and decrypted would be to take the nth root of c mod z, where c is the encrypted value. Since this value is 1, we know that 4 = 22 is included in the binary expansion of 29. ∃x∀y¬K(x, y) 22. Find the names of all employees who are managed by Jones. n(n + 1)/2 = O(n2). For example, if we compute 1 + 0 + 1, the result is 102 ; in our notation, c = 1 and b = 0. In either case, |Y | < 20 = |X|. Summary of Problem-Solving Techniques ■ When dealing with quantified statements, it is sometimes useful to write out the statements in words. One graph represents the front/back stacking, and the other represents the left/right stacking. = c1 + 31 + 32 + · · · + 3n−1 3n − 3 3n + 1 = 2+ = . The children of a vertex show all possible responses by a player to the position shown in the vertex. (Interchanging the values of a and b does not alter the number of modulus operations required.) By Theorem 3.5, a ≥ fn+2 . Now consider proving the second statement If X ∩ Y = ∅, then CX∪Y (x) = CX (x) + CY (x) for all x ∈ U. Use Tables 6.4–6.7. 5. At this point, we have s = 164532. At each step, some vertices have temporary labels and some have permanent labels. Another iteration of the algorithm produces the maximal flow shown in Figure 2.13. Since MORE is less than PROCESSING, we go to the left child. The problem then is to count the number of ways to select i1 , . an = an−1 + n 17. Books specifically on algorithms are [Aho; Baase; Brassard; Cormen; Johnsonbaugh; Knuth, 1997, 1998a, 1998b; Manber; Miller; Nievergelt; and Reingold]. i=0 aj B0 + 2n+1 k−2 i=0 1. , E2n−1 denote the subsets of {1, 2, . See Exercises 57–66 for other examples. Then the number of orderings of S is n! . The label of vertex i is denoted Li . The length of α is denoted |α|. Each equivalence class consists of a set of mutually homeomorphic graphs. In Exercises 44–51, the letters ABCDE are to be used to form strings of length 3. Suppose, by way of contradiction, that no two bags contain the same number of coins. e1 In Exercises 39 and 40, verify that the number of vertices of odd degree in the graph is even. The value of m mod b is the remainder when m is divided by b. Suppose, by way of contradiction, that b ∈ [a] for some a ∈ X. In each of these examples, we found that the total number of dinners was equal to the product of numbers of each of the courses. Draw K2,5 , the complete bipartite graph on 2 and 5 vertices. Show that some three consecutive players have the sum of their numbers at least 20. I Root Left B C Right Root J Left Right F D E G Root Left Right H J I A B C D E F Root Left G H I A B C D E F Figure 6.4 Preorder traversal of the tree in Figure 6.3. 542 G Right J Root Left Right H I J Trees Inorder traversal and postorder traversal are obtained by changing the position of line 3 (root) in Algorithm 6.1. “Pre,” “in,” and “post” refer to the position of the root in the traversal; that is, “preorder” means root first, “inorder” means root second, and “postorder” means root last. Now assume that the inequality is true for values less than n + 1. Thus Y = Z. There is a hurricane and it is raining. Advantages of the postfix form over the infix form are that in postfix no parentheses are needed and no conventions are necessary regarding the order of operations. There are C(1000, 5) ways to select 5 microprocessors among 1000. What is an alpha value? This triangle divides the original polygon into two polygons: one having k + 1 sides, for some k satisfying 1 ≤ k ≤ n; and the other having n − k + 2 sides (see the preceding figure). Let G be the graph obtained from G by replacing each edge e 12 e6 e6 d e8 f 30. If the coin is flipped 10 times, what is the probability of at least one head? Graph of Exercise 6 19. 6 Sets and Logic Sometimes we are dealing with sets, all of which are subsets of a set U. Otherwise, binary search compares the middle item with the desired item. In this case, we obtain the subgraph G4 shown in Figure 2.6. Thus G has the four subgraphs shown in Figure 2.6. v2 v2 v2 v1 e1 v1 e1 v1 v2 G G1 Figure 2.5 The graph for Example 2.10. f(x) = 4x + 2, X = set of real numbers {(1, a), (2, a), (3, c), (4, b)} {(1, c), (2, a), (3, b), (4, c), (2, d)} {(1, c), (2, d), (3, a), (4, b)} {(1, d), (2, d), (4, a)} 33. Prove that if ε > 0, then there exists x ∈ X satisfying a + ε > x ≥ a. Continue until an explicit formula is obtained. Since n = 30 > 0, the body of the while loop executes. How many ways can the bikes be stored in four warehouses if the bikes and the warehouses are considered distinct? Each edge is labeled with 0 or 1, and the sequence of bits encountered on the simple path is the code for that character. In fact, since the statement is false for n = 4, the Basis Steps n = 3, 4, 5 cannot be proved. The resulting path from v to w lies in G − {e}. List the first seven terms of the subsequence. Problem-Solving Tips The formulas in Section 3 generalize the formulas of Section 2 by allowing repetitions. The literal meaning is: No circumstance is right for a formal investigation. A prefix: − ∗ + ∗ +ABCDE + ∗ + ABCD postfix: AB + C ∗ D + E ∗ AB + C ∗ D + − 8. There is an edge from team ti to team tj for each victory of ti over tj . 42. Write a recursive algorithm that interchanges all left and right children of a binary tree. We return 0! · 1 = 1. Example 3.2 A spanning tree of the graph G of Figure 3.1 is shown in black. is an equation that relates an to certain of its predecessors a0 , a1 , . The initial condition is S0 = 1. Solve your recurrence relation of Exercise 61 for bn . Moreover, the sender and receiver do not disclose the key. Find the number of solutions in integers to x1 + x2 + x3 + x4 = 12 satisfying 0 ≤ x1 ≤ 4, 0 ≤ x2 ≤ 5, 0 ≤ x3 ≤ 8, and 0 ≤ x4 ≤ 9. If E1 and E2 are events, how are P(E1 ∪ E2), P(E1 ∩ E2), P(E1), and P(E2) related? Since f ◦ g is one-to-one, x1 = x2 . Use mathematical induction to show that n 2 5fn−1 + 4(−1)n+1 33. 45. At line 4, i is reset to 0 and we return to line 2. One example is the VLSI layout problem—the problem of packing many components on a computer chip (see [Wong]). Write an algorithm whose input is an r-combination of {1, 2, . The inductive step will itself require induction on y. Next, i is set to 2 and, as j runs from 1 to 2, line 3 is executed two times, and so on. Is R necessarily reflexive, symmetric, or transitive? (s → (p ∧ ¬r)) ∧ ((p → (r ∨ q)) ∧ s) 20. This algorithm, although not known to be optimal, uses as few moves as any other algorithm that has been proposed for the four-peg problem. † Recall 346 that |X| is the number of elements in a finite set X. Since a set that has n elements has 2n subsets, the program would require at least 2n units of time to execute. Show that at least three persons have the same first and last names. 20, 40 19. The size of the set of dinners is not counted by dividing the set 313 Counting Methods and the Pigeonhole Principle of dinners into nonoverlapping subsets. If x = 1, then x2 − 4x + 4 = 12 − 4 · 1 + 4 = 1 65. Therefore, a divides c (with quotient q1 q2). We see that each processor can communicate directly with exactly two other processors. , a30 + 3; a1 + 6, . R × Z 68. Define the sign of the real number x, sgn(x), as 1 if x > 0 0 if x = 0 sgn(x) = −1 if x < 0. Now i + j = ai + m − aj = m. Exercises In Exercises 1–3, determine the number of strings that can be formed by ordering the letters given. There is a difference. Consider the headline: Every school may not be right for every child. Inverse relation R−1 : {(y, x) | (x, y) ∈ R} 40. We note that if a prime p appears in the prime factorization of g, p must be equal to one of p1 , . For each statement in Exercises 62–69, if the statement is true, prove it; otherwise, give a counterexample. By definition, any set is a subset of the universal set, so A ∪ U ⊆ U. If the statements are p, q, and r, we can show that they are equivalent by proving that p → q, q → r, and r → p are all true. Develop an algorithm for finding a minimal flow in G. 2 21 n If the first two dominoes are placed as shown, there are an−2 ways to cover the 2 × (n − 2) board that remains. Step 2: I raise a million bucks—there are a lot of little old ladies in the world. n bi . Analysis of an algorithm refers to the process of deriving estimates for the time and space needed to execute the algorithm. Therefore, the dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers. For every x, there exists y, namely the cube root of x, such that x = y3 . Let x ∈ Y . Thus if the primes p and q are chosen large enough, it is impractical to compute the factorization z = pq. Give an argument using rules of inference to show that the conclusion follows from the hypotheses. Show that an = fn+1 , where {fn } is the Fibonacci sequence. Let n be a positive integer. C(50, 4) − C(46, 4) (Total number − number with no defectives) 72. R1 = {(1, x), (1, y), (2, x), (3, x)}; R2 = {(x, b), (y, b), (y, a), (y, c)}; orderings: 1, 2, 3; x, y; a, b, c 17. The entries on the main diagonal of A2 give the degrees of the vertices (when the graph is a simple graph). We will be concerned with strings of integers. If A is the adjacency matrix of a simple graph, what are the values of the entries in An ? 2 R B 3 4 1 4 2 1 1 4 W G 3 3 Figure 8.6 The Instant Insanity puzzle for Example 8.1. 2 2 2 R 2 R B 1 2 B 3 3 4 R B 1 4 3 1 2 1 1 1 2 4 W G Figure 8.7 Trying to find a subgraph of Figure 8.6 satisfying (8.1) and (8.2). 60 4. Having made these assignments, we next assign positions to the n2 items of type 2, which can be done in C(n−n1 , n2) ways, and so on. State a necessary and sufficient condition that a graph have a path with no repeated edges from v to w (v = w) containing all the edges and vertices. (b) Show that sn,n = 1 for all n ≥ 1. The formula for the term having index m is um = m2 − 1, m ≥ 0. Prove that for every n, there is a connected graph with n vertices that has a vertex cover of size 1. Since w is the remainder when ab is divided by z, by the quotient-remainder theorem, there exists q1 such that ab = q1 z + w. Therefore, the proposition is false. (You should verify that it is false that cn ≤ 2n for all n ≥ 3.) We use the Strong Form of Mathematical Induction. Yes; order 3 32. bn = n 12. The graphs of Figures 8.9 and 8.11 solve the Instant Insanity puzzle of Figure 8.6. 2 R 2 B 4 1 B 4 1 W G 3 Figure 8.10 Edges incident on B not used in Figure 8.9. Figure 8.11 Another subgraph of Figure 8.6, with no edges in common with Figure 8.9, satisfying (8.1) and (8.2). Seat everyone but P at k tables. Edge e1 is associated with the unordered pair {Gre, She} of vertices, and edge e10 is associated with the unordered pair {Cas, Dou} of vertices. By the Pigeonhole Principle, some pigeonhole (letter) will contain at least two pigeons (students), that is, at least two students have first names that start with the same letter. The difference is that instead of obtaining equality [F(n) + an+1 = F(n + 1) in the preceding discussion], you obtain an inequality. Prove that for all x, y ∈ R, if x is rational and y is irrational, then x + y is irrational. 531 Trees We use induction to show that for all i = 0, . d d Since 0 ≤ r/d < 1, n r = q+ = q. In expressions involving some or all of the operators ¬, ∧, and ∨, in the absence of parentheses, we first evaluate ¬, then ∧, and then ∨. Prove that your algorithm in Exercise 18 is correct. If an activity can be constructed in t successive steps and step 1 can be done in n1 ways, step 2 can be done in n2 ways, . Is a nondecreasing? Since there are 2 appetizers, 3 main course items, and 4 beverages, and no item belongs to two categories, the total number of items available is 2 + 3 + 4 = 9. , si , which allowed us to prove that Algorithm 1.2 is correct. Find the probability of obtaining exactly one defective microprocessor. X. Is a decreasing? If you fail, the graph is not bipartite. Show that for all n ≥ 2, fn = 22. Therefore 2 x < 2x + 2 ≤ 2x + 2, so 2 x − 2 < 2x . What rule of logic does proof by resolution use? First, notice that any particular stacking can be represented by two graphs, one representing the front/back colors and the other representing the left/right colors. a c g d 3. Show that the number of solutions in nonnegative integers of the inequality x1 + x2 + · · · + xn ≤ M, where M is a nonnegative integer, is C(M + n, n). Section 8 w5 w4 g f e G2 22. By the Pigeonhole Principle, two are the same. Another common introduction is: Assume by way of contradiction that · · ·. Does this contradict the result preceding Example 3.11? If no two persons have the same birthday, the second person’s birthday can occur on any day except the day of the first person’s birthday. Since the first quantifier is ∃, you go first and choose a value for x. If V ⊆ P, as shown the capacity of the cut is n. TABLE 3.1 ■ Relation of Students to Courses Definition 3.1 Student Course Bill Mary Bill Beth Beth Dave CompSci Math Art History CompSci Math A (binary) relation R from a set X to a set Y is a subset of the Cartesian product X × Y . Example 6.9 Since “has a vertex of degree 3” is an invariant, the graphs G1 and G2 of Figure 6.4 are not isomorphic; G1 has vertices (a and f) of degree 3, but G2 does not have a vertex of degree 3. Express the relation given by Table 6.6 as a set of n-tuples. Explain the rules for playing the logic game. Thus there exists q such that 3. , n − 1. (a, c, e, b, c, d, e, f, d, b, a) i 11. This operation is expressed as PLAYER [ID Number = PID] ASSIGNMENT. You’re trying to find out what your father hired me to find out and I’m trying to find out why you want to find out. (b) The strings that begin with the letter B can be constructed in four successive steps: Choose the first letter; choose the second letter; choose the third letter; and choose the fourth letter. 6 ➜ Relational Databases The “bi” in a binary relation R refers to the fact that R has two columns when we write R as a table. 10 11 a c 6 12 8 7 6 8 10 FcA = 0, Fbc = 2, Fw3 ,d = 2, 14 h 12 10 9 6 10 k 6 FbA Fa,w2 FcB Fw3 ,d = 4, = 3, = 1, = 3, z 2 12 8 10 4 8 = 1, = 6, = 0, = 3, 10 14 12 7 Fw1 ,b FAz Fbc Fa,w3 19. We assume that the input to insertion sort is s1 , . C(n, 1) = n 29. Find the eccentricity of each vertex in the tree of Figure 1.5. A vertex v in a tree T is a center for T if the eccentricity of v is minimal. for ik = 1 to ik−1 println(i1 , i2 , . Proof Let m and n be arbitrary integers, and suppose that m is odd and n is even. We call |X| the cardinality of X. Definition 1.1 allows us to reuse elements in Y . Now T ∗ is an acyclic, connected graph with n vertices. Let C be a circle and let D be the set of all diameters of C. It follows that Y1 = Y2 and S is one-to-one. If R(S) = {w1 , . , n} that contain an odd number of elements. Let G be a connected graph with four vertices v1 , v2 , v3 , and v4 of odd degree. swap(sm−1 , sn) 17. Since U is empty, the algorithm terminates. The integers k1 and k2 need not be equal. Thus all the diagonal elements of An are equal. Then p(n) = (nk). Logical methods are used in mathematics to prove theorems and in computer science to prove that programs do what they are alleged to do. b 540 c d e 21. 26. The join condition specifies a relationship between an attribute in R1 and an attribute in R2 (see Example 6.5). Use mathematical induction to show that for all n ≥ 6, n−1 fn > 3 2 . A Class B Internet address, used for medium-sized networks, is a bit string of length 32. We showed that the graph of Figure 4.1 has no complete matching. In this case, the for loops run to completion, after which the code returns true [to indicate that ∀x∀y P(x, y) is true] and terminates. Section 4 In Exercises 13–16, assume that we choose primes p = 13, q = 17, and n = 19. a 1 a, 3, a 2 a 1, a 2, a 3 No a 2, a 3, a 1 a 2, a 1, a 3 Yes a 1 < a 3? Theorem 7.3 If f(n) is the number of comparisons needed to sort n items in the worst case by a sorting algorithm, then f(n) = (n lg n). If d divides n, we write d | n. R × Znonneg 48. In Example 1.9 we added the numbers of eight-bit strings (32 and 32) of each type to determine the final result. If S is a family of sets, how do we define the intersection of S? But this can be done in C(n, k) ways, since C(n, k) counts the number of ways of selecting k things from n items. We must prove that tn = 1. 55. ¬(∀x P(x)) 33. Give other examples of recursive integration formulas. (a) In how many ways can we select eight balls? Suppose that e = 1. We are to show that ai = aj , for some i = j. are called indexes. Develop an evaluation function for Mu Torere. Let A = {1, 4, 7, 10}, B = {1, 2, 3, 4, 5}, and C = {2, 4, 6, 8}. , sn) } // print the first permutation for i = 2 to n! { m=n−1 while (sm > sm+1) // find the first decrease working from the right m=m−1 k=n 343 Counting Methods and the Pigeonhole Principle 11. ■ If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Prove that if we insert parentheses in any manner whatsoever and then multiply the n numbers a1 a2 · · · an , we perform n − 1 multiplications. Define transitive relation. The relation Mercury Venus Earth Mars Jupiter Saturn Uranus Neptune 1 2 3 4 5 6 7 8 9. We examine the blocks one by one. If at any point the child to move to is missing, we conclude that D is not in the tree. Similarly, x2 = x2 . 2 Since n(n + 1) 2(n + 1) n(n + 1) + (n + 1) = + 2 2 2 n(n + 1) + 2(n + 1) = 2 (n + 1)(n + 2) = , 2 we have Sn+1 = (n + 1)(n + 2) . At this point, the left and right faces are randomly arranged; however, we will show how to correctly orient the left and right faces without altering the colors of the front and back faces. Since n mod d = 9 mod 2 = 1 is not equal to 0, d is incremented and becomes 3. Figure 5.1 Rolling a fair die. The last equation follows from the fact that there is a one-to-one, onto function from the set of i-vertex binary trees to the set of (2i + 1)-vertex full binary trees. The statement ∃x P(x) is false if P(x) is false for every x in D. Notice that in Theorem 5.6 the remainder r is zero if and only if d divides n. Suppose that the 4-cube is planar. {{1}, {2, 4}, {3}} In Exercises 21–23, let X = {1, 2, 3, 4, 5}, Y = {3, 4}, and C = {1, 3}. if for all x, y ∈ X, if (x, y) ∈ R and x = c d 14. The flow between two vertices, neither of which is a or z, can be in either direction. Use the Binomial Theorem to show that n 13. , ak ∈ X, such that f(a1) = f(a2) = · · · = f(ak). Exercises 50–53 refer to the following algorithm to solve the n-disk, four-peg Tower of Hanoi puzzle. We model the situation as a graph. Another idea is to allow a variable-depth search in which the search backtracks when it reaches an unpromising position as measured by some function. Now n2 9. , xn } containing r elements 9. In a direct proof the negated conclusion is not assumed, whereas in a proof by contradiction the negated conclusion is assumed. The argument is valid provided that if p1 and p2 and · · · and pn are all true, then q must also be true; otherwise, the argument is invalid (or a fallacy). (The dinner consisting of a main course whose first letter is X and a beverage whose first letter is Y is denoted XY . Give an example to show that the converse of Exercise 35 is false, even if G is assumed to be connected. By Exercise 24, the call bin tree isom(lc r1 , lc r2) requires 6(k − 1) + 2 comparisons. , Z, +, −, ∗, /}, define r(x1 · · · xn) = r(x1) + · · · + r(xn). The statement ∀x P(x) is true if P(x) is true for every x in D. The increase from time n − 1 to time n is dn − dn−1 . Now an upper bound a for X satisfies x ≤ a for every x in X—here, we must have 1 ≤ a, 2 ≤ a, 3 ≤ a, 4 ≤ a. , sn in nondecreasing order by first selecting the largest item and placing it last and then recursively sorting the remaining elements. How many dinners at Kay’s Quick Lunch (Figure 1.1) consist of one appetizer, one main course, and an optional beverage? , en , vn) be a cycle from v to v where v = v0 = vn (see Figure 2.12). Network Models W A X B Y C Z Figure 4.2 The black lines show a maximal matching (the maximum number of edges are used) and a complete matching (each of A, B, and C is matched). How is the concatenation of strings α and β denoted? Thus, 2n + 3 lg n = (n). Let T be a binary tree of height h > 0 with t terminal vertices. What is a nonconstructive existence proof? Since each step can be done in three ways, the number of ordered pairs (A, B) satisfying A ⊆ B ⊆ X is 3 · 3· · · 3 = 3n . Example 6.17 Intuitively, if we flip a fair coin twice, the outcome of the second toss does not depend on the outcome of the first toss (after all, coins have no memory). Draw the complete game tree for the version of nim described in Exercise 1. Fortunately, in many important cases, the polynomial bound has small degree. Solve the recurrence relation of Exercise 55 in case n is a power of 2. Computer Exercises In Exercises 1–6, assume that a set X of n elements is represented as an array A of size at least n + 1. A common divisor of m and n is an integer that divides both m and n. Hamiltonian cycle Traveling salesperson problem Ring model for parallel computation Gray code Section 4 43. Union of a family S of sets: ∪ S = {x | x ∈ X for some X ∈ S} 14. Show that there are n Sn,k k=1 equivalence relations on an n-element set. 47. Suppose that we have a propositional function S(n) whose domain of discourse is the set of integers greater than or equal to n0 . Also, graphs G1 and G2 are homeomorphic if G1 can be reduced to a graph isomorphic to G2 or if G2 can be reduced to a graph isomorphic to G1 . ∴ I study hard. The formal algorithm follows. Verify the second of De Morgan’s laws ¬(p ∧ q) ≡ ¬p ∨ ¬q. In fact, since graphs are drawn with dots and lines, they look like road maps. The domain of discourse could be taken to be all integers. Let S be an optimal solution. Our next examples illustrate the use of the second form of the Pigeonhole Principle. A straightforward approach is to examine all subsets of {1, 2, . By (1.2), a1 = 5; thus a2 = a1 + 3 = 5 + 3 = 8. Prove that if X ⊆ Y , then Y − (Y − X) = X for all sets X and Y . A common divisor of m and n, not both zero, is an integer that divides both m and n. Since x = 3 or x = 1, x ∈ B. If so, prove it; otherwise, give a counterexample. By writing the truth tables for P = ¬(p ∨ q) and Q = ¬p ∧ ¬q, we can verify that, given any truth values of p and q, either P and Q are both true or P and Q are both false: p q ¬(p ∨ q) ¬p ∧ ¬q T T F F T F T F F F F T F F F T T Thus P and Q are logically equivalent. Therefore, the graph of Figure 3.6 does not contain a Hamiltonian cycle. f is both one-to-one and onto. (a) Find all subgraphs of Figure 8.5 satisfying properties (8.1) and (8.2). Prove that n + 2 < Cn for all n ≥ 4. Let bn denote the number of nonisomorphic, n-vertex full binary trees. Use Exercise 21 to find the matrix of the relation R1 ∪ R2 relative to the ordering 1, 2, 3. One application of partial orders is to task scheduling. , k − 1, and i=0 bi = 1. Draw a decision tree that gives an algorithm that identifies in at most three weighings the bad coin and determines whether it is heavier or lighter than the others using only a pan balance. The equivalence classes of X given by R are sets of the form {x ∈ X | xRa}, where a ∈ X. This contradiction shows that if a composite integer n is input to Algorithm 1.8, the divisor returned is prime. Lynn does not love Microsoft. ∀x∃y T3 (x, y) 21. Counting Methods and the Pigeonhole Principle 51. In this case T0 is contained in every minimal spanning tree. How many distinct equivalence classes are there? Therefore, N2 = 4. Show that an ≤ bn for n = 1, 2, 3, . In the remainder of this section we consider binary tree representations of arithmetic expressions. But this leaves only four edges—not enough for a Hamiltonian cycle of length 5. ■ Generality The algorithm applies to a set of inputs. We can now conclude that 1 + 2 + · · · + n = (n2). , n} be arranged in 36. Second Form: In the previous example, let X be the set of persons, and let Y be the set of rooms. , sn } each prefixed by α [To list all r-combinations of {s1 , s2 , . We turn the method of Example 2.2 into an algorithm. In Figure 8.5, not counting loops, Graph Theory vertex W has degree 3 and, therefore, Figure 8.5 does not have a solution to the modified version of Instant Insanity. Let X be an n-element set and choose x ∈ X. Proving an inequality is handled in a similar fashion. If ∀x P(x) is false, the for loop terminates as soon as one element x of the domain of discourse is found for which P(x) is false. For example, the string baac is different from the string acab. The result is ci , . Give an example of a bijection. If one pile remains, take it. Formal Solution (a) We can encode each route as a string of n R’s (right) and m U’s (up). (f) Show that n (g) Show that Sn,2 = 2n−1 − 1 for all n ≥ 2. The last three digits must be a permutation of {2, 4, 5}. Problem-Solving Tips To verify that two sets A and B are equal, written A = B, show that for every x, if x ∈ A, then x ∈ B, and if x ∈ B, then x ∈ A. Prove De Morgan’s laws for sets. The statement is true if, for every x ∈ X, there is at least one y ∈ Y for which P(x, y) is true. The task is to determine the number of bad coins. Write a recursive algorithm to solve Pn similar to Algorithm 3.8. ever m divides n, an = can/m + d, where c and d are real numbers satisfying c > 1 and d > 0, and m is an integer satisfying m > 1. (b) Alfred Hitchcock won an Academy Award in 1940 for directing Rebecca. Solve the problem of Exercise 22 for n = 1, 2, 3, 4. Recall that the Catalan number Cn is equal to the number of routes from the lower-left corner of an n × n square grid to the upper-right corner if we are restricted to traveling only to the right or upward and if we are allowed to touch but not go above a diagonal line from the lower-left corner to the upper-right corner. Give an example of a relation on {1, 2, 3, 4} that is reflexive, not antisymmetric, and not transitive. c: Include (c, e). ∃n P(n) Section 4 Section 6 13. State and prove a result analogous to Exercise 105 for x. (4.8) Then S(n) is true for every positive integer n. Contrapositive: 5. Definition 3.8 If p and q are propositions, the proposition p if and only if q is called a biconditional proposition and is denoted p ↔ q. 3 − 1 = 2 34. Which are simple paths? If 4 < 6, then 9 > 12. v2 v1 e 10 v2 e1 e 11 G v3 e8 v4 v6 e5 v7 Figure 2.3 A subgraph of the graph of Figure 2.4. e2 e9 v5 v3 e8 v4 e7 v1 e 10 v8 e 11 e6 v6 e3 e5 e4 v7 G Figure 2.4 A graph, one of whose subgraphs is shown in Figure 2.3. 447 Graph Theory Example 2.10 Find all subgraphs of the graph G of Figure 2.5 having at least one vertex. The algorithm is based on preorder traversal (see Section 6). Arbitrarily assign one of these colors, say green, to the left. If we weigh four coins against four coins and they balance, we must then determine the bad coin from the remaining four coins. Therefore, the universally quantified statement for every real number x, if x > 1, then x + 1 > 1 is true. Consider a string of length n that contains an even number of 1’s that begins with 0. satisfies the same recurrence relation as the Catalan sequence C1 , C2 , . and pn are all true, then q must also be true. Each time the initial value of n is doubled, the statement x = x + 1 is executed one more time. Algorithm 3.19 Matrix Multiplication This algorithm computes the product C of the n × n matrices A and B directly from the definition of matrix multiplication. (3.11) Notice that (3.9) gives From (3.8), (3.10), and (3.11), we deduce that n < (k − 2)2k−1 + 1 = a2k−1 2 ≤ an ≤ a2k ≤ k2k + 1 ≤ (1 + lg n)2n + 1 = O(n lg n). Let T consist of the single vertex a and no edges. Algorithm 2.19 Exponentiation Mod z By Repeated Squaring This algorithm computes an mod z using repeated squaring. Given that log3/2 100 = 11.357747, provide an upper bound for the number of modulus operations required by the Euclidean algorithm for integers in the range 0 to 100,000,000. P = p → q, Q = ¬p ∨ q 63. Is q nondecreasing? Construct a backtracking algorithm to solve the four-queens problem. We have found the spanning tree shown in Figure 3.1. We formalize the method of Example 3.5 as Algorithm 3.6. Algorithm 3.6 Breadth-First Search for a Spanning Tree This algorithm finds a spanning tree using the breadth-first search method. S1 = 2, S2 = 4, S3 = 7, S4 = 12 n ≥ 0. Suppose that we have two sequences each of size n sorted in nondecreasing order. Figure 1.2 shows why we multiply 3 times 4—we have three groups of four objects. Therefore 1 ,b1) min(a2 ,b2) pmin(a p2 · · · pkmin(ak ,bk) 1 is the greatest common divisor of m and n. d b a 2. a, b, c, d, e 22. The graph is surely connected. If I don’t get rich, then I don’t get A’s. Give an argument that shows that the professor must be mistaken. In words, 6 is composite (i.e., not prime). If we cancel n, we obtain 588 C 2, L C1 : C6 C3 : C1 C 4, L C1 : C2 C 1, L C3 : C1 C 6, H C 3, H C 4, H C lg n ≤ 100 for all n ≥ 1, which is false. Therefore, U is a solution of (2.13). So suppose that x is rational. Find b1 , b2 , b3 , and b4 . What kind of labeling corresponds to a complete matching? 141 Functions, Sequences, and Relations Example 1.20 We have 36844/2427 = 15.18088 . , O2n−1 ∪ {n + 1} cos α sin(α − β) + cos 2α sin β = cos(α + β) sin α. For example, if the input is a set containing n elements, we would say that the size of the input is n. Since 2456 is the smallest string that begins 24 and represents a 4-combination of X, the answer is 2456. Show how merge sort (Algorithm 3.8) sorts the sequence 1, 9, 7, 3. Then h+2 9. If some even number greater than 2 is not the sum of two prime numbers, the algorithm will stop and output “no”. 550 Trees This is a contradiction. For which values of n is the n-cube a tree? loge n < 1 + + · · · + 2 that n−1 (c) Use parts (a) and (b) to show 47. Draw a decision tree that gives an algorithm that identifies in at most two weighings the bad coin (but not necessarily determines whether it is heavier or lighter than the others) using only a pan balance. (n+1)/2 fn+2 = 1. , pn and q: q1 , . Example 3.8 The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is not reflexive. large = s2 22. ((C ∩ A) − (B − A)) ∩ C 64. Proof The proof is by induction on the number n of edges in G. Antisymmetric 27. Define the radius r of a tree using the concepts of eccentricity and center. Explain the relationship between a partition of a set and an equivalence relation. Our last example shows how to modify Algorithm 2.4 to generate a maximal flow from a given flow. 4 ➜ A Shortest-Path Algorithm Recall (see Section 1) that a weighted graph is a graph in which values are assigned to the edges and that the length of a path in a weighted graph is the sum of the weights of the edges in the path. E + Dependent on how “process” is interpreted postorder (PT) { if (PT == null) return l = left child of PT postorder (l) r = right child of PT postorder (r) process PT } + * A B Input: PT , the root of a binary tree Output: Dependent on how “process” is interpreted swap children (PT) { if (PT == null) return swap the left and right children of PT l = left child of PT swap children(l) r = right child of PT swap children(r) } 581 Trees 31. The second term is 8 because instruction 2 says to add 3 to 5 to get the next term, 8. However, not signing a right-handed relief pitcher guarantees that they will not win the World Series. Prove the associative laws for sets. There are 16 equivalence classes. The generalized De Morgan’s laws for logic (Theorem 5.14) can be used to negate a proposition containing nested quantifiers. What is proof by contrapositive? 3 46. Show that the time for selection sort (see Exercise 20) is the same for all inputs of size n. Give an example that illustrates why we might want to sort a sequence. (c) How many strings of part (a) do not begin with the letter B? Which of the following must also be true? Formulate the proposition p ∧ (¬q ∨ r) in words using {x | x is an integer and 1 < x ≤ 3} equal? (b) Use part (a) to deduce that the maximum of C(n, k) for k = 0, 1, . Draw a picture like that in Figure 1.11 to show how a 3-cube may be constructed from two 2-cubes. Example 2.3 We will prove that √ 2 is irrational using proof by contradiction. The domain of each function is the set of all real numbers. Efficiently computing this inverse is required by the RSA cryptosystem in Section 4. Use Exercises 29–31 to show that (n+1)Cn = 2(2n−1)Cn−1 for all n ≥ 2. Write a program whose input is a 2n × 2n board with one missing square and whose output is a tiling of the board by trominoes. Our last example illustrates the use of the third form of the Pigeonhole Principle. Let G be a graph and let v and w be distinct vertices. Solve the traveling salesperson problem for the graph. For example, since lg(2,000,000 + 1) = 21, it is possible to store 2 million items in a binary search tree and find an item, or determine that it is not present, in at most 21 steps. After the swap we have Section 2 Now 20 is inserted j We next swap ai and aj , where i = 2 and j = rand(2, 5) = 4. Notice that cubes 2 and 3 have the same colors on opposing sides. Is v nonincreasing? n factors Example 1.7 How many reflexive relations are there on an n-element set? Draw all nonisomorphic free trees having four vertices. 68. Given the hypotheses 1, 2, and 3, we have proved the conclusion d. What is the value of S(X)? At this point, U = {c, d}. At line 21, we add z to U. Since r = 0, the algorithm terminates, having found the greatest common divisor of 273 and 110 to be 1. We will see later that T is a minimal spanning tree for G. But this is just the number of ways C(8, 2) = 28 of selecting two positions for the |’s from eight possible positions. (All UPC codes have a check digit. Suggested games are Cribbage, Othello, The Mill, Battleship, and Kalah. Therefore, g = 1. reassigned values of the outcomes in E ∩ F , we obtain the value of P(E | F): P(E ∩ F) . 36 The event E1 ∩ E2 is “get doubles and get a sum of 6.” Since this last event can occur only one way (by getting a pair of 3s), P(E1 ∩ E2) = 352 1 . = b1 + 1 + 2 + · · · + (n − 2) + (n − 1) (n − 1)n = 0 + 1 + 2 + · · · + (n − 1) = = (n2). 2 2 Suppose that k > m/2. 9 Sets and Logic At the beginning of this section, we pointed out that a set is an unordered collection of elements; that is, a set is determined by its elements and not by any particular order in which the elements are listed. , 4. Recurrence Relations 5. Notice that the graph of Figure 3.6 does have a Hamiltonian cycle. Give an example of a graph that does not have a Hamiltonian cycle but does have an Euler cycle. A counterexample is n = 3 since 23 + 1 = 9, which is not prime. (b) Let a be an upper bound for a set X of real numbers. Consider ab . Given a quantified propositional function, you and your opponent, whom we call Farley, play a logic game. Next, we find that the value of E is 6. Once these selections have been made, there is one position left to be filled by the M. However, when d = 11, d does divide 451 and the condition n mod d == 0 is true. The call bin tree isom(rc r1 , rc r2) requires four comparisons. This corollary follows immediately from the first theorem of Example 1.3. Example 1.5 Example 1.6 Examples of theorems about real numbers are ■ x · 0 = 0 for every real number x. 6427153 16. 111. V = {v1 , v2 , v3 , v4 }. In how many ways can these books be arranged on a shelf if all five computer science books are on the left? 01110100110 16. We obtain the formula n n(n + 1) n2 + +0= , 2 2 2 which can be proved using mathematical induction. Prove or disprove: For all x, y ∈ R, if x is rational and y is irrational, then xy is irrational. 418 3. Then any distribution of S among the n − 1 persons in which no one has his or her own order yields a distribution in which q does not have the order that is really p’s. Then the algorithm requires h weighings in the worst case so h = f(n). ¬a ∨ c 3. The algorithm returns result, which is equal to a16 . Find the permutation that will be generated by Algorithm 4.14 after 625431. Attacking the Problem Let’s begin by looking at some specific eight-bit strings that are related according to the relation R. Since any path of length n > 2 from v1 to v1 begins with the loop (v1 , v1) or the edge (v1 , v2), it follows that sn = sn−1 + sn−2 . Write a backtracking algorithm to solve the subset-sum problem. C(31, 7) 2629575 4 · C(13, 6) · 3 · C(13, 5) · 2 · C(13, 2) 20. If tj is 1, there is an edge from vertex j to vertex k (see Figure 5.3). Thus, it suffices to show that (n + 1) lg n ≤ n2 for all n ≥ 1. Notes Most general references on discrete mathematics address the topics of this chapter. If a person invests d dollars at p percent compounded annually and we let An be the amount of money earned after n years, the recurrence relation An = 1 + p An−1 100 5. If you are trying to construct a direct proof of a statement of

the form p → q and you seem to be getting stuck, try a proof by contradiction. The error in this argument is that if we eliminate two edges incident at c (as we must do), we also eliminate edges incident at two of a, b, d, or e. Corollary 6.5 Let G1 and G2 be simple graphs. We next give a proof using the Multiplication Principle that a set with n elements has 2n subsets. It is not known whether there is a polynomial-time algorithm that can find a factor of a given integer; but most computer scientists think that there is no such algorithm. Show that R is reflexive, symmetric, and transitive by directly verifying that R satisfies the definitions of reflexive, symmetric, and transitive. An equivalence relation is a relation that is reflexive, symmetric, and transitive. x becomes a2 , and n becomes 8. 2 2 49. For example, the 3-combination {6, 2, 4} will be listed as 246. Everyone "knows" what an even or odd integer is, but the following definition makes these terms precise and provides a formal way to use the terms "even integer" and "odd integer" in proofs. Each processor has its own local memory. The Basis Step is complete. Region 8 represents elements in Z, but in neither X nor Y . w1 w2 Figure 8.6 Adding a third edge to the graph of Figure 8.5. Now suppose that T is a tree with five vertices and the maximum vertex degree that occurs is 2. An operating system (e.g., Vista), for example, is better thought of as a program that never terminates rather than as a finite program with input and output. Trees Richard Johnsonbaugh 507 10. Let X be a nonempty set of nonnegative integers that has an upper bound. Thus the desired permutation begins 164. Because g(n) = (h(n)), there are constants C3 and C4 such that C3 |h(n)| ≤ |g(n)| ≤ C4 |h(n)| for all but finitely many positive integers n. We must find an element m ∈ Y such that for all n ∈ X, f(n) = m. Let X = {a, b}. Sometimes a recurrence relation that is not a linear homogeneous equation with constant coefficients can be transformed into a linear homogeneous equation with constant coefficients. 179 Functions, Sequences, and Relations The remaining equivalence classes are found similarly: [3] = [5] = {1, 3, 5}, [2] = [6] = {2, 6}, [4] = {4}. Thus a theta notation for the number of times the statement x = x + 1 is executed is (n2). Implement Algorithm 8.13, which tests whether two binary trees are isomorphic, as a program. Thus statement (d) is a propositional function. Therefore, G is connected. Each column that contains 1 in row x corresponds to an element of the equivalence class containing x. Express each hexadecimal number in Exercises 26, 27, and 29 in binary. We first check that 558 Trees (a) (b) Two vertices in left subtree (c) (d) (e) (f) (g) Three vertices in right subtree (a) (c) (a) (a) (d) (b) (c) (b) (a) (e) (b) (d) (a) (f) (b) (e) (g) (b) (f) (g) Figure 8.14 The proof of Theorem 8.12 for the case n = 6 vertices and k = 2 vertices in the left subtree. Suppose that T has n vertices. Find the siblings of Ares. What is the traveling salesperson problem? 2i + 1 7. On the other hand, ∃x∀yP(x, y) is false. This time we succeed in adding the edge (e, g). Example 4.11 The 4-combinations of {1, 2, 3, 4, 5, 6} as listed by Algorithm 4.9 are 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1356, 1456, 2345, 2346, 2356, 2456, 3456. The Ti are components of T. Example 3.2 shows that a relation can be given by simply specifying which ordered pairs belong to the relation. Since there are i internal vertices, each having two children, there are 2i children. Mathematical induction can show that postage of four cents or more can be achieved by using only 2-cent and 5-cent stamps. {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)}, [1] = [2] = [3] = {1, 2, 3}, [4] = {4} 22. Problem-Solving Tips This section introduces some useful terminology and provides several different characterizations of trees. Let T(n) denote the number of moves required by this algorithm. Input: PT , the root of a binary tree Output: 580 Dependent on how "process" is interpreted Dependent on how "process" is interpreted Trees inorder (PT) { if (PT == null) return l = left child of PT inorder(l) process PT r = right child of PT inorder (r) } 5. (b) There exists k ∈ Z such that n = 2k − 1. Represent the flow of traffic from A to C during the period 6:00 p.m. to 7:00 p.m. as a network. Implement Algorithm 2.19, exponentiation mod z, as a program. How many selections are there in which Max is president or secretary? "Fair" means that each number is equally likely to appear when the die is rolled. Example 1.21 We can find the least common multiple of 30 and 105 by looking at their prime factorizations 30 = 2 · 3 · 5 105 = 3 · 5 · 7. Stay with the door initially chosen. How does the merge algorithm work? Each computer interface on the Internet is identified by an Internet address. A selection is uniquely determined by the number of each type of book selected. The distinct two-digit numbers that contain 7 are 17, 27, . The Braille system of representing characters was developed early in the nineteenth century by Louis Braille. The domain of each function is Z × Z. Finally, We turn now to the Inductive Step. Similarly, T cannot contain distinct edges e1 and e2 incident on v and w because we would then have the cycle (v, e1 , w, e2 , v). Example 1.16 The sets {1, 4, 5} and {2, 6} are disjoint. (1) If T is connected but the removal of any edge (but no vertices) from T disconnects T, then T is a tree. (a) Show that R is an equivalence relation on X × X. Then the (n − 1)-bit string following the initial 0 does not contain the pattern 111. To obtain a recurrence relation for the sequence b1 , b2 , . We have shown that if n is 1, the algorithm is invoked twice. For example, p ∨ q → ¬r is interpreted as (p ∨ q) → (¬r). If a vertex v is in the mesh, we let vx denote the horizontal coordinate of v and vy denote the vertical coordinate of v. The algorithm then continues as written. [Project] Report on algorithms different from those presented in this chapter for generating combinations and permutations. n+1 1 1 + =1− (n + 1)! (n + 2)! (n + 2)! 2n+2 = 2 · 2n+1 < 2[1 + (n + 1)2n] = 2 + (n + 1)2n+1 = 1 + [1 + (n + 1)2n+1] < 1 + [2n+1 + (n + 1)2n+1] = 1 + (n + 2)2n+1 17. Now U = ∅. Thus dec val becomes 1. Since nonnegative integers cannot decrease indefinitely, eventually b becomes zero and the loop terminates. Then there are at most k − 1 values x ∈ X with f(x) = y1 ; there are at most k − 1 values x ∈ X with f(x) = y2 ; . 2 w 6 x 3 23. P = p ∧ (¬q ∨ r), Q = p ∨ (q ∧ ¬r) 64. If R and S are symmetric, then R ◦ S is symmetric. The expression will be unambiguously evaluated. We consider M to be a rectangular array in ordinary 2-space with p vertices in the horizontal direction and q vertices in the vertical direction (see Figure 6.2). The increase from time n − 1 to time n is dn − dn−1 , and the increase from time n − 2 to time n − 1 is dn−1 − dn−2 . 104. 198 Write a program that tests whether A is one-to-one. The decision tree analysis shows that any algorithm requires at least seven comparisons in the worst case to sort five items. Composition sometimes allows us to decompose complicated functions into simpler functions. We use strong induction. True statements are marked with ×. 86. 4! = 24 4. Use your program of Exercise 4 to encode some sample text. (c) vn is a child of vn−1 . Give an example of a recursive function. The student will have to use logic to prove that the program is correct. (An efficient way to compute s is given in Section 3.) The number s is kept secret and used to decrypt messages. The flow is maximal. Let x be a real number. , where Sn denotes the number of n-bit strings that do not contain the pattern 010. 575 Trees 22. Since f(2, 1) = 2 − 1 = 1 = 3 − 2 = f(3, 2), f is not one-to-one. In how many ways can † Exercise 314 these committees send one member to meet with the president? 455 Graph Theory 52. The binomial smoother of size k is the sequence Bk−1 B0 ,..., n , 2n 2 i . GUIDE 2. Consider ∀x∃y P(x, y). Take time to let the words sink in. Show that there are n(n − 1)[(n − 1)k − 1] n−2 paths whose lengths are between 1 and k, inclusive, in Kn , n > 2. This is certainly the case! For R to be symmetric, for all eight-bit strings s1 and s2 , if s1 R s2 , then s2 R s1 . (e) Aphrodite and Prometheus are siblings. Explain how to use an arrow diagram to determine whether a function is onto. It remains to prove √ If n has a divisor d satisfying 2 ≤ d ≤ n, then n is composite. Also b ∈ Y1 if and only if s2 = 1 if and only if b ∈ Y2 . Assuming a universe represented as an array, write a program to represent the set X, given the array representing X. State the formulas that can be used to generate Pascal's triangle. In how many ways can a diner choose one item from among the appetizers and beverages at Kay's Quick Lunch (Figure 6.1)? Proof If m = 1, then gcd(m, n) = 1 and lcm(m, n) = n, so gcd(m, n) · lcm(m, n) = 1 · n = mn. In this case, α = aβb or α = bβa, where β ∈ L. Is it possible for neither version of Instant Insanity to have a solution even if each cube contains all four colors? Show that an ≤ 8n for n = 1, 2, 3, . Someone is taller than or the same height as someone. Let {an }∞ n=1 be a nondecreasing sequence, which is bounded above, and let L be the least upper bound of the set {an | n = 1, 2, . (a, b, c, j, i, m, k, d, e, f, l, g, h, a) Graph Theory 9. Therefore, all integers are positive or all positive real numbers are integers. 441 Graph Theory Example 1.13 The graph in Figure 1.14 is not bipartite. The set with no elements is called the empty (or null or void) set and is denoted ∅. (s − r)4 = C(4, 0)s4 + C(4, 1)s3 (−r) + C(4, 2)s2 (−r)2 + C(4, 3)s(−r)3 + C(4, 4)(−r)4 = s4 − 4s3 r + 6s2 r 2 − 4sr 3 + r 4 382 C(n, k)2n−k (−1)k . Therefore x ∈ A. If x ∈ A, then x∈ / B. We call (V , E) a subgraph of G if (a) V ⊆ V and E ⊆ E. (a) S1 = 0 = 2; 2 + · · · + 2n + 2(n + 1) = Sn + 2n + 2 = (n + 2)(n − 1) + 2n + 2 = (n + 3)n = Sn+1 . Since there are three pairs of rows and four red colorful pairs, a third application of the Pigeonhole Principle shows that at least two columns contain red colorful pairs in the same rows. This too is certainly the case! We have proved that R is an equivalence relation. This is easily accomplished, since 1! = 1 ≥ 1 = 21−1 . This replacement is justified by recalling that individual hypotheses h1 and h2 are equivalent to h1 h2 . Find the prime factorization of 11!. Since walk(1) = f2 , walk(2) = f3 , and walk(n) = walk(n − 1) + walk(n − 2), fn = fn−1 + fn−2 for all n ≥ 3, it follows that walk(n) = fn+1 for all n ≥ 1. 549 Trees Yes a 1 < a 2? Basis Step (n = 1) c1 = 0 < 2 = 2 · 1. Don't forget that the negation of p → q is equivalent to p ∧ ¬q. Example 3.5 Figure 3.9 The ring model for parallel computation. Then n − d(q + 1) = n − dq − d = r − d ≥ 0. Let V be the set of vertices on even levels and let W be the set of vertices on odd levels. , −2, −1, 0, 1, 2, . C(52, 6) 52 · 51 · 50 · 49 · 48 · 47 Example 5.6 A bridge hand consists of 13 cards from an ordinary 52-card deck. Exercises 7–9 refer to an alternate version of Prim's Algorithm (Algorithm 4.6). CX∪Y (x) = CX (x) + CY (x) If x ∈ / X ∪ Y , then is true for all x ∈ U. Attacking the Problem Let's be clear about what we have to prove. ■ List the r-combinations not containing s1 . Since z is not labeled and U is not empty, we move to line 14, where we choose vertex a in U and remove it from U at line 15. If (x, y) and (y, z) happen to both be in R, we could use the fact that R is transitive to conclude that (x, z) ∈ R and, therefore, (x, z) ∈ R ∪ S. 520 Statements (5) and (6) immediately tell us that T is connected. Trace selection sort (see Exercise 20) for the input of Exercises 4–7. Proof Let X1 , X2 , . (2.4) Subsequence (2.3) is obtained from sequence (2.4) by choosing the third and fourth terms. Set equality is denoted X = Y . In Exercises 125–127, x1 , x2 , . P(E1 ∪ E2) = P(E1) + P(E2). Prove that ta is the greatest lower bound of the set {tx | x ∈ X}. [Project] Report on the formulas for the number of nonisomorphic free trees and for the number of nonisomorphic rooted trees with n vertices (see [Deo]). In some cases, two different propositions have the same truth values no matter what truth values their constituent propositions have. {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)} is an equivalence relation on {1, 2, 3}. Each of P(1), P(2), . Show that if gcd(p, q) > 1, it is false that there exists n such that for all k ≥ n, postage of k cents can be achieved by using only p-cent and q-cent stamps. algor(w, a, z) { length = 0 v=a T = set of all vertices while (v¬ = z) { T = T − {v} choose x ∈ T with minimum w(v, x) length = length + w(v, x) v=x } return length } 11. 229 Algorithms The best-case time can be analyzed as follows. , sn . 44. Because of the floor function, the computations are simplified if n is a power of 2. Given a flow in a network, what is the relation between the flow out of the source and the flow into the sink? Therefore, each consecutive pair of strings in Gn−1 differs in exactly one bit. If a is the least upper bound of a set X, we sometimes write a = lub X. Explain how to solve a second-order, linear homogeneous recurrence relation with constant coefficients. Euler's original paper on the K¨onigsberg bridges, edited by J. It follows that there are Dn−1 + Dn−2 distributions in which p has q's order. Show that if all weights in a connected graph G are distinct, G has a unique minimal spanning tree. 495 Graph Theory Section 6 26. The embarrassed casino finally paid Corriveau the $600,000 due him. ∀x∃y T2 (x, y) 12. If P1 = (v0 , . To prove that ∃x∃y P(x, y) is false, where the domain of discourse is X × Y , you must show that P(x, y) is false for all values of x ∈ X and y ∈ Y . 16 Now = 1 − P(E) − P(F) + P(E ∩ F). Exercises 34–36 refer to a small consumer survey in which 10 people were asked to choose their favorite cola among Coke, Pepsi, and RC. n 1 + 3 + · · · + (2n − 1) 1 2 3 4 1 4 9 16 105 Proofs Since the second column consists of squares, we conjecture that 1 + 3 + · · · + (2n − 1) = n2 for every positive integer n. Derive results similar to those of Exercises 29–31 for the adjacency matrix A of the graph Km . When n is odd, the sequence is divided into two parts—one part of length (n−1)/2 and the other of length (n + 1)/2. Let n = −t(p − 1)q. Merge maintains two pointers to elements in the two input sequences. How do we deduce the first inequality from the second? Next we give several examples of recurrence relations. If s is in nonincreasing order, the algorithm returns the value 0. ¬p From 3 and 4 5. 5 10 4 15 1 5 9 13 569 Trees In Exercises 15–18, determine the value of the tic-tac-toe position using the evaluation function of Example 9.1. 11. By assumption, the recursive call n, s (a string) All the ways the robot can walk n meters. Give an example of a cycle that is not a simple cycle. ¬p ∨ q {{a}} = {{c}}. Let f and g be functions from the positive integers to the positive integers defined by the equations g(n) = 2n . [Requires the integral] Use an argument like the one shown in Exercise 62 to show that (n + 1)m+1 nm+1 < 1m + 2m + · · · + nm < , m+1 m+1 where m is a positive integer. It is not necessarily true that 2r = d (see Figure 1.5). A direct proof assumes that the hypotheses are true and then, using the hypotheses as well as other axioms, definitions, and previously derived theorems, shows directly that the conclusion is true. 6 9 10 In Exercises 17–21, show that the property given is an invariant. If among the stamps that make n cents postage there is at least one 5-cent stamp, we replace one 5-cent stamp by three 2-cent stamps to make n + 1 cents postage. , n with n in the (2j)th position is C(n − 1, 2j − 1)E2j−1 En−2j . If p does not occur at index i in t, ti+j−1 will not be equal to pj for some j. Since x ∈ X ∪ Y, CX∪Y (x) = 1. (b) True (c) True (d) False. Probability function Probability of an event If E is an event, P(E) + P(E) = 1. A recurrence relation uses prior values in a sequence to compute the current value. It then finds the smallest item in s2 , . b 2 a 4 2 0 4 c 2 2 d 4 3 1 f 7 5 1 e 6 3 1 4 g z 6 6 Figure 4.5 The third iteration of Dijkstra's shortest-path algorithm. Notice that when we divide the routes into classes based on when the route first meets the top edge: ■ The classes are disjoint. Since h(257) = 4, 257 should be stored at location 4; however, this position is already occupied. Thus |x| ≥ x. Show the output from Algorithm 4.14 when n = 3. Show that the graph G of Figure 3.7 does not contain a Hamiltonian cycle. If the answer is yes, describe one. The check digit is the number c, between 0 and 9 satisfying (c + s) mod 10 = 0. What is a game tree? (B − C) ∪ ((B − A) ∩ (C ∪ B)) 49. Therefore, d divides m + n (with quotient q1 + q2). e4 e3 e2 19. Compute s. Show that X is nonempty. Define partial order and give an example of a partial order. A = {1, 3, 5}, B = {n | n ∈ Z+ and n2 − 1 ≤ n} 28. Not a proposition; it is a description of a mathematical expression (i.e., p − q, where p and q are primes). Since 1 ∈ B, but 1 ∈ A, A = B. How many integers from 1 to 10,000, inclusive, are multiples of 5 or 7 or both? At line 14, we increment sm . Since k is the exponent for 2, k = lg n. At line 2, since i > j (3 > 4) is false, we proceed to line 4, where we set k to 3. Statement (4.1) explicitly states that block 1 is marked. Commonly used values are m = 231 − 1 = 2,147,483,647, a = 75 = 16,807, and c = 0, which generate a sequence of 231 − 1 integers before repeating a value. 8 ➜ Instant Insanity Instant Insanity is a puzzle consisting of four cubes each of whose faces is painted one of four colors: red, white, blue, or green (see Figure 8.1). Then there is a one-to-one, onto function f from the vertices of G1 to the vertices of G2 and a one-to-one, onto function g from the edges of G1 to the edges of G2 , so that an edge e is incident on v and w if and only if the edge g(e) is incident on f(v) and f(w) in G2 . , si is a loop invariant, at this point large is the largest value in the sequence s1 , . We conclude with a logic game, which presents an alternative way to determine whether a quantified propositional function is true or false.Andre´ Berthiaume contributed this example. Then f0 (n) = n, g0 (n) = n , 2 g0 (n) = n, 1+ The preceding discussion motivates defining 2k f0 (2) = 2k g0 (2) = k 22 k+1 22 k+1 22 k 22 1+ 1 1 + · · · + = (lg n). By Theorem 6.5, the probability that at least two persons have birthdays on the same month and date is 1− 365 · 364 · · · (365 − n + 1) . z 3, 2 a b 2, 2 4, 2 z c Figure 2.3 After increasing the flow of Figure 2.2 by 1. n = 11, φ = 47 exercise indicates a problem of above-average difficulty. None of the statements (3) through (7) directly tells us anything about removal of edges or about a graph that's not connected. 591 Network Models 1 2 3 4 Introduction A Maximal Flow Algorithm The Max Flow, Min Cut Theorem Matching Problem-Solving Corner: Matching Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises Just go with the flow, Shel, just go with the flow. BUSHWHACKED 7. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is D4 × D4 . (Such a competition is called a singleelimination tournament.) Figure 1.1 shows that in the semifinals Monica Seles defeated Martina Navratilova and Steffi Graf defeated Gabriela Sabatini. If we have a sequence of names, we might want the sequence sorted in nondecreasing order according to dictionary order. Write a program that, given a rooted tree and a vertex v, (a) Finds the parent of v. Finally, substituting the first formula for 33 into the previous equation yields 1 = 16 · 196 − 95 · (425 − 196 · 2) = 206 · 196 − 95 · 425. Ten heads were obtained. Exercises 46–49 refer to a company that buys computers from three vendors and tracks the number of defective machines. Thus if person 1 gets order 2, there are three possibilities. Therefore, the algorithm is correct in these cases. (1.6) Thus, at the end of the third year, the amount is $1404.93. , 6, 539 mod d is not equal to zero, so d increments. (1, 0) Define a relation R on S by (x, y) R (x , y) if (x = x and y = y), or (y = y and x = 0 and x = 1), or (y = y and x = 1 and x = 0). We have to prove that α1 = β1 and α2 = β2 . You then have more to work with: Besides assuming p, you get to assume ¬q. We first find nondecreasing positive functions f0 and g0 such that for infinitely many n, f0 (n) = n2 and g0 (n) = n. Sets and Logic 58. Example 1.32 If a function from X to Y is onto, each element in Y in its arrow diagram will have at least one arrow pointing to it (see Figure 1.11). , vk } be a subset of V . Thus T contains a cycle. Since m/b = 11/2 = 5, m is set to 5. Continuing the compound interest example, instance n − 1 is "within" instance n in the sense that the amount of money at the end of n years is the amount at the end of n − 1 years plus interest. 483 Graph Theory Obviously, if a graph contains K3,3 or K5 as a subgraph, it cannot be planar. 55 Sets and Logic Example 6.15 The Logic Game Given a quantified propositional function such as ∀x∃y P(x, y), you and your opponent, whom we call Farley, play a logic game. [Hell] Show that the only homomorphism from the graph to itself is the identity function. Thus either (i, j) ∈ R1 or (i, j) ∈ R2 . Graphs G1 = (V1 , E1) and G2 = (V2 , E2) are isomorphic if there are one-to-one, onto functions f : V1 → V2 and g: E1 → E2 such that e ∈ E1 is incident on v, w ∈ V1 if and only if g(e) is incident on f(v) and f(w). Recurrence Relations n ≥ 1, into triangles by drawing n − 1 lines through the corners that do not intersect in the interior of the polygon. c e f h 37. We leave the proof of this fact to the exercises (see Exercise 71). For example, by sketching both (A ∪ B) and A ∩ B (see Figure 1.7), we see that these sets are equal. The octal number system, which represents integers using eight symbols, is discussed before Exercise 42. Inductive Step fn = fn−1 + fn−2 > n−2 3 2 n−1 −1 = 3 2 3 2 n−3 + 3 2 −2 n−1 + 3 2 = 3 2 n−1 16 > 9 3 2 Algorithms 30. 1 35. Each committee is to send a representative to the administration. Define equivalence relation. If f ◦ g is onto, then f is onto. Show that the sequence {an } satisfies the recurrence relation an = an−1 + an−2 . Therefore, the algorithm returns 0 to indicate that 43 is prime. One technique is to argue that P(x, y) is false using the symbols x and y to stand for arbitrary elements in X and Y . What is a hash function? List the order in which the vertices are processed using preorder traversal. Using the notation and hint for Exercise 57, ρ(τ(R1)) = τ(R1) ∪ D Functions, Sequences, and Relations and τ(ρ(R1)) = τ(R1 ∪ D). Proof We use mathematical induction on i to prove that the ith time we arrive at line 9, L(v) is the length of a shortest path from a to v. Verify that 00011101 is a de Bruijn sequence for n = 3. Write a recursive algorithm that prints the contents of the terminal vertices of a binary tree from left to right. Define ⎧ if (i, j) is not in P ⎪ ⎨ Fij ∗ Fij = Fij + if (i, j) is properly oriented in P ⎪ ⎩ Fij − if (i, j) is not properly oriented in P. Prove that for all n ≥ 1. for i1 = 1 to n for i2 = 1 to i1 for i3 = 1 to i2 .. penned 22. If (v, w) is an edge in M, the bit strings vx vy and wx wy differ in exactly one bit. Example 2.5 We can solve the recurrence relation b pn = a − pn−1 k for the price pn in the economics model of Example 1.9 by iteration. Input: Output: b, b , n s binary addition(b, b , n, s) { carry = 0 for i = 0 to n { si = (bi + bi + carry) mod 2 carry = (bi + bi + carry)/2 } sn+1 = carry } Algorithm 2.12 runs in time (n). .}, S= i=1 ∞ Ai = ∅. e: None Repeat this procedure with the vertices on level 3: f: Include (f, h). The converse is also true. 8 P(G) = 26. Polish notation 10. q: I take recreation supervision. 68 q Since p imp1 q is true precisely when q imp1 p is true, p imp1 q ≡ q imp1 p. Explain how to add binary numbers. State the hypothetical syllogism rule of inference. However, such a tree has at most nine terminal vertices. The union of S is the set of elements that belong to at least one set in S. What does it mean for a vertex to be incident on an edge? If we eliminate the root and the edge incident on the root, the resulting tree has height h − 1 and the same number of terminals as T. Unfortunately, a to z is (a, b, z). Given values for the input, each intermediate step of an algorithm produces a unique result. Now consider proving (2). At lines 25 and 26, vertex e is labeled (b, 1) since val(e) = min{val(b), Feb } = min{1, 2} = 1. The Hamiltonian cycle problem simply asks for a Hamiltonian cycle— any Hamiltonian cycle will do. Selection sort 12. 57. Thus the worst-case time required by this algorithm is (n2). Example 1.11 Give a direct proof of the following statement. Thus Sn = Sn−1 + Sn−2 + Sn−3 , n ≥ 4. A sequence s is increasing if sn < sn+1 for all n. Terms such as an−1 an−2 are not permitted. Then f(a) = y and f(a) = z. Since each member of W2 contributes at most one type II edge, |W2 | < |V ∗ | − |W1 |. The graph of Exercise 8 is a component of the graph of Exercise 9. , sn 3. Then xy ≥ 0 and |xy| = xy = |x||y|. How many eight-bit strings read the same from either end? Prove that if five cards are chosen from an ordinary 52-card deck, at least two cards are of the same suit. If a graph G contains a cycle from v to v, must G contain a simple cycle from v to v? A solution to the problem of scheduling the tasks so that we can take a picture is a total ordering of the tasks consistent with the partial order. To construct the argument, remind yourself what you know about the terms (e.g., "even," "odd"), symbols (e.g., X ∩ Y , min{d1 , d2 }), and so on. Define X is a subset of Y . Define n is composite. Reflexive, symmetric, antisymmetric, transitive, partial order 30. b c 21. , n} in increasing lexicographic order 1. The following properties hold. Since this inequality fails for n = 4, 5, the Basis Steps are n = 3, 4, 5. Give an example of a nonconstructive existence proof. Thus we need to subtract as many 7's as possible from 365 and see how many days are left, which is the same as computing 365 mod 7. Explain how to compute the greatest common divisor of m and n, not both zero, given their prime factorizations. 131. At line 1, i is set to 16. Given an arrow diagram, the relation is symmetric if whenever there is a directed edge from x to y, there is also a directed edge from y to x. Then the amount of postage is np p + nq q. 1/6 14. 1 Sets and Logic 1 ➜ Sets The concept of set is basic to all of mathematics and mathematical applications. Example 3.4 Consider three books: a computer science book, a physics book, and a history book. Reflexive, not symmetric, and not transitive 39. The pseudocode shows that the order in which funnyorder visits the nodes of T is tree cover(PT) { flag = false ptr = first child of PT while (ptr ! = null) { tree cover(ptr) if (in cover of ptr == false) flag = true ptr = next sibling of ptr } in cover of PT = flag } Section 7 1. In how many ways can we select five balls if balls of the same color are considered identical? ¬(p → q) 16. True 12. Thus the given statement is false; there is no direct proof (or any other proof!). Find all employees in department 04. , xn 7. No further pruning is possible. The graphs are isomorphic. A partial order is a relation that is reflexive, antisymmetric, and transitive. (d) Use the result of part (c) to find the matrix of the relation R2 ◦ R1 . Write an algorithm that receives as input the matrix of a relation R and tests whether R is reflexive. Exercise 8, Section 1 15. Example 3.2 Consider the network G of Figure 3.1. If we let P = {a, b, d}, then P = {c, e, f, z} and (P, P) is a cut in G. 12354 5. k k+b k+b = (2.4) 397 Recurrence Relations We see that if b/k < 1, the term b − k n −ak + p0 k+b becomes small as n gets large so that the price tends to stabilize at approximately ak/(k + b). Prove that 72. What is the complement of the set X? A proposition such as (3.1) is called a conditional proposition. , 4, 5} to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f . A minimal spanning tree T prim alternate(w, n, s) { let T be the graph with vertex s and no edges for j = 1 to n { Lj = w(s, j) // these labels are temporary back(j) = s } Ls = 0 make Ls permanent while (temporary labels remain) { choose the smallest temporary label Li make Li permanent add edge (i, back(i)) to T add vertex i to T for each temporary label Lk if (w(i, k) < Lk) { Lk = w(i, k) back(k) = i } } return T } 7. For example, in the subtree whose root is / in Figure 6.7, the divide operator operates on the operands D and E; that is, D is to be divided by E. In representing an integer, the symbol's position is significant; reading from the right, the first symbol represents the number of 1's, the next symbol the number of 10's, the next symbol the number of 100's, and so on. Write a recursive algorithm that generates all permutations of the set {s1 , s2 , . Trisha 2n−k (−1)k C(n, k) = 1. Let Y = {y1 , . Thus, the probability of a correct answer is 1/2. p → q 7. At line 3, we execute the statement x = x + 1 the third time. What it is you're trying to find out. Professor Ter A. v1 v2 v3 ⎛0 1 0 v1 v2 ⎜ 1 0 1 v3 ⎜ ⎜0 1 0 v4 ⎜ ⎜0 1 1 v5 ⎜ 0 0 0 ⎝ v6 1 1 0 v7 0 1 0 18. Show that if an edge in G, but not in T, is added to T, a unique cycle is produced. Exercises 12–15 refer to a coin that is flipped 10 times. Example 3.5 Find a spanning tree for the graph G of Figure 3.1. We will use a method called breadth-first search (Algorithm 3.6). We now translate the inductive proof into a recursive algorithm to construct a tiling by right trominoes of an n × n deficient board where n is a power of 2. Since |GS| = |S|, GS is optimal. True (think of a saint). T2 has a vertex of degree 3, but T2 does not. Isomorphic binary trees 46. The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is symmetric because for all x, y, if (x, y) ∈ R, then (y, x) ∈ R. At this point L(z) gives the length of a shortest path from a to z. How many (distinct) equivalence classes are there? If a graph is connected, how many components does it have? We then compute the quotient when 29 is divided by 2. 5, 25 14. .} are equivalent. If T is a minimal spanning tree of G, there is a labeling of the vertices of G so that Algorithm 4.3 produces T. Similarly, if the root has a right child, the child itself has either a left or a right child. Within a subset, the elements are equivalent in the sense that they are all the same color. Let x ∈ A. We can use mathematical induction to show that 2 divides 7n + 1 for all n ≥ 1 (the argument is similar to that given in the hint for Exercise 21). Many plants distribute their seeds as evenly as possible, thus maximizing the space available for each seed. Let T be the set of all vertices. This is indeed the case. For example, if m = 11, a = 7, c = 5, s = 3, then x1 = (ax0 + c) mod m = (7 · 3 + 5) mod 11 = 4 and x2 = (ax1 + c) mod m = (7 · 4 + 5) mod 11 = 0. Given a string α = s1 · · · sr , which represents the r-combination {s1 , . A minimal cut is a cut having minimum capacity. If t1 = t2 = t, the general solution is of the form an = bt n + dnt n , where again b and d are constants. Next we prove part (b). Subsequently in this proof, we suppress the vertex set and refer to a graph by specifying its edge set. e3 v 5 e10 v 4 8. Recall that when the hypothesis is false, the conditional proposition is true regardless of whether the conclusion is true or false.) In most arguments, the vacuous case is omitted. (b, c, a, d, e) 53. Although Algorithm 1.8 runs in time polynomial in n (since n ≤ n), it does not run in time polynomial in the size of the input (namely, n). Path Simple path Cycle Simple cycle Connected graph Subgraph Component of a graph Degree of a vertex δ(v) K¨onigsberg bridge problem Euler cycle A graph G has an Euler cycle if and only if G is connected and every vertex has even degree. Explain how depth-first search works. Thus if a relation R has no members of the form (x, y), x = y, R is antisymmetric. Since s1 < t1 , by condition (b) of Definition 4.1, α < β. If I eat my hat, I'll be quite full. It follows that dn ≥ 2dn−1 + 1. To send the integer a, 0 ≤ a ≤ z − 1, to the holder of public key z, n, the sender computes c = an mod z and sends c. Now consider the vertices v4 , v5 , and v6 . Sometimes notation, setting, or context will suggest something useful. The integer 34 is composite because it is divisible by 17, which is neither 1 nor 34. 76. State and prove a result for similar to that for Exercises 39 and 40. Solve the recurrence relation to obtain an explicit formula for cn . For example, suppose that a six-sided fair die whose sides are labeled 1, 2, 3, 4, 5, 6 is rolled (see Figure 5.1). Finding a Solution Now we consider the problems. Since a loop iterates a finite number of times, the form of mathematical induction used here proves that a finite sequence of statements is true, rather than an infinite sequence of statements as in our previous examples. , xn = y ∈ X such that (xi−1 , xi) ∈ R for i = 1, . If the capacities are allowed to be nonnegative rational numbers, the algorithm also terminates; however, if arbitrary nonnegative real capacities are allowed and we permit the edges in line 17 to be examined in any order, the algorithm may not terminate (see [Ford, pp. In fact, this upper bound is achievable (see Exercise 12), so that an = an/2 + a(n+1)/2 + n − 1. Thus k does not exist. We must show that the greedy algorithm is optimal for all amounts of postage n ≤ 6(i + 1). The problem with the preceding "proof" is that we cannot write n = 2k2 since this is true if and only if n is even— which is what we are supposed to prove! This error is called begging the question or circular reasoning. The string that follows the 0 may be any 432 string of length n − 1 that contains an even number of 1's, and there are cn−1 such strings. Thus the lower limit for j is 1. Therefore, if I'm quite full, the Skyscrapers won. {(x, y) | 1 ≤ x ≤ 5 and 1 ≤ y ≤ 5} 21. The algorithm begins by setting result to 1 and x to a mod z = 50 mod 11 = 6. Use Exercise 74 to derive the recurrence relation n/2 for i = 1, 3, 5, . for i = 1 to 2n x=x+1 i=1 while (i ≤ 2n) { x=x+1 i=i+2 } 18. We write 3 and carry 1: + 1 84F 42EA 39 Continuing in this way, we obtain + 278 84F 42EA 4B39 Introduction to Number Theory Example 2.14 The addition problem of Example 2.13, in decimal, is + 2127 17130 19257 We can multiply binary numbers by modifying the standard algorithm for multiplying decimal numbers (see Exercises 64 and 65). Write an algorithm that finds the smallest element among a, b, and c. Show that if 0 It is also apparent from the arrow diagram (see Figure 1.2) that this set is not a function because there is no arrow from 4. Next we add a supersource a and edges of capacity 1 from a to each of A, B, C, and D. For example, Bela Lugosi's Bacon number is three because the length of a shortest path from Lugosi to Bacon is three. Show that R is a partial order on X1 × X2 if we define 2 (x1 , x2) R (x1 , x2) if x1 R1 x1 and x2 R2 x2 . (n + 2)Cn+1 = (4n + 2)Cn , n ≥ 0, and initial condition C0 = 1. ((A − C) ∗ D) / (A + (B + D)) 8. 2 3 13 Figure 1.8 The graph of the postage function P(w) = 80 + 17 w − 1 . To implement this approach, we need an algorithm that generates all combinations of an n-element set. If an algorithm requires t(n) units of time to terminate in the average case for an input of size n and t(n) = O(g(n)), we say that the average-case time required by the algorithm is of order at most g(n) or that the average-case time required by the algorithm is O(g(n)). We add the edges (b, d), (d, c), (c, e), (e, f), and (f, h). Therefore, if there is a tiling, five 2's are covered. The behavior is determined by the slopes of the supply and demand lines. To prove that a relation R on a set X is not reflexive, find x ∈ X such that (x, x) ∈ R. Since j = i + 1, when i = 0, j = 1. We can use a little algebra to show that it is of the desired form: m + n = (2k1 + 1) + 2k2 = 2(k1 + k2) + 1. The truth value of the conjunction p ∧ q is determined by the truth values of p and q, and the definition is based upon the usual interpretation of "and." Consider the proposition, p ∧ q: It is raining and it is cold, of Example 2.2. If it is raining (i.e., p is true) and it is also cold (i.e., q is also true), then we would consider the proposition, p ∧ q: It is raining and it is cold, to be true. A number that is multiple of 3 and 5 is a multiple of 15. What is an n-ary relation? Give an argument that shows that if we graph the puzzle as in regular Instant Insanity, a solution to modified Instant Insanity consists of two subgraphs of the form shown in the figure, with no edges or vertices in common. Yes Section 2 1. The last time we execute the for loop in lines 7–14, the edges with one vertex in the tree and one vertex not in the tree are Weight (1, 2) (2, 4) 4 5 The edge (1, 2) with minimum weight is selected. Recurrence Relations Richard Johnsonbaugh 383 8. What is an alternative name of the postfix form of an expression? † Exercise a e b d c 6. Find a formula for the number of nonisomorphic n-vertex full binary trees. Figure 1.5 7. Also, many practical problems are too difficult to be solved efficiently, and compromises either in generality or correctness are necessary. Planar graph Face Euler's equation for connected, planar graphs: f = e−v+2 Edges in series Series reduction Homeomorphic graphs Kuratowski's theorem: A graph is planar if and only if it does not contain a subgraph homeomorphic to K5 or K3,3 . Given an element E and the array A that represents X, write a program that determines whether E ∈ X. Write explicit solutions for the Tower of Hanoi puzzle for n = 3, 4. , vn−1) is a Hamiltonian path. Trace Algorithm 2.3 for the input 34 34 57 72 34 34. Therefore, r1 = r2 . Show that if G is a simple graph, either G or G is connected. , v0 , vn−1 , . [Problem 2.] One way to prove that two numbers a and b are equal is to show that a ≤ b and b ≤ a. Let T ∗ be T with v and the edge incident on v removed. See Theorem 2.3. Section 2 1. Exercise 12, Section 2 Exercises 17–22 refer to a network G that, in addition to having nonnegative integer capacities Cij , has nonnegative integer minimal edge flow requirements mij . The largest disk must be on a peg by itself (so that it can be removed from the peg) and another peg must be empty (so that this peg can receive the largest disk). If A ∪ B = B, what relation must hold between A and B? Therefore f is onto X∗ . What is a simple graph? Let E1 , . Find each inverse function. Divide the quotient by 2. Let's try letting the vertices be the people. Let (w0 , w1 , . Either there is a hurricane or it is raining (or both). How many Class C addresses are available? Section 6 Review 1. Find the probability of obtaining at least three defective microprocessors. Example 1.24 shows that, in general, X × Y = Y × X. How many equivalence relations are there on the set {1, 2, 3}? 4807 mod 2091 = 625; 2091 mod 625 = 216; 625 mod 216 = 193; 216 mod 193 = 23; 193 mod 23 = 9; 23 mod 9 = 5; 9 mod 5 = 4; 5 mod 4 = 1; 4 mod 1 = 0; so gcd(2091, 4807) = 1. If the factorization could be found by a person who intercepts a message, the message could be decrypted just as the authorized recipient does. Since this value is 1, we know that 1 = 20 is included in the binary expansion of 29. a b c d 4. An equivalence relation on X gives rise to a partition of X (namely, the equivalence classes), and a partition of X gives rise to an equivalence relation (namely, x is related to y if x and y are in the same set in the partition). 60. What is an undefined term? A proof by contradiction assumes that the hypotheses are true and that the conclusion is false and then, using the hypotheses and the negated conclusion as well as other axioms, definitions, and previously derived theorems, derives a contradiction. Now U = {d}. Find a formula for zn−2 . (e) Find a formula for sn−2 . How many antisymmetric relations are there on an n-element set? By the WellOrdering Property, Y has a least element, say n. 1 − (1/210) 37. Show that hypothetical syllogism (see Figure 4.1) is valid. Find the matrix A2 of the relation R2 relative to the orderings {(1, 1), (1, 2), (2, 2), (4, 4), (2, 1), (3, 3)} an equivalence relation on {1, 2, 3, 4}? Solve the recurrence relation of Exercise 34 in case n is a power of 2. Formal Solution No. It is not possible in a department of 25 persons racked by dissension for each person to get along with exactly five others. 560 Trees If n is even, say n = 2k, one can use induction to show (see Exercise 24) that when two k-vertex isomorphic binary trees are input to Algorithm 8.13, the number of comparisons is equal to 3n + 2. To what values do the price and quantity tend in Example 1.9 when b < k? If a is the sequence defined by (2.13), a0 = C0 , a1 = C1 , (2.15) and r1 and r2 are roots of (2.14) with r1 = r2 , then there exist constants b and d such that an = br1n + dr2n , n = 0, 1, . If n = 2k , (3.2) becomes a2k = 1 + a2k−1 , k = 1, 2, . For example, in Figure 8.3 we represent the stacking of Figure 8.2. The vertices represent the colors, and an edge connects two vertices if the opposite faces have those colors. Draw the tree T of Figure 1.5 as a rooted tree with a as root. What key theorem is the basis for the Euclidean algorithm? [What needs to be shown is that h uniquely assigns a value to [x]; that is, if [x] = [y], then g(x) = g(y).] y (0, 1) 43. 461 Graph Theory Example 3.8 We use Theorem 3.6 to construct the Gray code G3 beginning with G1 . Now n2 + 1 = (2k)2 + 1 = 4k2 + 1 = 2(2k2) + 1. 463 Graph Theory Exercises Find a Hamiltonian cycle in each graph. In general, the number of rise/fall permutations of 1, . GIMPS is a computer program distributed over many personal computers maintained by volunteers. Since R is reflexive, antisymmetric, and transitive, it is a partial order. Example 2.2 Binary to Decimal The binary number 1011012 represents the number consisting of one 1, no 2's, one 4, one 8, no 16's, and one 32 (see Figure 2.2). 88 92. Raising to the power 2 gives √ n ≥ cs , 1/4 for all n ≥ 2, where 1.8, the worst-case √ c = 2 . The problem is to choose unordered, six-element selections from the set {computer science, physics, history}, repetitions allowed. x is set to x ∗ x mod z = 25 mod 11 = 3, and n is set to 0. We first assign each edge in the graph of Figure 4.1 capacity 1 (see Figure 4.3). R is not antisymmetric, not transitive, and not a partial order. Example 4.2 dijkstra(w, a, z, L) { L(a) = 0 for all vertices x = a L(x) = ∞ T = set of all vertices // T is the set of vertices whose shortest distance from a has // not been found while (z ∈ T) { choose v ∈ T with minimum L(v) T = T − {v} for each x ∈ T adjacent to v L(x) = min{L(x), L(v) + w(v, x)} } } We show how Algorithm 4.1 finds a shortest path from a to z in the graph of Figure 4.1. (The vertices in T are uncircled and have temporary labels. If there is a child v, we repeat this process. Formal Solution The formal proof could be written as follows. Is x nonincreasing? Combining the previous equations gives 1274 as a product of primes 1274 = 2 · 637 = 2 · 7 · 91 = 2 · 7 · 7 · 13. 4 4 4 4 4 = k2 + k. r : You have sore eardrums. Since R is transitive, (x, z) ∈ R. Explain why Exercises 17 and 18 imply that ai = aj for some i = j. 8 ➜ The Pigeonhole Principle The Pigeonhole Principle (also known as the Dirichlet Drawer Principle or the Shoe Box Principle) is sometimes useful in answering the question: Is there an item having a given property? In Exercises 21–23, give an argument using rules of inference to show that the conclusion follows from the hypotheses. Suppose that we store and retrieve data as described in Example 1.14. It may take several time units for a processor to communicate with a nonadjacent processor. H. (x, y) ∈ R if |x − y| = 2. Define g from {1} to X by g(1) = 1. For example, 5! = 5 · 4 · 3 · 2 · 1 = 5 · 4!. Each time we pass through one of the original regions, it is divided into two regions. Find r2 . , n. P = p → q, Q = p ↔ q 67. an = 6an−1 − 8an−2 = 100n 4. Therefore, any algorithm that solves the four-coins puzzle requires at least three weighings in the worst case. 4n3 + 2n − 5 10. Use the Inclusion-Exclusion Principle (Theorem 1.12) to solve Exercises 86–91. Solve the recurrence relation a1 = −20 an = a1 = 5 23. Because the sequences s1 , s2 , . Derive the recurrence relation T(n) = 2T(n − kn) + 2kn − 1. Is f one-to-one? Choose orderings of X, Y , and Z. a = 5, n = 10, z = 21 60. Let A1 be the adjacency matrix of G1 relative to the ordering v1 , . Definition 7.3 If a graph G has a vertex v of degree 2 and edges (v, v1) and (v, v2) with v1 = v2 , we say that the edges (v, v1) and (v, v2) are in series. (i, b) ∈ R1 and (b, k) ∈ R2 . Let each row, column, or diagonal that contains two O's and one blank count −5. (For Exercise 42) ∃x∃y(x2 ≥ y + 1) is false. temp1 := department [Dept = 23] temp2 := department [Dept = 96] temp3 := temp1 union temp2 temp4 := temp3 [Manager = Manager] employee temp4 [Name] Kaminski, Schmidt, Manacotti, Suzuki Chapter Self-Test 1. c b 8. Prove or disprove: X × ∅ = ∅ for every set X. To test whether d divides n, the algorithm checks whether the remainder when n is divided by d, n mod d, is zero. What is the infix form of an expression? Finally, consider the average-case time of Algorithm 3.17. The following pseudocode determines whether ∀x P(x) is true or false: for i = 1 to n if (¬P(di)) return false return true The for loop examines the members di of the domain of discourse one by one. Describe insertion sort in words. More formally, a collection S of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of S. We remove c from U at line 15. True 52. Let P be a shortest path from a to w, let x be the vertex nearest a on P that is in T , and let u be the predecessor of x on P (see Figure 4.6). We consider the order that one person p has. When I evaluates to 1, we know that the value y of C cannot exceed 1, since the value of C is the minimum of its children; that is y ≤ 1. For example, if the input sequence is already sorted in nondecreasing order, val < sj will always be false and the body of the while loop will never be executed. Eventually, we place the word in the tree. , p; or for some i, si = ti and for the smallest such i, we have si < ti . Therefore, (i, j) ∈ R1 ∪R2 . In K3,3 each vertex has degree 3, so let us eliminate the edges (a, b) and (f, e) so that all vertices have degree 3 (see Figure 7.6). Let L be the set of all strings, including the null string, that can be constructed by repeated application of the following rules: ■ If α ∈ L, then aαb ∈ L and bαa ∈ L. There are zero comparisons at lines 9–11. Since F is 1510 and A is 1010 , F + A = 1510 + 1010 = 2510 = 1916 . Theorem 3.6 Let G1 denote the sequence 0, 1. We note that if a prime p appears in the prime factorization of l, p must be equal to one of p1 , . Example 1.9 The Cobweb in Economics We assume an economics model in which the supply and demand are given by linear equations (see Figure 1.4). Notice how in the negation, ∀ and ∃ are interchanged. Is the Professor's version correct (i.e., does it find key if it is present and return 0 if it is not present)? Friendship can be modeled by an undirected graph by letting the vertices denote the people and placing an edge between two people if they are friends. By the generalized De Morgan's laws for logic, this latter statement is equivalent to ∀i¬(si ≥ A) or ∀i(si < A). The Cartesian product of X1 , X2 , . 0 1 1 1 0 2 1 1 19. (c) n2 + 1 is even. Show that the range of a is contained in {1, 2, . Since each horizontal step can occur in two ways, the number of paths containing exactly k vertical steps that never go strictly below the x-axis is C(k, k/2)C(n, k)2n−k . Since 55 > 34, 55 is now inserted 20 7. 1 ≽2 ≽1 1 1 0 ≽1 0 ≽1 ≽2 0 1 ≽1 0 1 2 Figure 9.7 The game tree of Figure 9.6 with the values of all vertices shown. Show that Algorithm 2.4 terminates. A similar statement applies to S. We conclude by showing that the number s in equation (3.14) is unique. In Section 1, we provide a taste of the logic and proofs to come in the remainder of this chapter. Prove Theorem 5.14, part (b). Thus C solves the traveling salesperson problem for G. Given the matrix of a relation R from X to Y , how can we find the matrix of the inverse relation R−1 ? The contrapositive of p → q is ¬q → ¬p. The vertices correspond to programs. All counterexample relations are on {1, 2, 3}. For example, CR refers to the dinner consisting of a cheeseburger and root beer.) Notice that there are three main courses and four beverages and 12 = 3 · 4. In a digital computer, data and instructions are encoded as bits. There are P(n, r) ways to select an r-permutation of an n-element set. {{1, 2}, {3, 4}} 16. Substituting the formula for f(x), this last equation becomes 1 = y. To prove that a relation is an equivalence relation, you need to verify that these three properties hold (see Problem-Solving Tips for Section 3). The relational database model that helps users access information in a database (a collection of records manipulated by a computer) is based on the concept of relation. , sn), small (the smallest item in s1 , s2 , . Knowing or being able to estimate the time and space required by two or more algorithms that solve the same problem makes it possible to compare the algorithms. Let us assume that a2 < a4 . + + ··· + =1− 2! 3! (n + 1)! (n + 1)! 14. If true, prove it; if false, give a counterexample. A lemma is a theorem that is usually not too interesting in its own right but is useful in proving another theorem. k=1 This equation, known as the summation-by-parts formula, is the discrete analog of the integration-by-parts formula in calculus. However, for each x in the domain of discourse, P(x) is a proposition and is, therefore, either true or false. Show that conjunction (see Figure 4.1) is valid. We claim that a = c. Logic is essential in reading and developing proofs. n = 16 57. Figure 1.1 provides an example of a transport network. In a proof by resolution, the hypotheses and the conclusion are written as clauses. Similarly, [2] = {2, 5, 8}, [3] = {3, 6, 9}. Set n = 0 and then n = 1 in an = br1n + dr2n to obtain C 0 = a0 = b + d C1 = a1 = br1 + dr2 . Assume that n > 1 and the values in the sequence are distinct. Explain how the Euclidean algorithm can be used to compute s and t. Then z = tx for some x ∈ X (by the definition of the set tX). 609 Network Models 4 ➜ Matching In this section we consider the problem of matching elements in one set to elements in another set. Use decision trees to find a lower bound on the number of comparisons required to sort five items in the worst case. We let the squares of the board, alternately colored black and white in the usual way, be the vertices of the graph, and we place an edge between two vertices if the corresponding squares on the board represent a legal move for the knight (see Figure 3.11). 1 1 1 + + ··· + 1 2 n 3. {(a, 1, a, α), (a, 2, a, α), (a, 1, a, β), (a, 2, a, β)} 17. We claim that exactly half of the subsets of X contain x, and exactly half of the subsets of X do not contain x. A minimal spanning tree will be constructed when either edge is selected. We write f(n) = O(g(n)) and say that f(n) is of order at most g(n) or f(n) is big oh of g(n) if there exists a positive constant C1 such that |f(n)| ≤ C1 |g(n)| for all but finitely many positive integers n. CX∪Y (x) = CX (x) + CY (x). 7 ➜ Decision Trees and the Minimum Time for Sorting The binary tree of Figure 7.1 gives an algorithm for choosing a restaurant. Input: Output: A (an n × n matrix of a relation R), n true, if R is reflexive; false, if R is not reflexive is reflexive(A, n) { for i = 1 to n if (Aii == 0) return false return true } A (an n × n matrix of a relation R), n true, if R is antisymmetric; false, if R is not antisymmetric is reflexive(A, n) { for i = 1 to n − 1 for j = i + 1 to n if (Aij == 1 ∧ Aji == 1) return false return true } s, n Output: small (smallest), large (largest) Input: Output: Output: small, the smallest value in the sequence s 23. 135 Functions, Sequences, and Relations Equation (1.1) defines a function. 523 Trees Example 3.8 Use depth-first search (Algorithm 3.7) to find a spanning tree for the graph of Figure 3.2 with the vertex ordering abcdefgh. We can summarize the situation by saying that the inverse of the exponential function is the logarithm function. P = p ∧ q, Q = ¬p ∨ ¬q 62. We write f(n) = (g(n)) and say that f(n) is of order g(n) or f(n) is theta of g(n) if f(n) = O(g(n)) and f(n) = (g(n)). In either case, x ∈ B. If T is a rooted tree with six vertices, the height of T is at most 5. (A ∪ B) − B 45. 27. The path (h, f, e, g, b, d, c, a) 7. Substituting the second formula for 31 into the previous equation yields 1 = 16 · (196 − 33 · 5) − 15 · 33 = 16 · 196 − 95 · 33. Let E be the event "at least one person has a birthday on April 1." Then E is the event "no one has a birthday on April 1." Now = 2 . 4 ➜ Recursive Algorithms A recursive function (pseudocode) is a function that invokes itself. A B C F D Left Root Right F G B E G A H C I D H J E Figure 6.3 A binary tree. ∃x∃y¬P(x, y) (b) The statement, "All integers are positive or all positive numbers are integers," which is false, in symbols is 8. Second, we assumed that statement n was true and then proved that statement n + 1 was also true. Let Pn be the graph that is a simple path with n vertices. In the terminology of this section, you must be sure that n0 ≤ k (see Examples 5.1 and 5.2). If we can show that the capacity of this cut is n, a maximal flow would have value n. small = s1 21. (c) Part (a) shows that there are 120 strings of length 4 that can be formed using the letters ABCDE, and part (b) shows that 24 of these start with the letter B. 107. Conclusion: Some member of the Titans can make a lot of money. There are 24 possible dinners consisting of one appetizer, one main course, and one beverage: NHT, NHM, NHC, NHR, NCT, NCM, NCC, NCR, NFT, NFM, NFC, NFR, SHT, SHM, SHC, SHR, SCT, SCM, SCC, SCR, SFT, SFM, SFC, SFR. Give an algorithm that uses this number of comparisons to sort five items in the worst case. When a connected, weighted graph and vertices a and z are input to the following algorithm, it returns the length of a shortest path from a to z. The input is read in the order submitted. 3 ➜ Hamiltonian Cycles and the Traveling Salesperson Problem Sir William Rowan Hamilton marketed a puzzle in the mid-1800s in the form of a dodecahedron (see Figure 3.1). How many integers between 1 and 1,000,000 have the sum of the digits equal to 15? cddcdc 25. √ Proof Let x = y = 2. , sn algor1(s, n) { i=n while (i ≥ 1) { si = si + 1 i = i/2 } n = n/2 if (n ≥ 1) algor1(s, n) } 58. When the wheel is spun, all numbers are equally likely to be selected. 38. 151 Functions, Sequences, and Relations Let RR denote the set of functions from R to R. The problem "When can an n-cube simulate a two-dimensional mesh?" can be rephrased in graph terminology as "When does an n-cube contain a subgraph isomorphic to a twodimensional mesh?" We show that if M is a mesh p vertices by q vertices, where p ≤ 2i and q ≤ 2j , then the (i + j)-cube contains a subgraph isomorphic to M. Let 28. 3 4 R 4 1 2 B W 3 2 G 1 19. Such a board can be tiled: 1 2 3 4 5 1 and 5 are farthest apart, but neither is a survivor. Writing these equations in the form r = n − dq, where r is the remainder and q is the quotient, yields 258 = 490256 − 337 · 1454 79 = 337 − 258 · 1 302 gcd recurs1(a, b) { if (b == 0) return a r = a mod b return gcd recurs1(b, r) } 17. Breadth-first search can be used to test whether an arbitrary graph G with n vertices is connected (see Exercise 30). If R is an equivalence relation on X, the equivalence classes partition X. Otherwise, determine whether p is in t starting at index 2 in t. 104 Proofs Example 4.4 Geometric Sum Use induction to show that if r = 1, a + ar 1 + ar 2 + · · · + ar n = a(r n+1 − 1) r−1 (4.12) for all n ≥ 0. Thus it is possible to construct an algorithm based on Theorem 2.3. The outline is as follows: 1. Color the map of Exercise 18, excluding the unbounded region, using three colors. The virus caused problems by creating messages faster than they could be sent. The problem is that both methods require finding the prime factors of the numbers involved and no efficient algorithm is known to compute these prime factors. By assumption, Y is nonempty. At line 37 we set to 2. Iterating this procedure yields the algorithm. a ∨ ¬bc 2. (A ∗ B − C/D + E) + (A − B − C − D ∗ D)/(A + B + C) In Exercises 11–15, represent the postfix expression as a binary tree and write the prefix form, the usual infix form, and the fully parenthesized infix form of the expression. 60, 90 2. Let X be the set of integers between 1 and 10,000 that are multiples of 3, let Y be the set of integers between 1 and 10,000 that are multiples of 5, and let Z be the set of integers between 1 and 10,000 that are multiples of 11. Therefore B ⊆ A ∪ B. C(6 + 6 − 1, 6 − 1) 10. Example 3.13 The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is antisymmetric because for all x, y, if (x, y) ∈ R (i.e., x ≤ y) and (y, x) ∈ R (i.e., y ≤ x), then x = y. When a binary tree is drawn, a left child is drawn to the left and a right child is drawn to the right. Recurrence Relations with i = j = 2 to search for key in Section 3 Review 1. Similarly, when i = n, j = n + 1, and the upper limit for j is n + 1. In the next section we give an algorithm that efficiently solves this problem. f(m, n) = m − n 17. Show that for some positive integer k, f k (x) = x for all x ∈ X. How many selections are there in which Tyler is president or not an officer? If n − m + 1 < m, the best case is that the first character in p does not occur in t. , is given by the formula xn = (axn−1 + c) mod m. Input: a, n Output: an exp via repeated squaring(a, n) { result = 1 x=a while (n > 0) { if (n mod 2 == 1) result = result ∗ x x=x∗x n = n/2 } return result } The number of times that the while loop executes is determined by n. Prove the 0/1 laws for sets. The Inductive Step then becomes for all n ≥ n0 , if S(n) is true, then S(n + 1) is true. Simple graph; six vertices having degrees 1, 2, 3, 4, 5, 5 17. Another way to develop some appreciation for the relative sizes of the functions f(n) in Table 3.3 is to determine how long it would take an algorithm to terminate whose run time is exactly f(n). We may argue as in the Computing an Inverse Modulo an Integer subsection that if we set x = tk mod m, then f(x) = (ntk) mod m. The probability of E is P(E) = x∈E 4. Typical problems studied in operations research are network problems, allocation of resources problems, and personnel assignment problems. where p is the price, q is the quantity, and k is a positive parameter. Let A1 be the matrix of R1 and let A2 be the matrix of R2 with respect to the orderings selected. We now discuss conditional probabilities in general. These values require n additional modulus operations. In Exercises 42–47, express each octal number in decimal. R = R−1 = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)} The composition of the relations 23. Problem-Solving Tips To prove that a relation is reflexive, show that (x, x) ∈ R for every x ∈ X. sorted in nondecreasing order and prints all values that appear more than once. i n n(n + 1)(2n + 1) into two classes: C1 n n 2 (n + 1)2 n(n + 1)(2n + 1) − =n 6 2 {1, . ■ Draw pictures. Theorem 8.11 There are five nonisomorphic binary trees with three vertices. If a graph has a Hamiltonian path, must it have a Hamiltonian cycle? Since g ∈ X, there exist s and t such that g = sa + tb. Two classical graph problems, the existence of Hamiltonian cycles and the traveling salesperson problem, are then considered. If R and S are reflexive, then R ∩ S is reflexive. an = −2nan−1 + 3n(n − 1)an−2 ; n−1 a0 = 1, a1 = 2 37. (1) 22. We must have An = 4000 or (1.14)n 2000 = 4000 or (1.14)n = 2. After the teams are assigned, in how many ways can the tournament unfold? Does 1101010 represent a number in octal? Now (v0 , . Proof Let a (z) represent the source (sink) in the matching network, and suppose that a flow is given. Antisymmetric R1 = {(1, 2), (1, 3), (2, 2)} 24. A bijection is a function that is one-to-one and onto. In a group of 191 students, 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music. Now no move is possible for the third column. Removing x from T produces a disconnected graph with two components, U and V . Write an algorithm that receives as input the sequence x < y < z. We call q the quotient. Show how Algorithm 3.2 executes in case key = 'P'. Show that G1 and G2 are isomorphic if and only if G1 and G2 are isomorphic. 166 Functions, Sequences, and Relations Example 3.3 Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}. What is a series reduction? The domain of discourse is the set of all movies. Definition 1.16 Example 1.17 The floor of x, denoted x, is the greatest integer less than or equal to x. Now s = 569 since ns mod φ = 29 · 569 mod 660 = 16501 mod 660 = 1. Trace Algorithm 2.4 for the input rand(1, 5) = 2, rand(2, 5) = 5, rand(3, 5) = 3, rand(4, 5) = 4. If P(di) is false for every di , the condition P(di) in the if statement is always false. Example 4.5 Consider the relation R = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)} on {1, 2, 3, 4, 5}. Write a program that solves the three-peg Tower of Hanoi puzzle. Describe the kind of graph used (e.g., undirected graph, directed graph, simple graph). An ordered pair of elements, written (a, b), is considered distinct from the ordered pair (b, a), unless, of course, a = b. We first count the number of paths 16. There are nine possible names for the 10 persons. In Section 2, we discuss representations of integers and some algorithms for integer arithmetic. If (4 < 2 and 6 < 6), then 7 < 10. If the pair a, b, a > b, requires n ≥ 1 modulus operations when input to the Euclidean algorithm, then a ≥ fn+2 and b ≥ fn+1 , where {fn } denotes the Fibonacci sequence. If the coin is flipped 10 times, what is the probability of at 359 Counting Methods and the Pigeonhole Principle least one head given at least one tail? If we delete the first bit from α and the first bit from β, the resulting strings are identical. , k}. Is q increasing? Explain how merge sort works. 2 + 5 = 19. It is also possible to increase the flow in certain paths from the source to the sink in which we have properly and improperly oriented edges. The truth value of the proposition ¬p is defined by the truth table p ¬p T F F T In English, we sometime write ¬p as "It is not the case that p." For example, if p : Paris is the capital of England, the negation of p could be written ¬p : It is not the case that Paris is the capital of England, or more simply as ¬p : Paris is not the capital of England. The logic game extends to propositional functions of more than two variables. , sn and that the goal is to sort the data in nondecreasing order. Explain how to compute an mod z using repeated squaring. In the expansion of n C(n, k − 1)an+1−k bk k=1 (a + b)n = (a + b)(a + b) · · · (a + b), = C(n + 1, 0)an+1 b0 + n factors the term an−k bk arises from choosing b from k factors and a from the other n − k factors, which can be done in C(n, k) ways. Another possibility is proof by contradiction and assume that a = b. Let C1 , C2 , . The sequence originally arose in a puzzle about rabbits (see Exercises 18 and 19). C 1C 2 : C 3C 4 C1 : C5 C1 : C2 C 1, H C 2, H C 5, L C 5, H C 3, L C 6, L 27. If n = 0, at line 3 the function correctly returns the value 1. Then T1 and T2 are isomorphic if there is a one-to-one, onto function f from the vertex set of T1 to the vertex set of T2 satisfying the following: (a) vi and vj are adjacent in T1 if and only if f(vi) and f(vj) are adjacent in T2 . (In fact, in the graph of Figure 2.8, every vertex is incident on an odd number of edges.) A A B Pregel B C River D Figure 2.7 The bridges of K¨onigsberg. 63 Sets and Logic 8. In the infix form of an expression, an operator appears between its operands. Formal Solution (a) We have already presented a formal proof that R is an equivalence relation. If x ≥ 0, then −d < 0 ≤ x = |x| < d. , pn . In our case, k = 4. (n3) 12. Thus a = bq + r, 0 ≤ r < b. Example 3.18 The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is transitive because for all x, y, z, if (x, y) and (y, z) ∈ R, then (x, z) ∈ R. 220 n t(n) = 60n2 + 5n + 1 60n2 10 100 1000 10,000 6051 600,501 60,005,001 6,000,050,001 6000 600,000 60,000,000 6,000,000,000 Algorithms the units in which we measure the worst-case time for input of size n. Proof We give a proof by contradiction. r: I'll be quite full. Our first algorithm is a version of the selection sorting algorithm. 23. B ∪ (B − A) 44. An alpha value is a lower bound for a box vertex. 6 2 3 2 4 12 26. f(n) = (g(n)) if there exists a positive constant C2 such that |f(n)| ≥ C2 |g(n)| for all but finitely many positive integers n. Algorithm 1.8 does not run in time polynomial in the size of the input. Exercise 4, Section 1 11. ¬(∀x∃y∀zP(x, y, z)) ≡ ∃x¬(∃y∀zP(x, y, z)) ≡ ∃x∀y¬(∀zP(x, y, z)) ≡ ∃x∀y∃z¬P(x, y, z) Proofs c 2009 by From Discrete Mathematics, Seventh Edition, Richard Johnsonbaugh. Find a solution to the five-queens and six-queens problems. Notice that R is also reflexive and symmetric. For example, the graph G of Figure 3.4 does not have an Euler cycle since there are vertices of odd degree, yet Example 3.1 showed that G has a Hamiltonian cycle. Equivalence class containing a given by equivalence relation R: [a] = {x | x R a} 43. , n by n, we have 1k + 2k + · · · + nk ≤ nk + nk + · · · + nk = n · nk = nk+1 for all n ≥ 1; hence 1k + 2k + · · · + nk = O(nk+1). Therefore, cn = 2cn−1 + 1, n > 1. Lee takes computer science or Lee does not take mathematics. Therefore, 2n + 3 lg n = (n). At this point, the loop invariant becomes G = gcd(a, 0). 1 z a, 1 d a, 2 1 1 Figure 4.7 Initialization in Dijkstra's shortest-path algorithm. Figure 1.6 shows the result of storing 15, 558, 32, 132, 102, and 5, in this order, in initially empty cells. Reversing the order, if n − 1 is even, then n − 1 = 2k for some integer k; if n − 1 = 2k for some integer k, then n = 2k + 1 for some integer k; if n = 2k + 1 for some integer k, then n is odd, proves that if n − 1 is even, then n is odd. Suppose that G has a complete matching. Notice the difference between the two examples. k=1 Since 1 k=1 fk = f1 = 1 and f3 − 1 = 2 − 1 = 1, the equation is verified. (It's somewhat like looking up the answer in the back of the book!) You've shown that a1 + a2 + · · · + an+1 = F(n + 1), which is the Inductive Step. Compute an inverse modulo an integer Section 4 32. 5 ➜ Matrices of Relations A matrix is a convenient way to represent a relation R from X to Y . We devote the remainder of this section to the RSA public-key cryptosystem, named after its inventors, Ronald L. Inductive Step Assume that the algorithm correctly computes the sum when the input is n − 1. If j = 3, after the swap we have 5 9 ↑ i 17 23 21 ↑ j Next, i = 2. The point is that we found one value that makes the conditional proposition if n is prime, then n + 1, n + 2, n + 3, and n + 4 are not prime true. Prove √that for all√real numbers x and y, if xy ≤ 2, then either x ≤ 2 or y ≤ 2. Explain why they are not isomorphic. If E1 and E2 are mutually exclusive events, how are P(E1 ∪ E2), P(E1), and P(E2) related? The remainder gives the 2's bit. z is chosen as the product of primes p and q. How do we denote x is an element of the set X? Therefore, the number of equivalence classes is equal to the number of four-bit strings. How many strings do not contain the letter A, allowing repetitions? We generalize by allowing an arbitrary base b. Example 3.22 Task Scheduling Consider the set T of tasks that must be completed in order to take an indoor flash picture with a particular camera. Invalid (p ∨ r) → q q ∴ ¬p → r 6. Since these cases are disjoint, by the Addition Principle, there are 20 + 20 = 40 possibilities. Suppose that τ(R) = R. k=0 1. If X and Y are nonempty sets and X × Y = Y × X, what can we conclude about X and Y ? 7 7. , sr }, to find the next string β = t1 · · · tr , we find the rightmost element sm that is not at its maximum value. Let s equal the sum of the first digit plus three times the second digit plus the third digit plus three times the fourth digit, . Define parent in a rooted tree. a b 1. Given that there are 32 eight-bit strings that begin 101 and 16 eight-bit strings that begin 1101, how many eight-bit strings begin either 101 or 1101? a 33. How are the greatest common divisor and least common multiple related? By definition, the loop invariant is true the first time we arrive at line 5. The first chapter of the geometry book by [Jacobs] is devoted to basic logic. For example, the route in Figure 1.1 first meets the diagonal at point (3, 3). 360 Counting Methods and the Pigeonhole Principle TABLE 7.1 ■ Computing (a + b)3 . It follows that the greatest common divisor of 30 and 105, gcd(30, 105), is 15. Illustrate how your database would be used by posing and answering two queries. For each string in Exercises 1–3, explain (as in Example 4.10) exactly how Algorithm 4.9 generates the next r-combination. pn = a + spn−1 = a + s(a + spn−2) = a + as + s2 pn−2 = a + as + s2 (a + spn−3) = a + as + as2 + s3 pn−3 .. First, we place the root v1 at the top. If the middle item is the desired item, binary search terminates. Floppy disk storage is better than nothing. By the Multiplication Principle, there are 3 · 5 = 15 dinners. Suppose that G1 and G2 are simple graphs. an = −8an−1 − 16an−2 ; 32. Exercises 74–79 refer to the following situation. Prove that if x ≤ y + ε for every positive real number ε, then x ≤ y. Such a search may be carried out by people or by a computer. Replace any occurrence of e in P by P. If so, prove it; otherwise, give a counter- 47. From x ≤ d and d ≤ d2 , we may derive x ≤ d2 from the same previous theorem. In a game tree, each vertex shows a particular position in the game. Section 2 5. Nineteen persons have first names Zeke, Wally, and Linda; middle names Lee and David; and last names Yu, Zamora, and Smith. Let c1 = c 0 + 3 = 3 an = 1 . This form of the definition can often be used to prove that a function is onto. Is this possible? Since Farley can choose values that make x2 < y + 1 false (e.g., x = 2, y = 0), Farley can win the game. In case x ∈ Y , CX−Y (x) = 0 = CX (x)[1 − 1] = CX (x)[1 − CY (x)]. , cn , the value t, and n p(t) poly(c, n, t) { if (n == 0) return c0 return t ∗ poly(c, n − 1, t) + cn } Let bn be the number of multiplications required to compute p(t). If the number of contestants is not a power of 2, some contestants receive byes. By replacing O by and "at most" by "at least" in Definition 3.11, we obtain the definition of what it means for the best-case, worst-case, or average-case time of an algorithm to be of order at least g(n). Draw the complete game tree for nim in which the initial position consists of two piles of three tokens each. Show the output from Algorithm 4.14 when n = 2. If every upper bound b for X satisfies b ≥ a, we call a a least upper bound for X. Therefore, the number of ways the votes can be counted in which Wright is never behind Upshaw is Cr . Prove that if x1 ≤ x ≤ xn , then n |x − xi | = i=1 n−1 |x − xi | + (xn − x1), i=1 j=1 n2 (n2 − 1) . For a feature set F , we have P(F | Cj)P(Cj) P(Cj | F) = n . (4.2) 31 Sets and Logic This process of drawing a conclusion from a sequence of propositions is called deductive reasoning. Finally, we find that the value of A is 3. 3n2 + 2n lg n 7. Find a theta notation for the worst-case time required by the following algorithm: iskey(s, n, key) { for i = 1 to n − 1 for j = i + 1 to n if (si + sj == key) return 1 else return 0 } 234 29. (4.1) We first show that for all c, d ∈ X, if c R d, then [c] = [d]. Proof We must prove √ If n is composite, then n has a divisor d satisfying 2 ≤ d ≤ n, and If n has a divisor d satisfying 2 ≤ d ≤ √ n, then n is composite. We show that there exists k such that sk < A. Since b = −tan β and k = tan α, we conclude that the price stabilizes if and only if b < k. But we assumed that (b, b) ∈ R. Since X ∩ Y = X ∩ Z, x ∈ X ∩ Z. The preceding computation shows that the remainders, as N is successively divided by 2, give the bits in the binary representation of N. [Halmos; Lipschutz; and Stoll] are recommended to the reader wanting to study set theory in more detail. (Today is Monday and it is raining) and it is not the case that (it is hot or today is Monday). Therefore a1 is included in the product. ‡ "Divides" means "divides evenly." More formally, we say that a nonzero integer d divides an integer m if there is an integer q such that m = dq. i b Root 3. Next, we compute circle's values by minimizing over the children. Since each internal vertex has at most two children, such a tree can have at most four terminal vertices (see Figure 7.8). (The idea is that x is "free" to roam over the domain of discourse.) We call the variable x in the universally quantified statement ∀x P(x) (5.1) a bound variable. Then (x, y) ∈ R1 and (x, y) ∈ R2 . Figure 2.12 After increasing the flow on path (a, b, e, f, z) by 1. Show that a tree is a bipartite graph. Of course, if the worst-case time to solve the problem is proportional to a high-degree polynomial, the problem can still take a long time to solve. The Lucas sequence Ln = Ln−1 + Ln−2 , n ≥ 3; L1 = 1, L2 = 3 24. In this case we say that a collision has occurred. Therefore, the n × n board is correctly tiled. Example 1.13 There are frequently many different ways to prove a statement. i=1 The last equality follows from Definition 6.1, which states that the sum of the probabilities of all outcomes equals 1. Then (x, y), (y, z) ∈ R. The variable n is incremented and becomes 2. Let g(n) be a solution of (2.20). 22. Simple graph; five vertices having degrees 2, 3, 3, 4, 4 18. , 20} and the range Y is some subset of {0, 1, . Proof Let an denote the number of comparisons with null in the worst case required by Algorithm 8.13, where n is the total number of vertices in the trees input. If 4 megabytes is better than no memory at all, then we will buy a new computer. Higher Bacon numbers are defined similarly. Example 3.1 Since 105 mod 30 = 15, by (3.1) gcd(105, 30) = gcd(30, 15). Our next example shows that sometimes it is possible to define a relation by giving a rule for membership in the relation. If the statement is true, prove it. What is the probability that the student answers exactly five questions correctly? Notice that edge (e, b) is improperly oriented and so has its flow decreased by 1. Thus U ⊆ A ∪ U. Let X be a set with least upper bound a, and suppose that t > 0. (n lg n) = (−2 + lg n) Therefore, an = (n lg n), so merge sort is (n lg n) in the worst case. From the definition of max, it follows that d ≥ d1 and d ≥ d2 . On the Tao Spanish ? (The string vx vy is the string vx followed by the string vy .) Notice that f is one-to-one. If x ≤ 1, the conditional proposition if x > 1, then x + 1 > 1 is vacuously true. 2 Therefore, |P(X)| = 2|P(Y)| = 2 ·

2n = 2n+1 . ¬(p ∨ q) ∧ r 47. What is a collision for a hash function? Finally, at some point the n − 1 disks were moved on top of the largest disk, which required at least dn−1 additional moves. Prove that for all n ∈ Z, n is even if and only if n + 2 is even. Write each proposition in Exercises 28–33 in words. {(a, b), (a, c), (b, a), (b, d), (c, c), (c, d)} a b c 6 2 11 32. This time is called the worst-case time for inputs of size n. Let the five cards be the pigeons and the four suits be the pigeonholes. 396 Recurrence Relations Example 2.4 Find an explicit formula for c , the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved (see Example 1.9). ∀x(x > 1 → x/(x2 + 1) < 1/3) 54. Example 3.6 If a > 1 and b > 1 (to ensure that logb a > 0), by the change-of-base formula for logarithms, logb n = loga n logb a for all n ≥ 1. , vk) 499 Graph Theory of length k = 2. Example 1.4 Let A = {1, 2, 3} and B = {y, 4}. Basis Step (k = 1) If k = 1, the 2 × 2 deficient board is itself a tromino and can therefore be tiled with one tromino. (3.13) Note that s is almost the desired value; the problem is that s may not satisfy 0 < s < φ. Exercise 66 shows that 1+ 1 n n i=0 n n ai an−i i=0 5. In the California Daily 3 game, a contestant must select three numbers among 0 to 9, repetitions allowed. The program should find a shortest path and its length. Then, either 1. He was professor of astronomy at the University of Dublin, where he published articles in physics and mathematics. There are C(10 + 3 − 1, 3) = 10 compact discs to Mary, Ivan, and Juan. By Exercise 32, the 5 × 5 subboard with a corner square removed has a tiling. ∀x ¬P(x) 20. (Exercise 16 asks for an algorithm that generates all r-permutations of an n-element set.) 342 Counting Methods and the Pigeonhole Principle Example 4.12 To construct the permutation of 12 1 0. Since 163542, we should keep as many digits as possible at the left the same. Gray codes have been used in telecommunications Find all simple cycles in the following graph. sn , a, and key (the value to search for) Output: The index of key, or if key is not found, 0 i=1 0. am + am+1 + · · · + an 12. Show that n > 0 has an inverse modulo φ > 1 if and only if gcd(n, φ) = 1. Evaluate each term in the game tree. Similarly, Figure 7.6 shows that if we begin by comparing one coin against one coin and the outcome is equal, the decision tree can account for only three outcomes. Show that any graph having four or fewer vertices is planar. i=2 for all n ≥ 3. Now f29 = 514229, f30 = 832040, and f31 = 1346269. ¬p ∨ ¬(q ∧ r) 20. Show that an = 1 n [4 + (−1)n+1] n. In (b), P (shown in color) is a path in G from v to v1 , and P (shown in heavy color) is the portion of P starting at v whose edges are also in G . The Egyptians of antiquity expressed a fraction as a sum of fractions whose numerators were 1. A full binary tree is a binary tree in which each vertex has either two children or zero children. But now T has more than n − 1 edges. Any value for np ≥ 0 will do, so we take np = 0. To compute a and t, the proof of Theorem 3.7, which is illustrated in Example 3.8, first finds gcd(a, b) and then works backward from the last equation of the computation of gcd(a, b) to the first equation obtained. Else increase n by 2 and continue with step 2. Some proofs of p → q involve combining the proofs of p → q and q → p. In how many ways can the 8 bulbs be illuminated? (The vertices denote actors.) {x, y} ∈ R if x − y = 2. Let A be a matrix whose ijth entry is 1 if the ijth entry of either A1 or A2 is 1. Let P(x, y) be the propositional function "x ≤ y" with domain of discourse Z × Z. C(9 + 2 − 1, 9) 21. The statement p is not a proposition, because whether p is true or false depends on the value of p. gcd(243, 100) = gcd(100, 43) = gcd(43, 14) = gcd(14, 1) = gcd(1, 0) = 1, s = 226 40. (x, y) ∈ R if x = y. Since the comparison succeeds, the algorithm returns i = 1 to indicate that p was found in t starting at index 1 in t. We may then use existential generalization to conclude that ∃x P(x) is true. Finally, we introduce a supersink z and edges of capacity 1 from each of J1 , J2 , J3 , J4 , and J5 to z. The codomain of each function is also the set of real numbers. Suppose, for example, that we want to count the total number of items available at Kay's Quick Lunch. †A starred exercise indicates a problem of above-average difficulty. If we define a relation R on X by (x, y) ∈ R if x divides y, then R = {(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}. If we define f(X) = X, where X ∈ P(U), then f is a unary operator on P(U). To decrypt the message, compute cs mod z, which can be shown to be equal to a. ¬(∃x P(x)) 17. 275 Algorithms 6. 150 7. Prove that if a relation R on a set X is not symmetric but transitive, the collection of pseudo equivalence classes (see Example 4.15) does not partition X. Suppose that Algorithm 3.1 is used (5.5). Finally, consider R ◦ S. 2 Thus the time required by Algorithm 3.1 is (n2). Moreover, each child is designated as either a left child or a right child. The domain of each function is the set of real numbers. By inspection, gcd(15, 0) = 15. If we list all possible dinners consisting of one main course and one beverage, HT, HM, HC, HR, CT, CM, CC, CR, FT, FM, FC, FR, we see that there are 12 different dinners. If the robot begins by taking a 1-meter step, a distance of n − 1 meters remains; but, by definition, the remainder of the walk can be completed in walk(n − 1) ways. CX∪Y (x) = CX (x) + CY (x) is true if X ∩ Y = ∅ if X ∪ Y . Example 1.9 Let X = {x | 3x2 − x − 2 = 0}. By definition, n is a solution, so no one can fake from this optimal solution, so cn ≤ 2n . , (Cn) are n simple cycles of length k in G2 . In the example of Figure 4.1, the relation is "is the same color as"; hence equivalent means "is the same color as." Each set in the partition consists of all the balls of a particular color. Draw a decision tree that gives a reasonable strategy for playing blackjack (see, e.g., [Ainslie]). The property expressed by equation (1.1) is called conservation of flow. As an example, personal data are frequently stored and retrieved by hashing on employee identification numbers. To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. ex = i! i=0 In particular, ∞ (−1)i 1 1 1 e−1 = = − + ···. (2) Prove if q then p. One of the earliest problems to be proved unsolvable is the halting problem: Given an arbitrary program and a set of inputs for the program, will the program eventually halt? You could always use the Strong Form of Mathematical Induction. Suppose that m and n be positive integers. This function is not onto Y because there is no arrow pointing to c. This fact can be used, just as in the previous examples for recurrence relations of order 2, to solve a linear homogeneous recurrence relation of order k with constant coefficients. The following figure shows the grid for k = 4: 22. Since ns mod φ = 1, 297 Introduction to Number Theory cs mod z = (an mod z)s mod z = ans mod z = a. Show that the conclusion, Lynn loves Apple, follows from the hypotheses. We then move to lines 7 and 8, where we set a to 36 and b to 0. Example 8.2 The trees T1 and T2 of Figure 8.3 are not isomorphic because T2 has a vertex (x) of degree 3, but T1 does not have a vertex of degree 3. Suppose that X contains an element less than or equal to n + 1. | | | | [s1 , s2 , . Write an algorithm that finds the length of a shortest path between two given vertices in a connected, weighted graph and also finds a shortest path. Then there exist integers p and q such that x = p/q. What is the interpretation of ∀x∃yP(x, y)? Explain why Algorithm 1.8 is not considered to be a polynomial-time algorithm. In this situation, F becomes the new sample space. For every real number x,√ if x is an integer, then x is rational number. If D is equal to the data item at the current vertex, we have found D, so we stop. Therefore, the ijth entry of A is 1. Explain how to construct an optimal Huffman code. Since we are primarily concerned with estimating the time of an algorithm rather than computing its exact time, as long as we count some fundamental, dominating steps of the algorithm, we will obtain a useful measure of the time. Geometric sum: n ai = am + am+1 + · · · + an i=m ai = am · am+1 · · · an i=m n ar i=0 25. By Exercise 92, 00 333. A sequence is a decreasing if an > an+1 for all n. Tune 19. Decrypt the message UTWR ENKDTEKMIGYWRA using the key of Example 4.1. 3. Example 6.5 Suppose that P is a propositional function whose domain of discourse is {d1 , . , You do not run 10 laps daily, you do not take multi-vitamins, and you are not healthy. , Xt are sets and the all the set Xi has n elements. Define bipartite graph. (The set D3 is defined before Exercise 1.) 27. Now suppose that pn is true; that is, all pi are false. In the 1990s, David Corriveau of Quebec won three straight games of a computer keno game in Montreal, each time choosing 19 of 20 numbers correctly. The distinct three-digit integers that contain 7 are 107 and 1xy, where xy is one of the two-digit numbers listed above. We obtain the recurrence relation An = An−1 + (0.12)An−1 . Definition 1.28 Example 1.29 If f is a function from X to Y and the range of f is Y, f is said to be onto Y (or an onto function or a surjective function). Solve Exercise 6 for the fourteen-coins puzzle. For example, in Figure 1.5, edges e1 and e2 are both associated with the vertex pair {v1 , v2 }. We note that if n and d are positive integers and d | n, then d ≤ n. The months with Friday the 13th in year x are found in row y= x+ x−1 4 − x−1 100 + x−1 400 mod 7 in the appropriate column: y 0 1 is one-to-one and onto. This can be done in sn,k ways. Input: A network with source a, sink z, capacity C, vertices a = v0 , . Next, we show that Ti is contained in a minimal spanning tree T . Then m m m−k m/2 or j ≤ m/2 and i = 1, sn = sn−1 + 2n (0) Give a proof using mathematical induction that your algorithm for part (a) is correct. 369 Counting Methods and the Pigeonhole Principle Notes An elementary book concerning counting methods is [Niven, 1965]. , n have among themselves orders 1, 3, 4, . As a concrete example, consider computing 5!. Kronos 4. Example 5.4 After Example 5.2, we showed that if A is the matrix of the graph of Figure 5.2, then A2 = c d e (a b c d e) 2 0 2 0 1 | 0 3 1 2 1 | | 2 1 3 0 1 . More precisely, if n = 2k , the statement x = x + 1 is executed k + 1 times. Example 1.24 If X is the set of persons who have social security numbers, and ∈ X his or her social security number SS(x), we obtain a one-to-one function since distinct persons are always assigned different social security numbers. Prove that your graph is the Petersen graph; that is, prove that it is isomorphic to the graphs in Exercise 4. If you succeeded, the graph is bipartite and you have discovered the sets V1 and V2 . Selection sort selects the largest element, places it last, and then recursively sorts the remaining sequence. Notice that Definition 1.11 states that if e is an edge in a bipartite graph, then e is incident on one vertex in V1 and one vertex in V2 . Example 5.16 Write the statement Every rock fan loves U2, symbolically. A relation is reflexive if and only if its matrix has 1's on the main diagonal. (x, y) ∈ R if x ≥ y. 78. Moreover, since f is one-to-one, f(C1), f(C2), . 2 · 4 · 6 · · · (2n) n+1 14. ∨ p2 G1 40. How many edges does G contain if K = 2n + 2? 2 ∀ Solving Recurrence Relations To solve a recurrence relation involving the sequence a0 ,a1 , . 2 Example 2.5 Hexadecimal to Decimal Convert the hexadecimal number B4F to decimal. See Theorem 1.17. Proof If 2m2 = 40, we must have 2m2 ≤ 40. Solving for x and n, we obtain a = 1, 1 b=− . If ∅ is not a subset of some set Y , according to the discussion preceding Example 1.9, there would have to be at least one member of ∅ that is not in Y . What can you say about a vertex in a rooted tree that has no ancestors? But for the tree to be balanced, the root must have two children. Three squares in a row form the only other type of polyomino of order 3. (a) The only positive integers that divide‡ 7 are 1 and 7 itself. The smallest i for which si = ti is i = 3. We next compute a8 = a4 · a4 , which uses 1 additional multiplication. The 7 × 7 matrix whose ijth entry is 1 if i + 1 divides j + 1 or j + 1 divides i + 1, i = j; whose ijth entry is 2 if i = j; and whose ijth entry is 0 otherwise 19. The argument is by contradiction, (p , then T denoting true and F denoting false, and for each such combination lists the truth value of P. If the domain of discourse is Z+ × Z+ , ∃x ∀y P(x, y) is true; however, ∀x ∀y P(x, y) is false. How many commutative binary operators are there on {1, 2, . (e) Is there a complete matching? Since the hypothesis p is false, the conditional proposition is true. Find a recurrence relation and an initial condition for the sequence {bn }. Answer 8. Prove that P* is reflexive. Prove that the vertices in a connected, weighted graph and label the rows and columns of a matrix with its vertices. Then P* is the reverse of P. By completing the following steps, give a proof by induction on p to show that every fraction p/q with 0 < p/q < 1 may be expressed in Egyptian form. Figure 1.5 5, 1 a b d 605 Network Models 6. In the Maryland Lotto game, to win the grand prize the contestant must match six distinct numbers, in any order, among the numbers 1 through 49 randomly drawn by a lottery representative. Then xy ≤ 0 and |xy| = −xy = (x)(−y) = |x||y|. Subtracting |X∩Y | from both sides of the preceding equation gives the desired result. Develop a game-playing algorithm for Kalah based on the evaluation function of Exercise 32. M A SEVEN YEARS 3. [Kurosaka] gave an algorithm to find the bad coin and determine whether it is heavier or lighter than the others using only a pan balance in n weighings in the worst case. If a coin is removed, its label is reversed. We give a proof by contradiction. Draw all nonisomorphic simple graphs having four vertices. The real numbers furnish another example of a mathematical system. 4 103. Pick a vertex v ∈ V2 , v = v1. Let m be the worst-case time to solve Pn by the algorithm of Exercise 74. Consider, for example, vertex c. Professor Euclid is paid every other week on Friday. , sr }. By Theorem 4.7 there is a complete matching. The following is one way to show that two simple graphs G1 and G2 are not isomorphic. Another spanning tree of the graph G of Figure 3.1 is shown in Figure 3.2. a a e1 e 12 b e7 c d f e4 e 11 c d e f e3 e 10 e 9 e 8 e6 b e5 e2 h g g Figure 3.1 A graph and a spanning tree shown in black. Define a relation R on a set X as S R if |domain s| = |domain t| and if the domain of s is {m, m + 1, . We say that Algorithm 1.8 is not a polynomial-time algorithm. 172 Functions, Sequences, and Relations Given a relation R from X to Y, we may define a relation from Y to X by reversing the order of each ordered pair in R. The geometric sum is the sum a + ar + ar 2 + · · · + ar n . A formal proof would show that for every x, if x ∈ (A ∪ B), then x ∈ A ∩ B, and if x ∈ A ∩ B, then x ∈ (A ∪ B). Eventually, we will reach a vertex, with degree 1, and we have a cycle or a vertex to leave (see Figure 7.9). This contradicts the choice of fk1 as the largest Fibonacci number satisfying n ≥ fk1 . (b) For each improperly oriented edge (j, i) in P, 0 < Fij . In this case, the for loop runs to completion, after which the code returns true [to indicate that x% P(x) is true] and terminates. Addition Principle 3. 18. Since m = 5, the condition m > 0 is true; so we execute the body of the while loop. Thus, if n = 2k , an = 2 + lg n. Thus there are at most m(k − 1) members in the domain of f . Find the probability that among n ≥ 2 persons, at least three people have birthdays on the same month and date (but not necessarily in the same year). Prove or disprove: Where numbers p ≥ 1 and 1 and b such that ab is irrational. x1 = 1, x2 ≥ 0, x3 ≥ 0 25. Prove that the greatest lower bound for X. Water for cities A and B is delivered from wells w1 , w2 , and w3 . Since a string is a sequence, order is taken into account. Since the while loop in Algorithm 2.19 executes (lg n) times, the total time for Algorithm 2.19 is O(lg n lg2 z). At lines 19 and 20, we label vertex b as (a, 3) since predecessor(b) = a and val(b) = min{, 3 − 0} = min{∞, 3 − 0} = 3. Trace Algorithm 1.8 for the input n = 539. Example 1.25 If a function from X to Y is one-to-one, then each element in Y in its arrow diagram will have at most one arrow pointing to it (see Figure 1.9). Which are simple cycles? Y × X × Y × Z In Exercises 65–72, give a geometric description of the set of points in the Figure 8.1. As in the case of graphs, we can show that two trees are not isomorphic if we can exhibit an invariant that the trees do not share. C + C + D * D * ? What is a clause? Let X denote the set of n elements obtained by S by considering the ni objects of type i distinct for i = 1, . Give an example of a simple, connected, nonplanar graph for which e ≤ 3v − 6. From p → r and ¬r, we may use the modus tollens to conclude ¬p. preorder ABDCE 4. An optimal vertex cover is a vertex cover having the smallest possible number of vertices. Given a list of primes and the number n = 1 in N, where a ≠ 1 or ab has a nonempty suitable that the library has at least six copies of each of these books. The next combination is 1357. Write a depth-first search algorithm that finds all solutions to the four-queens problem. The proof explicitly constructs Hamiltonian cycles for certain smaller boards and then pastes smaller boards together to obtain Hamiltonian cycles for the larger boards. The converse is similar and is left as an exercise (see Exercise 30). We try to construct a counterexample in which R and S are transitive but R ◦ S is not transitive. Since T acyclic, P contains no cycles. 12876543 7. Is T decreasing? Similarly, there are 2000 multiples of 5 between 1 and 10,000 and 909 multiples of 11 between 1 and 10,000. (p ∧ q) → ¬(r ∨ q) Given that proposition p is false, proposition q is true, and proposition r is false, determine whether each proposition in Exercises 16– 21 is true or false. If the statement is true, explain; otherwise, give a counterexample. Every student is taking a math course. Evaluation of game tree, or even a part of a game tree, can be a time-consuming task, so any technique that reduces the effort is usually welcome. Find an expression, which is the and of clauses, equivalent to (p ∨ ¬q) → ¬rs. A query is a request for information from a database. , an = bn . 135. Similarly, the only edge of the form (w, z) having flow is (v, w). Use decision trees to find a lower bound on the number of comparisons required to sort six items in the worst case. .}. Let (v0 , v1 , . } predecessor(w) = v val(w) = min{, U is the set of unexamined, labeled vertices U = {a} // continue until z is labeled while (val(z) == null) { if (U == ∅) // flow is maximal return F choose v in U U = U − {v} Δ = val(v) for each edge (v, w) with val(w) == null if (Fvw < Cvw) { predecessor(w) = v val(w) = min{, Cvw − Fvw } U = U ∪ {w} } for each edge (w, v) with val(w) == null if (Fwv > 0) { predecessor(w) = v val(w) = min{, Fwv } U = U ∪ {w} } } } // end while (val(z) == null) loop // find path P from a to z on which to revise flow w0 = z k=0 while (wk ¬= a) { wk+1 = predecessor(wk) k =k+1 } P = (wk+1 , wk , . The least common multiple of m and n is the smallest positive common multiple of m and n. Therefore, C(n + 1, k) = C(n, k + 1) + C(n, k). We obtain 5n−1 = 6n−2 = 5(2n−1 + 4 · 3n−1) − 6(2n−2 + 4 · 3n−2) = (5 · 2 − 6)2n−2 + (5 · 4 · 3 − 6 · 4)3n−2 = 4 · 2n−2 + 4 · 36 · 3n−2 = 22 2n−2 + (4 · 32)3n−2 = 2n + 4 · 3n = sn . Therefore, C is a subset of A and we write C ⊆ A. Develop a framework for an expert system to detect Exercise 11 for S1 , S2 . . [Project] Prepare a report on a commercial relational database such as Oracle or Access. Use the Euclidean algorithm to find integers s and t satisfying s · 396 + t · 480 = gcd(396, 480). Find the maximum height of a full binary tree having t terminal vertices. Applying (3.2) to expressions 3 and 4, we derive 5. bn = bn−1 + 1, b0 = 0 10. , an 1, takes order into account; that is, (a1 , a2 , . Recursion is a powerful, elegant, and natural way to solve a large class of problems. We state many useful properties of sets in Theorem 1.21. Prove that the pseudo equivalence class [b] (see Example 4.15) is empty. For example, 3 ∈ {1, 2, 3, 4}, but 3∉ { x | x is a positive, even integer}. State the universal generalization rule of inference. ∃x∀y T2 (x, y) 13. List the elements of the set with blanks between them: x1 − x2 − x3 − x4 − x5 − x6 − x7 − x8 − x9 − x10 If we first compute a2 = a · a, which uses 1 multiplication. What does it mean for two free trees to be isomorphic? There was at least A(m, n) > 1 for all m ≥ 1, n ≥ 0 3 0 1 0 1 0 Example 5.2 The matrix of the relation R of Example 5.1 relative to the orderings 2, 3, 4 and b, a, c is (f) b a c (d b a c) 2 0 0 0 1 | 3 [1 0 1 0] | 4 ... Therefore, it is minimal. Since p → 3 < 4 = q and si = ti for i = 1, 2, 3, condition (a) of Definition 4.1 is satisfied. A judge ruled that the ordinance was "vague." Presumably, the intended meaning was "It shall be unlawful for any person to keep more than three [3] dogs or more than three [3] cats upon his property within the city." 75. The idea is that whenever we compute the greatest common divisor, we also compute the values of s and t. We show that X is nonempty using proof by cases. Prove or disprove: Every rational numbers a and b such that ab is rational. First we solve the preceding recurrence relation in case n is a power of 2, say n = 2k . For example, (b), (c) and (c), b) are in R, but (b, b) is not in R. A vertex, such as vertex v4 in Figure 1.5, that is not incident on any edge is called an isolated vertex. 209 Algorithms Algorithm 1.2 Finding the Maximum Value in a Sequence This algorithm finds the largest of the numbers s1 , . leaden 21. , and step t can be done in nt ways, then the number of different possible activities is n1 · n2 · · · nt . 10. State the universal generalization rule of inference. (For the degenerate cases k = 1 and k = n, we set P0 = 1.) Therefore, the total number of ways is 426 for all y ≥ 0. Show how Algorithm 3.2 executes in case key = 'Z'. Notice that Figure 1.1 has this property. A network consists of n nodes. Definition 1.9 440 The complete graph on n vertices, denoted Kn , is the simple graph with n vertices in which there is an edge between every pair of distinct vertices. How many ways are there taking business and neither French nor music? Give an example of a proof by cases. Show the bottom-up evaluation of the mean by the smoothing sequence by an approaching from end, Example 2.12 Let An = s1 + s2 + · · · + sn n be the average of the real numbers s1 , . We conclude that X = ∅. Choose the largest integer n satisfying 1 1 p + + · · · + k . The number of x's between the first and second 1 represents n1 copies of the first element in the set. Input: Output: A connected, weighted graph with vertices 1, . Therefore 1/n < b − a. We give the rewritten Fermat's We next introduce a new variable. Since we have n ≥ 2 then we ith the and add the clauses to the hypothesis. } Let bn be the number of comparisons (lines 10 and 14) required for an input of size n. Similarly, g with b. Explain, in words, the meaning of the statement. At line 8, z is not circled. A A 2. Suppose that we choose b. Each subsequent vertex we have is incident on n − 1 ways (since it must be different from its predecessor). Theorem 1.12 Inclusion-Exclusion Principle for Two Sets If X and Y are finite sets, then |X ∪ Y | = |X| + |Y | − |X ∩ Y |. (The idea is that x is "bound" by the quantifier ∀.) We previously pointed out that a propositional function does not have a truth value. Consider the reverse problem—converting a decimal number to base b. Give an example of a relation that is not transitive. To obtain a solution, we first prove that the algorithm is fast. Prove that a graph G that represents all of the cubes. v1 v1 v2 v2 v3 v4 v3 T1 v4 T2 Figure 8.11 Nonisomorphic binary trees. In how many ways can 10 balls be selected if exactly one red ball and at least one blue ball must be selected? Write an algorithm that returns the greatest common divisor of two integers given two integers. The following exercises show how to find all solutions in nonnegative integers to x1 + x2 + x3 = n. Let X be a nonempty set. Using Exercise 16, write an algorithm to compute the gcd that uses subtraction, but not the modulus operation. 3 2 5 23 24. Basis Step (n = 0) If n = 0, X is the empty set. 7 30. Thus [1] = {x ∈ X | 3 divides x − 1} = {1, 4, 7, 10}. Prove that a relation R on a set X is antisymmetric if and only if , then (x, y) ∈ R. How many eight-bit strings have either the second or the fourth bit 1 (or both)? (For Exercise 35) ∃x P(x) ∧ ¬Q(x)). For d = 2, 3, . We require that P(H | Pos) = 0.5 = (0.95)P(H) . Write a program that tests whether two logical expressions in p and q are logically equivalent. E | F 13. Now suppose that we want to store 257. // rest of elements are successors of sm for j = m + 1 to r sj = sj−1 + 1 21. 3 + 5 < 2 if and only if 1 + 3 = 4. Similarly, ba = ab. (If, for example, 2m2 = 40, then we must add 2m2 to 2m2 , the sum 2m2 + 3m2 will exceed 40.) If 2m2 ≤ 40, then m2 ≤ 20 and m can be at most 4. Now gcd(fn+1 , fn+2) = gcd(fn+1 , fn+1 + fn) = gcd(fn+1 , fn) = 1. Assuming that the events "Joe passes" and "Alicia passes" are independent, find the probability that Joe or Alicia, or both, passes the final exam. We now know that gcd(r, x) = 2. 8 8, but the fact that R is transitive does not allow us to infer (x, y) ∈ R. We see that we can generate all of the permutations of {1, 2, . ∃x∀y P(x, y) 82. Each player, if a direct proof seems unpromising, try a proof by contradiction. Our next theorem shows that Dijkstra's algorithm is (n2) in the worst case. Since R is [c]. Thus the permutation following the given permutation will begin 16. One example is a binary search tree. Rewrite the sum 83. The root v1 in T1 has a right child, but the root w1 in T2 has no right child. We next evaluate the children of D and then D itself. , e13 } of edges. 109. In this chapter, "increasing sequence" means strictly increasing; that is, the sequence a is increasing if an < an+1 for all n. 11011 10. k=1 Now n+1 k=1 fk = n fk + fn+1 k=1 = (fn+2 − 1) + fn+1 by the inductive assumption = fn+1 + fn+2 − 1 = fn+3 − 1. Thus the height of the given tree is 4. Suppose that the height of the tree is 4. Find the probability that the number of ways that a given rook is 30 and explain your answer. Moreover, the symbol ∀ is called a universal quantifier. Hypotheses: Ken, a member of the Titans, can hit the ball a long way. Assume that the following are previous theorems: If a, b, and c are real numbers, then b = 0 = b and a(b + c) = ab + ac. (n2) 11. 109 Proofs Now assume that the statement is true for n; that is, assume a1 + a2 + · · · + an = F(n). Moreover, no additional vertices can be added to 1, 3, 5, and so the resulting vertex set has a directed edge between each two vertices. We assume that T is a tree. 100000 12. Therefore, x < R. Then the argument symbolically is p → q ¬r ∴ ¬p ∨ ¬r ∴ r 8. Prove? 7 For example, a person searching for the exit in a maze could randomly make a choice at each intersection. We have illustrated how to write any integer greater than 1 as a product of primes. Let G be a graph and let v be a vertex in G. By the Pigeonhole Principle, some pigeonhole will contain at least two pigeons (the matched socks). Since p can have any one of n − 1 values, for each ε > 0, compute X, there is a 1-to-1 function for the 3-valued. If n = 2 in the recurrence relation follows. If not, either n > c or a < c. Assume that G is connected. When better cars are built, Buick will build them. In this case, the Mathematics Department might or might not hire an additional faculty member. Draw K2,3 and K3,3 . Since each cube must appear in each subgraph exactly once, we cannot select any of the edges labeled 1 or 2 since we already have selected edges with these labels. Since each edge in P is properly oriented, at line 41 we increase the flow in each edge in P by = 2 to obtain Figure 2.8. We then return to the top of the while loop (line 3). , sn small index } small index = j swap(sj , ssmall index) } 3. ∃y = 4A + 2B + C. Write a program that uses a graphics display to show a tiling with trominoes of a 2n × 2n board with one square missing. Example 1.8 The integer n = 12 is even because there exists an integer k (namely k = 6) such that n = 2k; that is, 12 = 2 · 6. At this point, U = {d, z}. Can the first proof be extended to the case k = 4? In the number case when we replace x with, the proposition of is called the and of P. We show that Q(x) (3.4) that results is true. Example 5.1 The matrix of the relation R = {(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)} from X = {1, 2, 3, 4} to Y = {a, b, c, d} relative to the orderings 1, 2, 3, 4 and a, b, c, d is (a b c d 1 0 1 0 1 | 2 [0 0 1 0] | ... If there is no child, we create one, put in an edge incident on the root and new 537 Trees NEVER JUST DIE PROGRAMMERS LOSE OLD MEMORIES THEIR THEY Figure 5.5 Another binary search tree that stores the same words as the tree in Figure 5.4. vertex, and place the word in the new vertex. Again, we'll approach this by contradiction. The subset of rational numbers less than 2 is bounded above, but does not have a rational least upper bound. 4 1 C e 5 4 6 G2 G1 3. Continuing the discussion in the previous paragraph, suppose that we compute |X| + |Y |. The solution, with one ai coin removed, is optimal for n = . return large 8. , Xn : 28. To show how mathematical induction can be used in a more profound way, let Sn denote the sum of the first n positive integers: Sn = 1 + 2 + · · · + n. Character Frequency ! @ # $ % 2 3 7 8 12 The algorithm begins by repeatedly replacing the smallest two frequencies with the sum until a two-element sequence is obtained: 2, 3, 7, 8, 12 → 2, 3, 7, 8, 12, 5, 7, 8, 12, 12 → 8 → 12, 12, 15 → 27. Next, the algorithm then constructs Huffman trees working backward beginning with the two-element sequence 12, 20 as shown in Figure 1.13. It follows that p2 is even. 11111111 13. By contradiction, we find the Euler cycle (v6 , v4 , v7 , v5 , v1 , v3 , v4 , v1 , v2 , v5 , v4 , v2 , v3 , v6). Define a function from X* to X* as f(α) = αab. To decode a bit string, we begin at the root and move down the tree until a character is encountered. The theorem generalizes De Morgan's laws for logic (Example 3.11). All teachers had at least one student attend their office hours. This table expresses the relationship among identification numbers, names, positions, and ages. This latter fact can be used to solve certain problems. To verify the preceding equation, use algebra to manipulate the left-hand side of the equation [F(n) + an + 1] until you get [F(n + 1). Thus the Basis Step is n = 1. f(n) = O(g(n)) 41. Write an algorithm that uses the output of your algorithm in Exercise 33 to print each vertex and its parent. Then x ∈ X or x ∈ Y − X. There are 2 3 n This area is less than the area under the curve, which is equal to 1 252 n 1 dx = logen n. Basis Step (n = 1) J(2) = 1. 2 32. Rm, n) ≤ n + 1 22. While this technique becomes to time-consuming as n grows larger, it suffices for relatively small values of n. 48 94. Suggest additional properties for Example 1.7 that might be useful in comparing the results. This fact is not trivial to prove. That P(X ∩ Y) = P(X) ∩ P(Y) for all sets X and Y. 4 outcomes Figure 7.8 A sorting algorithm that makes at most two comparisons. (Here, by "diameter" we mean a line segment through the center of the circle with its endpoints on the circumference of the circle.) † 95. 3 e2 e3 2 e1 1 e4 e5 5 4 e6 7 e8 e7 6 G Figure 2.1 A connected graph with paths (1, e1 , 2, e2 , 3, e3 , 4, e4 , 2) of length 4 and (6) of length 0. σ(R1 ∩ R2) = σ(R1) ∩ σ(R2) 32. How many strings can be formed if we allow repetitions? Thus the total number of times line 3 is executed is (see Example 3.7) 1 + 2 + · · · + n = n(n + 1) 2. The edges now known to be in H form a cycle C. Another way to visualize a function is to draw its graph. The statement is false because if x = 1 and y = 0, then the conditional proposition is false. p(x) → (y < 0) → (x + y = 0) is false. For these choices of x and y, P(x , y) is false. Automata, Grammars, and Languages Richard Johnsonbaugh 665 Appendix: Matrices Richard Johnsonbaugh 713 Appendix: Algebra Review Richard Johnsonbaugh 733 References Richard Johnsonbaugh 741 Index 747 Sets and Logic 3 4 5 6 Sets Propositions Conditional Propositions and Logical Equivalence Arguments and Rules of Inference Quantifiers Nested Quantifiers Problem-Solving Corner: Quantifiers Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises Logic, logic, logic. If n = 1, T consists of one vertex and zero edges, so the statement is true for n = 1. The entire xy-plane 68. The Fibonacci sequence.) The following myth was also created to accompany the puzzle (and, as one assumes, to help market it). These four rooted trees are shown in Figure 8.9. (a) (b) (c) (d) Figure 8.9 The four nonisomorphic rooted trees with four vertices. The algorithm receives the three values a, b, and c as input and produces the value large as output. a 8. Show that e is contained in some minimal spanning tree. If the door hides a car, the game is over. We compute a4 using the formula a4 mod z = a2 a2 mod z = [(a2 mod z)(a2 mod z)] mod z, and so on. n = 7, φ = 20 36. What is the probability of choosing the winning numbers? Exercise 47 shows that the equation is true for x = 0, 1, 2 and for all x. The first term is 3. What is an index, and why is one used? , Xn gives this exactly if (x1 , x2 , . Case ··· If CX∪Y (x) = CX (x) + CY (x) for all x ∈ X, then X ∩ Y = ∅. Since ∃x ∀y P(x, y) is false, for every x there exists y such that P(x, y) is false. Thus 140 students are taking French or business or music. We shall verify our solution as follows. Any input A sequence si , . the graph of Exercise 18. Show that every graph has an even number of vertices of odd degree. The first and second columns. In how many ways could the distinct events if they appeared in at least one film together. ⎛4 3 3 3 ⎞ | 3 4 3 3 | | 3 3 4 3 3 | ⎝ ⎠ 3 3 3 3 3 4 3 2 3 Suppose that f is a function from X to X, f is also one-to-one. If m and n are even, all we can conclude is that m = 2k1 and n = 2k2 for some integers k1 and k2 . Draw the tree free of Exercise 1 as a rooted tree with root f . The vertices of the dual graph G consist of one point in each face of G, including the unbounded face. 1/[C(50, 5) · 36] 4 · C(13, 5) · 3 · C(13, 4)C(13, 2)2 28. For each of these arrangements, we can place the Martians in five of the eight in-between positions, which can be done in P(8, 5) ways. If we choose sets A and C so that there is an element that is in C, but not in A, we will have a counterexample. Example 8.3 Show that if we select 151 distinct computer science courses numbered between 1 and 300 inclusive, at least two are consecutively numbered. Repeat until reaching the first index or encountering an element that is less than or equal to the element to insert. By the inductive assumption, the number of vertices visited is the number of nodes of T1 is in preorder(T1) , and the order in which funnyorder visits the nodes of T2 is revpost(T2). The statement |∀x ∃y P(x, y) is true, where the domain of discourse is X × Y. But now |f(n)| > 2|g(n)| > 4|f(n)|, a 28. Hardy (1877–1947), used number theory as an example of a beautiful, but impractical, branch of mathematics. No 48. 225 Algorithms Example 3.10 Show that if f(n) = (g(n)) and g(n) = (h(n)), then f(n) = (h(n)). Definition 4.9 Example 4.10 Let R be an equivalence relation on a set X. Algorithm 3.8 Merge Sort This recursive algorithm sorts a sequence into nondecreasing order by using Algorithm 3.5, which merges two nondecreasing sequences. For the second statement, we assume that X ∩ Y = ∅ and prove that CX∪Y (x) = CX (x) + CY (x) for all x ∈ X. In the path (2.1), vertex 2 appears twice. Suppose that we are at line 9 for the ith time and we choose vi in T with minimum value L(v). Example 5.3 A tree that defines a Huffman code is binary tree. Explain why 1 ≤ bi ≤ n and 1 ≤ ci ≤ n. One type of existence proof exhibits a value of x that makes the statement true (and proves that the statement is indeed true for the specific x). = 1 = n+1 4. m = 2 or m = n − 2, or m = 3, or m = n − 3 and the order of the vertices. Write these codes as a sequence of numbers (b, c, d), (e, f), (g, h) and (f, g), 3). A vertex with degree 3 is not in a binary tree, but is valid in R. Now let us consider the situation when P(x) is false. If T is a full m-ary tree with i internal vertices, how many vertices does T have? Find the length of a shortest path from a to z. Suppose that we roll two dice and we seek the sum is 5. Now the market forces drive the price up to p2 , as we can see by moving vertically to the demand curve. Example 2.2 In the graph of Figure 2.1, (1, e1 , 2, e2 , 3, e3 , 4, e4 , 2) (2.1) is a path of length 4 from vertex 1 to vertex 2. In graph terminology, we are asking when the n-cube contains a simple cycle with 2n vertices as a subgraph or, since the n-cube has 2n processors, when the n-cube contains a Hamiltonian cycle. There are eight of them. i!j!k! (C(k, k/2 (C(n, k)) k + 1, Ah-ha-k . Many proofs in mathematics and computer science are deductive arguments. Write an algorithm that encodes and decodes text given a Huffman code. Discussion Consider the Inductive Step, where we want to prove that n-cents postage can be achieved using only 2-cent and 5-cent stamps. Define a sequence {gn } as g1 = c1 and g2 = c2 for constants c1 and c2 , and gn = gn−1 + gn−2 for n ≥ 3. and 3,7,7. Each row of the table can assign the function the value T or F. 1 1 n+ · · + 2 3 n−1 Counting Methods and the Pigeonhole Principle (f) Show S4,3 = 6. Let X be a set with greatest lower bound, as a guide, we need to find a k to 2 in a region. Since n ≥ 1, the condition m > 0 is true, so we execute the body of the while loop. Samantha Alexandra Attacking the Problem Where do we start? Show that if a > b ≥ 0, then From the 6 one-to-one and onto if gcd(m, n) = 1. Rivest, Adi Shamir, and Leonard M. A reasonable question is: How does one come up with the formulas? Figure 3.2 The graph of Hamilton's puzzle. We show that the weights are positive integers and that we want to find a shortest path from a vertex a to vertex z. Next, we place A, C, and E in these positions. What is a Hamiltonian cycle? If n ≥ 2 and k ≥ n/2, inequality (5.1) is satisfied. 11011 + 1101 22. Region 3 represents A ∩ B, the elements in both A and B. If g is onto, then f ◦ g is onto. Since there are a finite number of vertices, we will eventually return to a previously visited vertex. Let Cij denote the capacity of the directed edge (i, j). Find a formula for the sequence c defined by 18. The digraph of a reflexive relation has a loop at every vertex. Take 17, key sk (Z' < 'Z') is false, so at line 10 we set i to 4. For example, the partition {x1 }, {x2 }, {x3 , x4 }, {x5 , x6 }, {x7 , x8 }, {x9 }, {x10 } would be represented as x1 | x2 | x3 x4 | x5 x6 | x7 x8 | x9 | x10 . G 23. Steffi Graf, being the sole undefeated player, became Wimbledon champion. Show that each level of the tree is an element greater than or equal to n + 1, X must contain n + 1, which is the least element of the set of nonnegative integers over {0, 1, 2} of length n that contain an even number of 1's. The situation is analogous to the inductive step in mathematical induction when we must find a smaller case (e.g., case n) within the larger case (e.g., case n + 1). Find r0 . The argument in the preceding paragraph can now be repeated to show that the greedy solution is optimal. Use proof by cases to prove that sgn(xy) = sgn(x)sgn(y) for all real numbers x and y (sgn is defined in Exercise 34). defined by the equations c1 = 0, p 1 1 1 + + ··· + , q n1 n2 nk cn = 4cn/2 + n for all n > 1. ∃x∀y(x2 < y + 1) 27. Since c1, 9) 51. A ∩ B 41. The method usually used to generate pseudorandom numbers is called the linear congruential method. For example, suppose that we have verified that blocks 1-5 are marked, as shown in Figure 4.1. If A is marked, we then show that block 6 is also marked, by (4.2) (taking n = 5), block 6 is also marked. An input-output table. 9. Suppose that the kth entry in all Xi is nonzero, then (i, k) ∈ R2 ◦ R1 . Therefore, the bit string (1, 1) corresponding to the power set of the index set. The method is not included. Under the conditions of Exercise 42, find the minimum number of persons required so that the probability of at least three people having birthdays on the same month and date is greater than or equal to 1/2. a c b d Figure 3.2 The digraph of the relation of Example 3.5. 167 Functions, Sequences, and Relations We list the exceptional things we are going to inspect. Suppose first that, in this, there are relatively few important properties that relations may have. p Negation of conclusion 5. Next, we compare MORE with PROCESSING. A query is a request for information from the database. A simple cycle 8. Thus our problem is to count the number of such orderings. ABDCD∗←∗+ 22. Looking for a solution to a puzzle or for an optimal move in a game tree can be stated as a searching problem. 1 · 2 + 2 · 3 + 3 · 4 + · · · + n(n + 1) = 3 3. Similarly, we can solve for b. , an) ∈ Xi } 62 Logic Proposition Conjunction: p or q, p ∨ q Negation: not p, ¬p Truth Table Exclusive-or of propositions p, q: p or q, but not both Section 3 Conditional proposition: if p, then q: p → q Hypothesis Conclusion Necessary condition Sufficient condition Converse of p → q: q → p Biconditional proposition: p if and only if q, p ↔ q Logical equivalence: P ≡ Q De Morgan's laws for logic: ¬(p ∨ q) ≡ ¬p ∧ ¬q, ¬(p ∧ q) ≡ ¬p ∨ ¬q Contrapositive of p → q: ¬q → ¬p Section 4 Deductive reasoning Hypotheses Premises Argument Valid argument Invalid argument Rules of inference for propositions Modus ponens Modus tollens Addition Simplification Conjunction Hypothetical syllogism Disjunctive syllogism Resolution of all the statements. For each of the Is simple cycles of length 8 in G1 . Theorem 5.4 If T is a full binary tree with i internal vertices, then T has i + 1 terminal vertices and 2i + 1 total vertices. 4 → 2, 3 → Applications to the Analysis of Algorithms In this section we use recurrence relations to analyze the time algorithms require. Mathematical system Axiom Definition Undefined term Theorem Proof Lemma Direct proof Even integer Odd integer Subproof Disproving a universally quantified statement Begging the question Circular reasoning Section 2 15. Example 1.7 showed that there are n2 = n entries off the main diagonal. The flow that corresponds to the matching of Figure 4.2. We show that this flow is maximal by exhibiting a minimal cut whose capacity is 3. We will derive a formula for f −1 (y). Suppose that (X−Y)∩ (Y − X) is nonempty. Use induction to show that A(2, n) = 3 + 2n, n = 0, 1. Begin with 00. a = a + as + as2 + · · · + asn−1 + sn p0 a − asn + s n p0 1−s −a a n=s + p0 + 1−s 1−s b n − ak ak n − + p0 + . 48. (Explain.) Otherwise, √suppose that xy is irrational. Set a = 1 and b = x and replace n by n − 1 in the Binomial Theorem to obtain (1 + x)n−1 = (n − 1, k)xk . The last quantifier is ∃, so you choose a value for z. At this point, the reader may want to read the Problem-Solving Corner that follows this section. Notice that it is practical to implement an algorithm that requires 2n steps for an input of size n only for very small input sizes. The equivalence class containing an arbitrary permutation p consists of all permutations of X that are identical if we forget the numbers of multiplications (lines 6 and 10) required to compute an . A person invests $1000 at 12 percent interest compounded annually. Consider the point in an optimal solution to the n-disk problem when the largest disk is moved for the first time. P(11, 3) = 11 · 10 · 9 10. The team won the 2006 National Basketball Association championship proposition? (Touching does not count; the states must share a border.) The graph is not connected because there is no path from Hawaii to California (or from Hawaii to any other state). Give a solution of the Tower of Hanoi puzzle. Imitating the specific case discussed previously, we would like to assume that we can make k − p cents postage and then add a p-cent stamp to make k cents postage. 0 0 1 relative to the ordering 1, 2, 3. Since there are n2 entries in A and A2 , the worst-case time to compute the entries of (n2). 0 0 0 1 Its square is ⎛ 1 0 0 0 | 0 2 2 0 | ⎡A2 = ⎛ 0 2 2 0⎞ . (b) Arguing as in part (a), if Alice is chairperson, we have 5 · 4 · 20 = 20 ways to select the remaining officers. For example, the relation R defined on the set of integers by (x, y) ∈ R if x ≤ y orders the integers. There are n P(n − r, r ")" r (r may be any proposition whatever). We define Cin in terms of Gn−1. Let GRn−1 denote the sequence Gn−1 written in reverse. Dijkstra (1930–2002) was born in The Netherlands. a b c d e 12 , . En are events, then P(E1 ∪ E2 ∪ · · · ∪ En) = P(Ei). Since every full binary tree has an odd number of vertices, it is a full binary tree. Exercise 26, (n + 2)Cn+1 = (4n + 2)Cn. Then, m is n = n + 2. If R, S ⊆ R ∩ S and R is symmetric? Exercise 26, n ≥ 1, n + 2 20. The given graph is a path P from a to b in T. The number 11 that appears in the numerator is the total number of letters. A card is selected at random from an ordinary 52-card deck. According to Exercise 14, not counting loops, every vertex has at least degree 2. 3 X 3 O X OX X X O X X O X X O X O X X X O X X X X O X X O X X O X 5 X X 5≥1 4 4≥1 3 X X 3≥1 2 4 4≥0 4 4≥1 4 4 3≥1 3 2 2 5≥2 3 O X O X X 5≥1 4 4≥1 3 X X 3≥1 2 4 4≥0 4 4≥1 4 4 3≥1 2 X X X 3 X 3 X X 4 3 3 3≥1 2 3 4 ≥ 4 X X X 2 3 X 2 4≥0 4 4≥0 5 4 4 4 3 ≥0 3 3 3 ≥ 0 2 to b in T. Prove that in the front/back graph. 124. The number 11 that appears in the numerator is the total number of letters. A card is selected at random from an ordinary 52-card deck. An odd number of vertices, it is a full binary tree. Why can't a maze be solved. We will show you to a corner. A counterexample is x = y = 2. Well-Ordering Property The Well-Ordering Property for nonnegative integers states that every nonempty set of nonnegative integers has a least element. , vn−1 , vn)), v1 = vn , is a Hamiltonian cycle, (v1 , . The set of all subsets (proper or not) of a set X, denoted P(X), is called the power set of X. Since this value is 0, we know that 2 = 21 is not included in the infinite sequence. Prove that the binary tree is also a full binary tree in this case, and note that x, y ∈ X. A sufficient condition such that relation R on a set X is a partial order. Give a partial order that is not a total order. Suppose that X is a set of employees. Prove that X ⊆ P(X) ∩ P(Y). a b c d e f g T1 (a) (b) 3. Suppose that p and P and Q are both false. In this case, the program works through the element that is less than the integer and then the proof. Prove that there are two propositions that are equivalent to p → q and q → p. 2 0 relative to the orderings 1, 2, 3 and a, b, c. Notice we that k ≥ 5. Such an argument is called a proof. Since there are n + 1 positions in a row of n + 1 coins, we are done. P(1) is true. Since b is a real number, it follows that x is a real number. Let n be a positive integer. Draw the computer science course tree. Define the predecessor of k − 2 − 2 + 2 + 2 4 4 4 4 4 3 3 3 3 3 2 2 1 2 3 2 1 1 1 3 3 5 4 4 3 3 ... It is a good idea to check that the binary tree is at least one element. For example, the sequence (0, 1, 2) of 0's and 1's is a set of size n. Let s1 , s2 , . . , sn be integers that are divisible by 5. Recursive algorithm that finds the maximum and minimum of a sequence. It begins with s1 and ends with sn . Give an example of a set X. Then P(X) = P(Y). If the tree is a perfect binary tree, then it is full. Let f be defined by f (n) = n2 . Then there exist integers s and t with gcd(a, b) = sa + tb. These are the only nontrivial things that can be said about the problem, but we can also combine the chance of getting at least one ace. We use the previous two examples to show that the set of real numbers is uncountable. Suppose that X and Y are sets. Similarly, by the definition of S, (b, c) ∈ S. Since gcd(1, 0) = 1, we have gcd(7, 1) = 1. Since the intersection of two sets is a subset of each set, we have A ∩ B ⊆ A. Draw a graph that represents the same relation. So the entire binary tree is full. By Theorem 8.8, it has at least (2i + 1) vertices. We consider a recursive algorithm to compute the nth Fibonacci number. To find the next string in the standard order, we begin at the right and move left, looking for the first element that is not at its maximum value. This gives a full binary tree. A basic example is the truth table for the proposition p ∧ q. In this case, a is the greatest common divisor. Suppose that S is a set of integers. Every integer greater than 1 has a prime factorization. No two distinct terminal vertices in a binary tree may have the same parent. The tree in the figure is an example of a binary search tree. We can show that for all n, the statement is true. Give a recursive algorithm that computes the sum of all integers between 1 and n. Example 1.13 Let X = {1, 2, 3}. A string over a set X is a finite sequence of elements from X. Then (m + 1)/2 is the median, so the sequence is divided into two parts. Exercises 1–4 refer to the relation R on the set {1, 2, 3, 4}. Determine whether each relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order. Definition 2.1 Let G be a graph, and let v and w be vertices in G. A path from v to w is a finite alternating sequence of vertices and edges. Show that the algorithm is correct. How many ways are there to distribute 10 identical balls into 4 distinct boxes? Find the recurrence relation for the number of n-bit strings that do not contain the pattern 00. Prove that if X is a set, then ∅ ⊆ X. Let A be the adjacency matrix of a graph G. Show that the ijth entry of A2 is the number of paths of length 2 from vertex i to vertex j. What is the minimum number of comparisons needed to find the largest element in a list of n elements? A tree is a simple graph T such that there is a unique simple path between any two vertices. Show that any tree with two or more vertices has at least two vertices of degree one. An ordered rooted tree in which each vertex has at most two children is a binary tree. Let T be a binary tree with height h and t terminal vertices. Then lg t ≤ h. Find the height of the binary search tree. Write an algorithm that inserts a word into a binary search tree. If the word is already in the tree, do nothing. Show that the Euclidean algorithm computes gcd(a, b) in O(lg b) steps. How many ways can we choose a committee of three from a group of ten persons? In how many ways can we select three persons to serve on a committee if there are eight candidates? Let n be a positive integer. Prove that C(2n, n) is even. 52 Exercise 25 If a graph G has a Hamiltonian cycle, then G has n edges. Write an algorithm to test whether a graph is connected. 76 Show that if every vertex of a graph has even degree, then the graph has an Euler cycle. Show that if G is a connected graph with n vertices and n − 1 edges, then G is a tree. Show that the number of edges in a tree with n vertices is n − 1. Let T be a minimal spanning tree of the graph G. Prim's algorithm finds a minimal spanning tree. 8 Trees Definition 1.1 A tree T is a simple graph satisfying the following: If v and w are vertices in T, there is a unique simple path from v to w. A rooted tree is a tree in which a particular vertex is designated the root. Example 1.3 Figure 1.3 shows a tree T. Let T be a tree with root v0. Suppose that x, y, and z are vertices in T and that (v0, v1, . . . , vn) is a simple path in T. Then (a) vn−1 is the parent of vn. (b) v0, . . . , vn−1 are ancestors of vn. (c) vn is a child of vn−1. (d) If x is an ancestor of y, y is a descendant of x. (e) If x and y are children of z, x and y are siblings. (f) If x has no children, x is a terminal vertex (or a leaf). (g) If x is not a terminal vertex, x is an internal (or branch) vertex. (h) The subtree of T rooted at x is the graph with vertex set V and edge set E, where V is x together with the descendants of x and E = {e | e is an edge on a simple path from x to some vertex in V}. In this chapter, we discuss trees and their applications. (Exercise 17). The tree in Figure 1.4 has height 4. Let A = {1, 2, 3, 4} and R be the relation on A. Each element of A is related to itself, so R is reflexive. Figure 4.2 The digraph of the relation of Example 4.2. The relation R on a set X is called reflexive if (x, x) ∈ R for every x ∈ X. The relation R on a set X is called symmetric if for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R. The relation R on a set X is called antisymmetric if for all x, y ∈ X, if (x, y) ∈ R and x ≠ y, then (y, x) ∉ R. The relation R on a set X is called transitive if for all x, y, z ∈ X, if (x, y) and (y, z) ∈ R, then (x, z) ∈ R. A relation R on a set X is called a partial order if R is reflexive, antisymmetric, and transitive. Example 2.7 The relation R on the set X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is reflexive, antisymmetric, and transitive; hence it is a partial order. Example 4.1 Consider the relation R = {(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 4), (4, 2), (4, 4), (6, 6)} on X = {1, 2, 3, 4, 5, 6}. 73 Proofs 1 2 3 4 5 74 Mathematical Systems, Direct Proofs, and Counterexamples More Methods of Proof Problem-Solving Corner: Proving Some Properties of Real Numbers Resolution Proofs Mathematical Induction Problem-Solving Corner: Mathematical Induction Strong Form of Induction and the Well-Ordering Property Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises I'm going to need to see some hard proof.

However, the Dutch authorities said that there was no such profession, and he had to change the entry to "theoretical physicist." He won the prestigious Turing Award from the Association for Computing Machinery in 1972. Prove that for all sets A and B, A ⊆ B if and only if B ⊆ A. (It is assumed that each person has a unique identification number.) The attribute Name is not a key because different persons can have the same name. Assume that the capacity of the minimal cut (P, P) is less than n. The formal definition follows. More precisely, ⎧ ⎨ a if a < b min{a, b} = a if a = b ⎩ b if b < a. To denote the nonnegative numbers that belong to one of the three sets, we use the superscript nonneg. Thus the number of selections in which both Alice and Dolph are officers is 3 · 2 · 4 = 24, that is, |X ∩ Y | = 24. In this section, we look at some algorithms that support secure communication. Since (Beth, History) ∈ R, we may write Beth R History. If you do not run 10 laps daily or you do not take multi-vitamins, then you will not be healthy. In Exercises 17–23, we use the notation of Definition 6.1. 17. As an illustration, we present an example that shows the usefulness of allowing an algorithm to make random decisions, thereby violating the requirement of determinism. Our first example illustrates one possibility. 51 45. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is D3 × D3 . The number of outcomes in the event divided by the number of outcomes in the sample space Counting Methods and the Pigeonhole Principle Section 5 5. Also m+1 m 1 = + < a + (b − a) = b. All matrices of relations are with respect to these orderings. Thus, suppose that q > n. We begin with the formal definition of cut. The security of the RSA encryption system relies mainly on the fact that currently there is no efficient algorithm known for factoring integers; that is, currently no algorithm is known for factoring d-digit integers in polynomial time, O(d k). Many interesting games, such as chess, have game trees so large that it is not feasible to use a computer to generate the entire tree. In a cryptosystem, the sender transforms the message before transmitting it, hoping that only authorized recipients can reconstruct the original message (i.e., the message before it was transformed). Show that G has two components. It follows that there are Dn−1 + Dn−2 ways for person 1 to get order 2 and for all the others to get the wrong orders. (For Exercise 1) If Joey passes the discrete mathematics exam, then he studied hard. 93. , n} to the real numbers is represented as an array A, indexed from 1 to n. Increase the flow through the path by , where is defined as in Theorem 2.3, and go to line 2. First, suppose that G contains no cycles. Write a program that lists all permutations of ABCDEF in which A appears before D. Solve the preceding Sudoku puzzle. Example 6.9 Consider the statement ∃x∀y(x ≤ y). If j = rand(4, 5) = 5, after the swap we have 5 21 17 9 23 ↑ i ↑ j Notice that the output (i.e., the rearranged sequence) depends on the random choices made by the random number generator. 2 9. Problem Solution 0! 1! 2! 3! 4! 5! 1 1 · 0! = 1 2 · 1! = 2 3 · 2! = 3 · 2 = 6 4 · 3! = 4 · 6 = 24 5 · 4! = 5 · 24 = 120 Recall that if n ≥ 1, n! = n(n − 1) · · · 2 · 1, and 0! = 1. The graph of Figure 3.2 has a Hamiltonian cycle, but not an Euler cycle. Write an algorithm that finds the lengths of the shortest paths between all vertex pairs in a simple, connected, weighted graph having n vertices in time O(n3). It follows that P(E1 ∪ E2) = i P(xt) + t=1 j P(yt) − t=1 k P(zk) t=1 = P(E1) + P(E2) − P(E1 ∩ E2). Converse: If Leah studies hard, then Leah gets an A in discrete mathematics. No a 2, a 1, a 3 No Yes a 1 < a 3? If we now input n = 637 to Algorithm 1.8, the algorithm returns the prime 7 because 7 divides 637, specifically 637 = 7 · 91. It does not state that if v1 is a vertex in V1 and v2 is a vertex in V2 , then there is an edge between v1 and v2 . x4 + 4x3 y + 6x2 y2 + 4xy3 + y4 6. Theorem 6.4 Graphs G1 and G2 are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal. Since there are sj paths of length n from vertex i to vertex j, there are sj tj = sj paths of length n + 1 from i to k, where the last edge is (j, k) (see Figure 5.3). sn < A. , 2n + 1} having the property: For no distinct i, j ∈ X do we have i + j ∈ X. (d) Show that sn,n−1 = C(n, 2) for all n ≥ 2. c 3. Reflexive, symmetric, and not transitive 15. In Exercises 57–63, if the statement is true for all relations R1 and R2 on an arbitrary set X, prove it; otherwise, give a counterexample. 20 12 8 12 6 12 20 8 2 8 16 6 10 16 20 1 7 20 12. The one way for two persons to get the wrong orders is denoted 2,1. A graph with neither loops nor parallel edges is called a simple graph. How many Class B addresses are available? The relation 31. Again, the domain of discourse is R × R. The first element of the sequence is a, the second element of the sequence is a, and so on. The inside of a circle represents the members of that set. Thus the Basis Steps are n = 2 and n = 3. 53 47. At line 5, since key ('G') is not equal to s1 ('C'), we proceed to line 7. uncircled vertices, b and f , adjacent to a. A restaurant chain advertised a special in which a customer could choose one of five appetizers, one of 14 main dishes, and one of three desserts. for i = 1 to n for j = 1 to n x=x+1 233 Algorithms 19. State and prove a generalization of Exercise 42 where the number of vertices of odd degree is arbitrary. Assume that it is equally probable for a boy or a girl to be born. ∃x∃y T1 (x, y) 7. Second Form: If f is a function from a finite set X to a finite set Y and |X| > |Y |, then f(x1) = f(x2) for some x1 , x2 ∈ X, x1 = x2 . Thus c is a common divisor of b and r. How many faces are there? SCHOOL 3. Since y belongs to exactly one member of S, we must have S = T . , where f denotes the Fibonacci sequence. Find a theta notation in terms of n for the number of times algor2 is called when invoked as algor2(1, n). Prove that a base b integer m has 1 + logb m digits. and V2 = {v4 , v5 }, each edge is incident on one vertex in V1 and one vertex in V2 . f7 = 13 > 11.39 = (3/2)6 . Since the two strings consisting of all 0's or all 1's are not allowed as a hostid, there are 224 − 2 hostids. = C(n + 1, 0)an+1 b0 + 1 1 2 1 3 4 5 3 10 k=1 5 1 4. 011000010 15. Divide the routes into classes based on when the route first meets a vertical line, and use the Addition Principle to derive a formula like that proved in this section. Let's turn to definition (b), which says, in words, that a is a least upper bound for a set X if, among all upper bounds for X, a is smallest. Figure 1.4 with flows 7. At lines 530 Trees 15 and 16, vertex 5 is added to the minimal spanning tree and edge (1, 5) is added to E. The solution to that problem encoded each route as a string of n R's (right) and n Combinations U's (up). What is the least upper bound of the set {tx | x ∈ X} if t = 0? Proof Since X = (X − Y) ∪ (X ∩ Y) and X − Y and X ∩ Y are disjoint, by the Addition Principle |X| = |X − Y | + |X ∩ Y |. The relation {(1, 2), (2, 1)} is not an antisymmetric relation on {1, 2}. Proof Suppose that a graph has a path P with no repeated edges from v to w containing all the edges and vertices. sudoku(s) { i=0 j=1 // advance advances i and j to the next position in which // a value is not specified. We discuss an algorithm to solve the text-searching problem. A set is simply a collection of objects. The first remainder gives the 1's coefficient, the next remainder gives the b's coefficient, and so on. The formal definitions follow. Example 7.6 The graphs G1 and G2 of Figure 7.5 are homeomorphic since they can both be reduced to the graph G of Figure 7.5 by a sequence of series reductions. At line 2, since i > j (1 > 2) is false, we proceed to line 4, where we set k to 1. The following graph represents a pumping network in which oil for three refineries, A, B, and C, is delivered from three wells, w1 , w2 , and w3 . Example 4.8 We use a loop invariant to prove that when the pseudocode i=1 fact = 1 while (i < n) { i=i+1 fact = fact ∗ i } terminates, fact is equal to n!. Define the sequence c0 , c1 , . The technique is to develop a recurrence relation and initial conditions that define a sequence a1 , a2 , . Basis Step (n = 1) The board is a tromino. 140 Functions, Sequences, and Relations Example 1.18 … … Figure 1.7 shows the graphs of the floor and ceiling functions. What is the probability of getting doubles? We use this function to compute 1!. , p, or (b) for some i, si = ti , and for the smallest such i, we have si < ti . The argument then proceeds using the symbol x. Let G be a connected, weighted graph. Specify the Euler cycle. We must compute |X ∪ Y |. Assign each person (pigeon) that person's name (pigeonhole). For example, there is an edge between California and Oregon and between Illinois and Missouri. In sum, a statement with free (unquantified) variables is not a proposition, and a statement with no free variables (no unquantified variables) is a proposition. Output: large (the largest element in the sequence), small (the smallest element in the sequence) 45. 27. The component of G containing v5 is the subgraph G3 = (V3 , E3), V3 = {v5 , v6 }, E3 = {e4 }. Prove that the n-cube is bipartite for all n ≥ 1. ISBN 10: 1-292-02261-2 ISBN 13: 978-1-292-02261-1 British Library Cataloguing-in-Publication Data A catalogue record for this book is available from the British Library Printed in the United States of America P E A R S O N C U S T O M L I B R A R Y Table of Contents 1. Headline over a story about subsidized housing: Everyone can't afford home. large small(s, n, large, small) { 2. , vn) descendants of vi−1 . This modification is known as the method of nearest neighbors (see [Gose]). A string of length n that contains an even number of 1's that begins with 1 can be followed by any string of length n − 1 that contains an odd number of 1's. Exercise 1, Section 1 8. We call the variable x in the propositional function P(x) a free variable. The second hypothesis is ¬P(Lynn). For the ISBN 978-1-59448-950-1, the group code is 1, which identifies the book as one from an English-speaking country. , 7. We remove an edge (but no vertices) from this cycle. Give an example of a connected graph in which, for every vertex cover V and every edge disjoint set E , we have |E | < |V |. In our case, if we 399 Recurrence Relations define the sequence U by the equation Un = bSn + dTn = b2n + d3n , U is a solution of (2.9). 210 P(E ∩ F) = Since P(F) = 1 − (1/210) = 0.999023437, P(E ∩ F) 0.65625 P(E | F) = = = 0.656891495. Give an example of a subgraph. Proofs [Problem 4.] Let tX denote the set {tx | x ∈ X}. A complete matching finds jobs for everyone. We may solve the recurrence relation by iteration: cn = cn−1 + 3n−1 = cn−2 + 3n−2 + 3n−1 .. P(1) ∧ P(2) ∧ P(3) ∧ P(4) 24. Let S(n, k) denote the number of functions from {1, . Using the generalized De Morgan's laws for logic, we find that the negation is ¬(∀y ∈ Y ∃x ∈ X(f(x) = y)) ≡ ∃y ∈ Y ¬(∃x ∈ X(f(x) = y)) ≡ ∃y ∈ Y ∀x ∈ X¬(f(x) = y) ≡ ∃y ∈ Y ∀x ∈ X(f(x) = y). Define CX (x) = 0 if x ∈ / X. We first swap ai and aj , where i = 1 and j = rand(1, 5) = 1. This accounts for the presence of Dn−2 in the recurrence relation. ∀x∃y P(x, y) 39. When writing up a proof by contradiction, alert the reader by stating, "We give a proof by contradiction, thus we assume · · · ," where · · · is the negation of the conclusion. ¬p → (q ∨ r) 53. The inverse of this relation is R−1 = {(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)}. (For Exercise 19) In symbols: ∃x∃y¬T3 (x, y). Suppose that the sequence {an } satisfies a1 = 0, a2 = 1, and an = (n − 1)(an−1 + an−2) for all n ≥ 3. (0.95)(0.15) + (0.02)(0.85) Section Review Exercises 1. Similarly, the number of selections in which Dolph is an officer is 60, that is, |Y | = 60. Region 2 represents A − B, the elements in A but not in B. Notice that if a composite integer n is input to Algorithm 1.8, the divisor returned is prime; that is, Algorithm 1.8 returns a prime factor of a composite integer. q : You heard the "Y2K" rock concert. From Newsweek: Formal investigations are a sound practice in the right circumstances, but every circumstance is not right. Note that n n−i ≥ , k−i k for i = 0, 1, . In Figure 8.5 we have drawn the graph that represents the cubes of Figure 8.1. Then, by inspection, we find two subgraphs of G satisfying properties (8.1)–(8.3). Y − X = X ∪ Y for all sets X and Y . I study hard. The graph is connected and v and w are the only vertices having odd degree. There is one comparison at line 6b and we return before reaching line 10 or 14. a-2, b-1, c-0, d-3, e-2, f -3, g-4, h-5, i-4, j-5, k-5, l-5 c f e g h i a b c 3. Set n = 0 and then n = 1 in an = br n + dnr n to obtain C 0 = a0 = b C1 = a1 = br + dr. If x < 0, by definition |x| = −x. Let G be a triangulation of The contribution of Appel and Haken was to show that only a finite number of cases involving the vertex of degree 5 needed to be considered and to analyze all of these cases and show that all could be colored using four colors. (a) R is reflexive. , xn), then q(x1 , x2 , . , and let n1 , n2 , . Such graphs are said to be isomorphic. 8. Simulate the execution of the algorithm and count the time required by the various steps. The domain of an attribute is a set to which all the elements in that attribute belong. = n−1 1.2i · 100(n − i) i=0 n−1 1.2i − 1.2 · 100 3. If E is an event, how are P(E) and P(E) related? Therefore there exists k such that sk < A. [The numbers Sn,k are Stirling numbers of the second kind (see Exercise 88).] 90. Write an algorithm that receives as input the matrix of a relation R and tests whether R is transitive. Write the negation of ∃x∃yP(x, y) using the generalized De Morgan's laws for logic. Therefore, Algorithm 3.3 is correct. for all n ≥ 2. Prove that if X ⊆ Y , then CX (x) ≤ CY (x) for all x ∈ U. (In Section 5, we will discuss such "for all" statements formally and in detail.) In the following discussion, we let P(x) denote x > 0 and Q(x) denote x2 > 0. M I S S I S S I P P I Because of the duplication of letters, the answer is not 11!, but some number less than 11!. The first string is 12345, which is followed by 12346 and 12347. The meaning of this definition is that sets X and Y have the same elements if and only if X and Y are equal. An example is shown. If key is not in the sequence, line 3 will be executed n times, so the worst-case time of Algorithm 3.17 is (n). Therefore, m ≥ 4. The last r-combination precedes the first r-combination. (c) Since q is false, q → r is true (regardless of whether r is true or false). , ik) 336 Counting Methods and the Pigeonhole Principle Notice that each line of output consists of k integers i 1 i 2 · · · ik , (3.2) n ≥ i1 ≥ i2 ≥ · · · ≥ ik ≥ 1, (3.3) where and that every sequence (3.2) satisfying (3.3) occurs. k=0 82. q ↔ (p ∧ ¬r) p imp1 q q imp1 p T T F F T F T F T F F T T F F T F T p q p→q ¬p ∨ q T T F F T F T F T T F T T T Since p → q is true precisely when ¬p ∨ q is true, p → q ≡ ¬p ∨ q. If there are sj paths of length n from i to j and tj is 1 if edge (j, k) exists and 0 otherwise, the sum of sj tj over all j gives the number of paths of length n + 1 from i to k. A ∪ ∅ 10. Find 4 i=1 i=1 n i=1 77. Give an example of a graph that has a Hamiltonian cycle but not an Euler cycle. One technique to derive a formula is to experiment with small values and try to discover a pattern. How can we quickly determine whether a relation R is antisymmetric by examining the matrix of R (relative to some ordering)? Such functions can be constructed in two steps: Pick the positions for the j 1's; then place the other numbers. Different values of k give distinct n-vertex binary trees, so by the Addition Principle, the total number of n-vertex binary trees is n−1 ak an−k−1 . Players alternate moves. Thus the theorem is true if n = 1. By continuing this process, we obtain the "cobweb" shown in Figure 1.5. Price Supply p0 p2 p4 p6 p5 p3 p1 Demand Quantity q 2 q4 q6 q 7 q 5 q 3 q 1 Figure 1.5 A cobweb with a stabilizing price. x The domain is the set of all numbers that can be entered into the calculator and whose reciprocals can be computed and displayed by the calculator. Show that there is no solution to the two-queens or the threequeens problem. What is a weighted graph? In this case, the statement x = x + 1 is executed once. Definition 2.9 The negation of p, denoted ¬p, is the proposition not p. 301 Introduction to Number Theory 21 = 258 − 79 · 3 62. Suppose that {an } is a nondecreasing sequence and that whenever m divides n, an = an/m + d, 72. [(b) → (c).] We assume that A ∩ B = A, and prove that A ∪ B = B. Prove that T is a tree if and only if T is connected and when an edge is added between any two vertices, exactly one cycle is created. C(11 + 4 − 1, 4 − 1) 13. Similarly, in statement (b), for each real number x, we obtain a proposition; therefore, statement (b) is a propositional function. Let A(x, y) be the propositional function "x attended y's office hours" and let E(x) be the propositional function "x is enrolled in a discrete math class." Let S be the set of students and let T denote the set of teachers—all at Hudson University. Next we turn our attention to R∩S. I get rich. Notice that there is no solution even though each cube contains all four colors. How long will it take for a person to double the initial investment? Squaring 2 = p/q gives 2 = p2 /q2 , and multiplying by q2 gives 2q2 = p2 . Since concatenating a string α with the null string λ does not change α, every string in X∗ is the concatenation of two strings, each in X∗ . A recursive algorithm is an algorithm that contains a recursive function. Since T contains all the vertices of G, T is a spanning tree of G. True 29. 90. (p ∨ (¬p ∧ ¬(q ∨ r))) → (p ∨ ¬(r ∨ q)) In Exercises 56–59, write each conditional proposition symbolically. (Hypotheses) ··· m + n is odd. 3−1 2 9. It follows that the size of the event "no two persons have the same birthday" is 365 · 364 · · · (365 − n + 1). e i d g d j a e b Root f g b h a c i f c T♁ T h j Figure 1.5 A tree T and a rooted tree T . State the 0/1 laws for sets. Write a recursive algorithm to compute tn , n ≥ 1. c b a 12. [We leave to the exercises the question of when an n-cube can simulate a ring model with an arbitrary number of processors (see Exercise 18).] We first observe that if the n-cube contains a Hamiltonian cycle, we must have n ≥ 2 since the 1-cube has no cycles at all. a a b c 4. We call f −1 (B) the inverse image of B under f . , sn in nondecreasing order. Give an example of a proof by contrapositive. Since this area is greater than the area under the curve, the given inequality follows immediately. In words: For all x, if x is an integer, then x is positive; or if x is positive, then x is an integer. We are given n coins, some of which are bad, but 552 s2k to sort in nondecreasing order. We assume that |x| < d. x becomes a4 , and n becomes 4. Since m/b = 5/2 = 2, m is set to 2. Let x ∈ B. Let X = {1, 2}, Y = {1}, and Z = {2}. What does the shaded area represent? Thus there are at most two additional comparisons (of the form sk == key). This is surely one of the oldest algorithms. Use mathematical induction to show that if E1 , E2 , . 259 Introduction to Number Theory Example 1.2 Since 21 = 3 · 7, 3 divides 21 and we write 3 | 21. Let k = n/m . The graph of a function f , whose domain and codomain are subsets of the real numbers, consists of the points in the plane that correspond to the elements in f . To prove that the universally quantified statement ∀x P(x) is false, find one value of x (a counterexample) in the domain of discourse for which the proposition P(x) is false. Example 2.4 If p : A decade is 10 years, q : A millennium is 100 years, then p is true, q is false (a millennium is 1000 years), and the conjunction, p ∧ q: A decade is 10 years and a millennium is 100 years, is false. The matching corresponding to this maximal flow would be a complete matching. Proof Let w = ab mod z, x = a mod z, and y = b mod z. If no tree is input (i.e., PT equals null), nothing is processed since, in this case, the algorithm simply returns at line 2. Show that |XT | = 2(2n − 1) for all T ∈ X1 . Then u is not in T , so u was chosen at line 9 during a previous iteration of the while loop. Let bi be the length of a longest increasing subsequence starting at ai , and let ci be the length of a longest decreasing subsequence starting at ai . What do we mean by the time and space required by an algorithm? , sr) // print the first r-combination for i = 2 to C(n, r) { m=r max val = n while (sm == max val) { // find the rightmost element not at its maximum value m=m−1 max val = max val − 1 } // the rightmost element is incremented sm = sm + 1 // the rest of the elements are the successors of sm for j = m + 1 to r sj = sj−1 + 1 println(s1 , . Suppose that the formula holds for connected, planar graphs with n edges. In the first, we assume that the graph has a Hamiltonian cycle. (Each hi is one of 0–9 or A–F.) 10. 539 Trees 7. If we had already proved this result, to prove that merge sort is (n lg n) in the worst case, it would have been sufficient to prove that merge sort is O(n lg n) in the worst case. Since n mod 2 is equal to 1, result becomes result ∗ x = a14 ∗ a16 = a30 . f(n)/g(n) ≤ max{f(1)/g(1), f(2)/g(2), . Would a two-level search program based on the evaluation function E of Example 9.1 play a perfect game of tic-tac-toe? The graph is planar: a d c g f e h b 14. a b f 1 d c 4 f e d e b 3 14. Since G has one more edge than G , one more vertex than G , and the same number of faces as G , it follows that (7.3) also holds for G. What is the least common multiple? α2 , f is not one-to-one. What is the time required by selection sort? and We must show that q1 = q2 and r1 = r2 . If T contains two or more cycles, we would be able to produce a connected graph T by deleting two or more edges from T . Prove that CX∪Y (x) = CX (x) + CY (x) for all x ∈ U if and only if X ∩ Y = ∅. 361 Counting Methods and the Pigeonhole Principle The Binomial Theorem can also be proved using induction on n (see Exercise 16). We prove the inequality by using mathematical induction. The join operation on relations R1 and R2 begins by examining all pairs of tuples, one from R1 and one from R2 . To prove that ∀x∃y P(x, y) is true, show that for all x ∈ X, there is at least one y ∈ Y such that P(x, y) is true, where the domain of discourse is X × Y. To make it convenient to refer to individual lines within a function, we sometimes number some of the lines. At this point, the largest value, m, has been found. To prove (1.1), we assume that x1 , x2 , . Since a1 = 2, we obtain the explicit formula an = 2 + 3(n − 1) for the sequence a. We suspect that (b) also could be false. Suppose that f(α) = f(β). If Joey studies hard, then he will pass the discrete mathematics exam. Suppose that an n-bit string begins with 0 and does not contain the pattern 111. An edge disjoint set for G is a subset E of E such that for every pair of distinct edges e1 = (v1 , w1) and e2 = (v2 , w2) in E , we have {v1 , w1 } ∩ {v2 , w2 } = ∅. Figure 4.1 shows a pine cone with 13 clockwise spirals and 8 counterclockwise spirals. 157 Functions, Sequences, and Relations One way to form a new sequence from a given sequence is to retain only certain terms of the original sequence, maintaining the order of terms in the given sequence. So suppose that f(α) = f(β). The truth values of propositions such as conjunctions and disjunctions can be described by truth tables. Therefore, the expressions x < 10 || x > 20 and !(x >= 10 && x 2, then 1 + 3 = 4. If x ∈ B, then x ∈ / A. We set to 2 [= val(c)] at line 16. Show that the ordered pairs (bi , ci), i = 1, . Let X = {a, b, c}. Since n mod 2 is not equal to 1, result is not modified. We first note that if A ⊆ B, then τ(A) ⊆ τ(B). 1 1 1+ 1 ≤ 1+ 1 n n for every positive integer n. Multiplying d > n and q > n gives √ √ n = d q > n n = n, √ which is a contradiction. an = 2nan−2 − an−1 15. In Sections 3–5, we discuss resolution and mathematical induction, which are special proof techniques. Constructing an Optimal Huffman Code This algorithm constructs an optimal Huffman code from a table giving the frequency of occurrence of the characters to be represented. tn = 2n − 1, 41. v2 is incident on e1 , e2 , and e3 . No Euler cycle 31. Thus the set of common divisors of a and b is equal to the set of common divisors of b and r. If 4 megabytes is better than no memory at all, then we will buy more memory. 36 9 By Corollary 6.11, P(E1 ∪ E2) = P(E1) + P(E2) = 1 1 5 + = . You know, it's a funny thing. After the swap we have 77 44 77 77 64 44 64 The sequence is sorted. Suppose that we want to prove a statement for all n ≥ 2 involving cn . However, if we argue by contradiction and assume that the removal of some edge (but no vertices) from T does not disconnect T, then when we remove that edge from T, the graph T that results is connected. † For C D Figure 2.8 A graph model of the bridges of K¨onigsberg. Simple graph; five vertices having degrees 2, 2, 4, 4, 4 v2 v5 v8 v 10 19. Example 3.27 Suppose that R and S are transitive relations on a set X. 19 Sets and Logic Exercises Determine whether each sentence in Exercises 1–11 is a proposition. We require that an is strictly less than an+1 for every n. To solve the recurrence relation an = c1 an−1 + c2 an−2 , first solve the equation t 2 − c1 t − c2 = 0. Class A addresses are used for large networks; Class B addresses are used for mediumsize networks; and Class C addresses are used for small networks. We show how to construct a Gray code for each positive integer n, thus proving that the n-cube has a Hamiltonian cycle for every positive integer n ≥ 2. When the Pigeonhole Principle is successfully applied, the principle tells us only that the object exists; the principle will not tell us how to find the object or how many there are. [Even, 1973; Hu; and Reingold] treat combinatorial algorithms. A complete matching for G is a matching E having the property that if v ∈ V , then (v, w) ∈ E for some w ∈ W . Exercises 49–51 concern a variant of the Monty Hall problem, in which a contestant chooses one of four doors; behind one of the doors is a car, and behind the other three are goats. We first show that f is one-to-one. Let A be the adjacency matrix of a graph G with n vertices. Show all solutions to the six-queens problem in which one queen is in row 2, column 1, and a second queen is in row 4, column 2. X X X X X O X O O X X O X X O O X X X O O X O O O X O Figure 9.6 The game tree for tic-tac-toe to level 2 with symmetric positions omitted. Given edges of the form (v, v1) and (v, v2), where v has degree 2 and v1 = v2 , a series reduction deletes vertex v and replaces (v, v1) and (v, v2) by (v1 , v2). Write a program that solves the four-peg Tower of Hanoi puzzle in fewer moves than does the solution to the threepeg puzzle. The results are given in Table 3.2. Recall that the Fibonacci sequence {fn } is defined by the equations f1 = 1, f2 = 1, fn = fn−1 + fn−2 , n ≥ 3. In this case, the pair di , dj is a counterexample. The nth element of the sequence is 2n. Let's try to show that T is connected and acyclic. In what order are the edges added by Prim's Algorithm for the graph of Exercise 13 if the initial vertex is 1? Show that universal generalization (see Figure 5.1) is valid. We require that an isomorphism of binary trees preserve the left and right children. Prove that the algorithm B E F C D G In Exercises 6–10, represent the expression as a binary tree and write the prefix and postfix forms of the expression. Of course, the algorithm may do work in addition to the halving that will increase the overall time. If x ∈ X, then x ∈ X ∪ (Y − X). Prove that CX−Y (x) = CX (x)[1 − CY (x)] for all x ∈ U. A Hamiltonian cycle would have 12 edges. Let p : Jerry receives a scholarship, q : Jerry goes to college. Example 3.3 v2 Show that the graph of Figure 3.5 does not contain a Hamiltonian cycle. If r is a root of t 2 − c1 t − c2 = 0, (2.14) then the sequence r n , n = 0, 1, . n + 1 18. Use the Binomial Theorem to show that using the Binomial Theorem. Let α = ccddc and β = c3 d 2 . Then α ∈ L and the first rule states that ab = aαb ∈ L. Figure 1.8 Part of a graph that models Bacon numbers. Thus every element in X belongs to at least one member of S. What is a parallel algorithm? Give an example of a network G, in which mij ≤ Cij for all edges (i, j), for which no flow exists. f(e) = E, If we define a relation R on a set of graphs by the rule G1 R G2 if G1 and G2 are isomorphic, R is an equivalence relation. A database management system is a program that helps users access information in a database. Can the permutation following the given permutation have the form 163−−− ? , sj where all the zeros precede all the ones sort(s, i, j) { if (i == j) return if (si == 1) { swap(si , sj) sort(s, i, j − 1) } else sort(s, i + 1, j) } 59. To determine whether 451 is prime, Algorithm 1.8 checks whether any of √ 2, 3, . If R is an equivalence relation on a finite set X and |X| = |R|, what must the relation look like? Use breadth-first search (Algorithm 3.6) with the vertex ordering hgfedcba to find a spanning tree for graph G of Figure 3.1. 8. 20. As a specific case, consider an initial distribution consisting of two piles: one containing three tokens and one containing two tokens. s1 + s2 + · · · + sn > nA. Given statement: p → q; false. Put v ∈ V1 . 421 Recurrence Relations Exercises 50–54 refer to Algorithm 3.14. (c) We show that if the given statement is true, statement (c) is necessarily true. S1 = 2 (there are two 1-bit strings), S2 = 4 (there are four 2-bit strings), and S3 = 7 (there are eight 3-bit strings but one of them is 000). Show that any graph having five or fewer vertices and a vertex of degree 2 is planar. Example 4.4 } // put vertex and edge in mst v(add vertex) = 1 E = E ∪ {e} } } return E Show how Prim's Algorithm finds a minimal spanning tree for the graph of Figure 4.1. Assume that the start vertex s is 1. Trees v1 v2 w1 w2 v3 v4 w3 T1 w4 T2 Figure 8.10 Isomorphic binary trees. 1 Inductive Step Assume that the statement is true for n. Therefore, the graph in Figure 1.14 is not bipartite. r: I take popular culture. To model this situation, we should have P(2) = P(3) = P(4) = P(5) = P(6) and P(1) = 3P(2). Describe the algorithm for generating r-combinations. ¬q 3. Then there exists q such that 15 − 6sq = 1. Write a program that prints the amount accumulated yearly if a person invests n dollars at p percent annual interest compounded m times yearly. A clause consists of terms separated by or's, where each term is a variable or the negation of a variable. Find a formula for the number of edges in Kn . The domain of f is X, the codomain of f is Y , and the range of f is {a, b}. p q p ∧ ¬q T T F F T F T F F T F F T F 70. Since Kn has n(n−1)/2 edges (see Exercise 15, Section 1), its worst-case run time must be at least n(n−1)/2 = (n2). False 6. We may summarize the Addition Principle by saying that we add the numbers of elements in each subset when the elements being counted can be decomposed into pairwise disjoint subsets. Section 8 58. But if P(x) is false for at least one x in the domain of discourse, ¬P(x) is true for at least one x in the domain of discourse. abc, acb, bac, bca, cab, cba, abd, adb, bad, bda, dab, dba, acd, adc, cad, cda, dac, dca, bcd, bdc, cbd, cdb, dbc, dcb 7. To prove that X is a subset of Y , let x be an arbitrary element of X and prove that x is in Y . To increase the value of a given flow, we must find a path from the source to the sink and increase the flow along this path. How many functions are there from an n-element set onto {0, 1}? Basis Step 1/(1 · 3) = 1/3 12 + · · · + n2 + (n + 1)2 = Inductive Step Assume true for n. Lexicographic order 13. Exercise 4; a0 = 0 14. What is the Tower of Hanoi puzzle? Assuming that p and r are false and that q and s are true, find the truth value of each proposition in Exercises 13–20. Divide the problem into n subproblems: 20. 5. The selection operator chooses certain n-tuples from a relation. 49. Then x ∈ A−B or x ∈ B − A. Then ∀x∃yP(x, y) is true since, for every integer x, there exists an integer y (e.g., y = x) such that x ≤ y is true. The 0 can go in any one of eight positions; thus there are eight such strings. We will end when the permutation n(n − 1) · · · 21 is generated. If p and ¬p ∨ r are true, then r is true. C(n, r) Section 2 1. How many strings can be formed by ordering the letters SALESPERSONS if the four S's must be consecutive? We next define the capacity of a cut. is also valid. Prove that for all n ∈ Z, n is odd if and only if n + 2 is odd. Trace Algorithm 4.6 for n = 5. [Johnsonbaugh] has additional information on Huffman codes and a proof that Algorithm 1.9 constructs an optimal Huffman tree. am am+1 · · · an 13. If x = − 2, then P(−2) is false (−2 > 0 is false) and Q(−2) is true [(−2)2 > 0 is true]. 506 1 4 G3 1 4 4 3 3 G4 G5 3 2 1 G6 We denote the two edges incident on B and G labeled 1 in the graph of Exercise 29 as 1 and 1 here. Thus R is not symmetric if ¬[∀x∀y[(x, y) ∈ R] → [(y, x) ∈ R]]. Example 6.1 Restate ∀m∃n(m < n) in words. If x ∈ X ∩ Y , then CX∩Y (x) = 0. Moreover, if the primes are written in nondecreasing order, the factorization is unique. d w6 29. You will find that any sorting algorithm that compares elements and, based on the result of a comparison, moves items around in an array is (n lg n) in the worst case; thus merge sort is optimal within this class of sorting algorithms. Explain how the selection operator works and give an example. Use breadth-first search (Algorithm 3.6) with the vertex ordering hfdbgeca to find a spanning tree for graph G of Figure 3.1. j g 5. 1 0 1 1 1 17. Write an algorithm that returns the smallest and secondsmallest values in the sequence s1 , . Give an alternative proof of the existence of q and r in Theorem 5.6 for the case n ≥ 0 by first showing that the set X consisting of all integers k where dk > n is a nonempty set of nonnegative integers, then showing that X has a least element, and finally analyzing the least element of X. So if we pick one x (and we know we can since the domain of discourse is nonempty), the given statement assures us that there exists at least one y for which P(x, y) is true. In how many ways can a diner choose one item from among the appetizers and main courses at Kay's Quick Lunch (Figure 6.1)? No 30. How many selections are there in which either Alice or Dolph or both are officers? What is a minimal cut? If you are healthy, then you run 10 laps daily or you take multivitamins. Explain how the Binomial Theorem is derived. If A = {1, 2, 3} and B = {2, 3, 4, 5}, find A B. Inequality (5.1) then becomes 2 ≤ n/2 < n. By Theorem 5.14, part (a), the negation of the preceding proposition ¬(∀x P(x)) is equivalent to ∃x ¬P(x). When we finally backtrack to the root and cannot add more edges, depth-first search concludes. Otherwise, Algorithm 3.9 computes r = a mod b and returns the greatest common divisor of b and r, which is correct since Theorem 3.2 tells us that gcd(b, r) = gcd(a, b). Write a recurrence relation for bn assuming that cm,k = m + k. ∃x∃y P(x, y) Assume that ∃x∃y P(x, y) is false and that the domain of discourse is nonempty. p q p→¬q T T F F T F T F T F T F T T 56. Draw all nonisomorphic, cycle-free, connected graphs having five vertices. Since block 2 is marked, by (4.2) (taking n = 2), block 3 is also marked. Yes; order 1 4. (Before January 2007, an ISBN was a code of 10 characters.) An ISBN consists of five parts: currently the first part is 978, a group code, a publisher code, a code that uniquely identifies the book among those published by the particular publisher, and a check character. charges $1 for the first mile and 50 cents for each additional mile. The code 950 uniquely identifies the book among those published by Riverhead Books, Penguin Group (Hosseini: A Thousand Splendid Suns, in this case). The two 0's must go in two of the blanks shown: 111111 · Thus the two 0's can be placed in C(7, 2) ways. Since n mod 2 is equal to 1, result becomes result ∗ x = a2 ∗ a4 = a6 . Example 3.14 shows that for size n, repeated halving takes time (lg n). The matrix of the relation R2 ◦ R1 is obtained by replacing each nonzero term in A1 A2 by 1. Show that the maximum occurs when i = 1. ∀x∀y P(x, y) 38. Let v be a vertex of degree 2. Example 4.4 A Matching Network Model the matching problem of Example 4.1 as a network. To solve problems (a) and (b), we could have used the statement in Example 6.6. Sometimes, an existing example can be modified to solve a given problem. If either p or q is 1, we can make k cents postage for all k ≥ 1 by using k 1-cent stamps. Consider the well-known quotation from Shakespeare's "The Merchant of Venice": All that glitters is not gold. Thus the number of strings and number of routes are equal to C(2n, n). Is t nondecreasing? Move each vertex so that its new angle is half the current angle, maintaining edge connections. n = 1007 6. , vn) is an Euler cycle in G1 , then, since g is onto, (f(v0), f(v1), . The 64 disks were to be transferred by monks according to the rules set forth previously. Thus n − dn = n(1 − d) ≥ 0. As an example, consider the following algorithm that finds the maximum of three numbers a, b, and c: 1. (The double quotes " mark the start and end of the string.) Within a computer, bit strings (strings of 0's and 1's) represent data and instructions to execute. Adding an edge incident on any of v1 , v2 , or v3 gives a tree isomorphic to the tree T2 of Figure 8.3. v2 v3 v v1 Figure 8.4 Vertex v has degree 3. , pn are the cases). If the Professor's version is correct, what is the worst-case time? i=1 dn = rn = 3 · 2n − 4 · 5n , i=1 n 3 zn = n(n − 1). If a is a nonnegative integer, b is a positive integer, and r = a mod b, then gcd(a, b) = gcd(b, r). If (i, j) is an edge, w(i, j) is equal to the weight of (i, j); if (i, j) is not an edge, w(i, j) is equal to ∞ (a value greater than any actual weight). Sample space 18. And then Boston—Boston is the cradle of the Revolution. (A full binary tree is a binary tree in which each internal vertex has two children.) 21. 2 c f d In Exercises 10–15, determine whether the graphs G1 and G2 are isomorphic. First 64 is inserted in Now 3 is inserted 44 3 Since 64 > 44, it is immediately inserted to 44's right 44 64 Next 77 is inserted. The height of a decision tree that represents an algorithm is proportional to the worst-case time of the algorithm. Give an example of a proof of equivalence. Suppose by way of contradiction that every increasing or decreasing subsequence has length n or less. Since these sets partition X, |X| = |X1 | + |X2 | + · · · + |Xk | = r + r + · · · + r = kr and the conclusion follows. (Algorithm 2.19 provides an efficient way to compute an mod z.) To decrypt the message, the recipient computes cs mod z, which can be shown to be equal to a. Introduction to Number Theory In Exercises 12–16, assume that we choose primes p = 59, q = 101, and n = 41. Classical sorting algorithms are thoroughly covered in [Knuth, 1998b]. are the subsets of {1, . Proof Suppose that G has an Euler cycle. Such an induction can be justified by considering the first statement to be about the smallest integer in X, the second statement to be about the second smallest integer in X, and so on. Is ϒ nondecreasing? Therefore x ∈ (A ∪ B) − (A ∩ B) = A B. Definition 3.25 Let R1 be a relation from X to Y and R2 be a relation from Y to Z. compound interest(n) { 2. An r-combination is an unordered selection of r elements, repetitions not allowed, from among n items. For statement (b), ∃x∀y P(x, y), to be true, for some x, P(x, y) would need to be true for every y. i=n while (i ≥ 1) { x=x+1 i = i/2 } First, we examine some specific cases. Then, by the choice of m, a < (m + 1)/n. 1 + 2 + ··· + n = 33. 12!/(3!)4 12. The argument given extends to an arbitrary number of dimensions (see Exercise 11); that is, if M is a p1 × p2 × · · · × pk mesh, where pi ≤ 2ti for i = 1, . Show that f(n) = (lg n). For each vertex whose value is computed, write the value in the vertex. It is straightforward to verify that R is an equivalence relation on the set of all permutations of X. Suppose that the available denominations are 1 = a1 < a2 < · · · < an . Therefore, the argument is valid. Illustrate Exercise 42 using the following graph. The resulting graph G is said to be obtained from G by a series reduction. At this point we give some formal definitions. Exercise 5, Section 1 b Fw1 ,b = 2, Fa,w2 = 0, FBz = 4, l m 2 n 3 ➜ The Max Flow, Min Cut Theorem In this section we show that at the termination of Algorithm 2.4, the flow in the network is maximal. A linear congruential random number generator uses a formula of the form xn = (axn−1 + c) mod m. A surprising pattern develops in Table 3.2: The a column is the Fibonacci sequence starting with f2 and, except for the first value, the b column is also the Fibonacci sequence starting with f2 ! We are led to conjecture that if the pair a, b, a > b, when input to the Euclidean algorithm requires n ≥ 1 modulus operations, then a ≥ fn+2 and b ≥ fn+1 . Using Euler's formula for graphs, we find that 2e ≥ 4(e − v + 2). Otherwise, let fi and fj denote the smallest frequencies, and replace them in the list by fi +fj . U − C 7. Some terms are not explicitly defined but rather are implicitly defined by the axioms. In how many ways can 15 identical computer science books and 10 identical psychology books be distributed among five students? Then there exist √ integers p and q such that 2 = p/q. What is the probability of exactly two girls? 9450 cannot represent a number in binary because 9, 4, and 5 are illegal symbols in binary. Show that there exists n such that for all k ≥ n, postage of k cents can be achieved by using only p-cent and q-cent stamps. 2 Section Review Exercises 1. Problem-Solving Tips The validity of a very short argument or proof might be verified using a truth table. Definition 1.34 A function that is both one-to-one and onto is called a bijection. Since Example 7.5 (x + y + z)9 = (x + y + z)(x + y + z) · · · (x + y + z) 1 1 1 1 1 1 2 3 4 5 1 1 3 6 10 1 4 10 1 5 1 we obtain x2 y3 z4 each time we multiply together x chosen from two of the nine terms, y chosen from three of the nine terms, and z chosen from four of the nine terms. p q ∴p∧q 10. (a) Model this situation as a matching network. Does any algorithm use fewer comparisons for n = 4, 5, 6? Cryptosystems are important for large organizations (e.g., government and military), all Internet-based businesses, and individuals. So, suppose that f(n1) = f(n2). Now suppose that the input to this algorithm is n > 1. Let Ti be the subtree rooted at vi and suppose that Ti has height hi and ti terminal vertices, i = 1, 2. In how many ways can these books be arranged on a shelf if all books of the same discipline are grouped together? The six-coins puzzle can be solved in at most three weighings in the worst case, which, together with the initial weighing, requires four weighings in the worst case. Query Project 56. 3168 111. Define a function from X∗ to the set of palindromes over X as f(α) = ααR . if (si+1 > large) 27. When a randomized algorithm executes, at some points it makes random choices. , si , si+1 is sorted. 305 Counting Methods and the Pigeonhole Principle APPETIZERS Nachos 2.15 Salad 1.90 MAIN COURSES Hamburger 3.25 Cheeseburger 3.65 Fish Filet 3.15 BEVERAGES Tea .70 Milk .85 Cola .75 Root Beer .75 Figure 1.1 Kay's Quick Lunch menu. During a search, if we stop the search at an empty cell, we may not find the item even if it is present. Figure 1.4 6. Describe a graph model appropriate for solving the following problem: Can the permutations of {1, 2, . Thus Cn does not divide n + 2. (Why?) Let a = xy and b = 2. The domain of discourse is D. As examples, … 284 97 P(3.7) = 80 + 17 3.7 − 1 = 80 + 17 2.7 = 80 + 17 · 3 = 131, ( ] 114 P(2) = 80 + 17 2 − 1 = 80 + 17 1 = 80 + 17 · 1 = 97. In each of these two cases, we consider the two cases: b ≤ c and b > c. Let G be a bipartite graph with disjoint vertex sets V1 and V2 , as in Definition 1.11. (1.2) We already know that m = 2k1 + 1 and n = 2k2 . Taking logarithms to the base 2, we have k − 1 ≤ lg n < k. 1 S 1 0 0 1 1 A 1 1 D 0 0 0 0 29. Randomized Algorithms It is occasionally necessary to relax the requirements of an algorithm stated in Section 1. Find a least-expensive road system that connects all the cities. We may summarize the Multiplication Principle by saying that we multiply together the numbers of ways of doing each step when an activity is constructed in successive steps. 4 bi . (In this problem, we extended a sum that ends at index i − 1 to a sum that ends at index i by adding an additional term rather than by computing the sum that ends at index i from scratch. In Figure 1.2 the (undirected) graph G consists of the set V = {Gre, She, Wor, Buf, Gil, Sho, Cas, Dou, Lan, Mud} of vertices and the set E = {e1 , e2 , . Prove or disprove: There exists i such that si > A. In Section 3, we present the Euclidean algorithm that efficiently finds the greatest common divisor even for large values of a and b. If the terminal vertex is a box, the second player lost. Choose another element x2 ∈ X that is not related to x1 . The induction proof of the p = 2, q = 5 problem can be summarized as follows. To what does "cryptology" refer? an = a2k = 3a2k−1 + 2k = 3[3a2k−2 + 2k−1] + 2k This last inequality implies that = 32 a2k−2 + 3 · 2k−1 + 2k .. A finite sequence x indexed from i to j can j be denoted {xn }n=i . Now s1 , s2 , s3 is sorted. Merge copies the smaller element to the output and moves the pointer to the next element in the sequence that contains the element just copied. A function f from X to Y is not onto Y if ∀y ∈ Y ∃x ∈ X(f(x) = y) is false or, equivalently, its negation is true. In this section, we give examples of several useful algorithms. Then the root is incident on one edge. Graphs G1 and G2 are isomorphic if there is a one-to-one, onto function f from the vertices of G1 to the vertices of G2 and a one-to-one, onto function g from the edges of G1 to the edges of G2 , so that an edge e is incident on v and w in G1 if and only if the edge g(e) is incident on f(v) and f(w) in G2 . The proof might be informal or formal using the techniques presented in this chapter, but a proof will be required. Give an example of a transitive relation. P ≡ Q 70. Since Professor Sabic's algorithm uses at most 100n comparisons, we must have Cn lg n ≤ 100n for all n ≥ 1. Thus Inductive Step Assume true for i. Then the first bag contains at least one coin; the second bag contains at least two coins; and so on. Most of the statements in mathematics and computer science use terms such as "for every" and "for some." For example, in mathematics we have the following theorem: For every triangle T , the sum of the angles of T is equal to 180◦ . Show that if, as long as possible, we remove an edge from G having maximum weight whose removal does not disconnect G, the result is a minimal spanning tree for G. P(X), the power set of X: set of all subsets of X 11. Give an example of a set X and a relation R on X that is not reflexive, not symmetric, and not transitive, but for which the collection of pseudo equivalence classes (see Example 4.15) partitions X. , 3333. q : I get A's. Several applications to the analysis of algorithms are presented in [Johnsonbaugh]. Exercises 52–62 refer to the integers from 5 to 200, inclusive. A cryptosystem is a system for secure communications. We then move to lines 7 and 8, where we set a to 396 and b to 108. We must have U0 = 1 = U1 . Let P(x, y) be the propositional function x ≥ y. Show that in a triangulation of a simple, planar graph, 3f = 2e. Therefore, n! n(n − 1) · · · (n − k + 1) C(n, k) = = (n − k)!k! k(k − 1) · · · 1 nn−1 n−k+1 = ··· k k−1 1 nn n ≥ ··· kk k n k = . In general, the adjacency matrix of a graph changes when the ordering of its vertices is changed. 101. are called the Euler numbers. Therefore, f is onto Y . See the proof of Theorem 4.3. Section 4 1. and c > 1. x is set to x ∗ x mod z = 81 mod 11 = 4, and n is set to 3. Since both y and x belong to S, y R x and R is symmetric. if repetitions are not allowed? For example, Huffman codes are used with other techniques to compress data for fax machines. +3 +3 (2.2) Replacing n by n − 2 in (2.1), we obtain an−2 = an−3 + 3. Let X = {1, 2, . The root of T1 has a left child but the root of T2 does not. The statement is a propositional function. (A Hamiltonian path begins and ends at different vertices.) 25. We also say that g is an asymptotic lower bound for f . Define the difference of sets X and Y . 577 Trees 26. Many persons would guess that n would have to be considerably larger than 23 for the probability to be greater than 1/2. 2 12 12 n It follows that an = an−1 + an−2 . However, algorithms are known that can determine whether a graph having n vertices is planar in time O(n) (see [Even, 1979]). A path is an alternating sequence of vertices and edges 43. Proof We use proof by contradiction and assume the negation of the conclusion ¬∃i(si ≥ A). Definition 2.14 If {ai }ni=m is a sequence, we define n n ai = am + am+1 + · · · + an , i=m ai = am · am+1 · · · an . No student can represent two committees. Let P denote the set of integers greater than 1. 64 77 Finally 3 is inserted. d. Put the books in some fixed order. Therefore, using the fact that r = a − bq, we have g = s b + t r = s b + t (a − bq) = t a + (s − t q)b. Let's take an arbitrary string 01111010 and find strings related to it. Explain how Algorithm 3.14 sorts the sequence. If f is onto, then f ◦ g is onto. Find a formula for the sequence d defined by 103. Therefore, τ(R1 ∪ D) ⊆ τ(R1) ∪ D. Explain why sj = sk for some j = k. Write a program that prints all n-bit strings that do not contain the pattern 010. Prove that for all k ∈ Znonneg , n nk+1 ik = i=1 k+1 + Ck nk + Ck−1 nk−1 + · · · + C1 n + C0 , for all n ∈ Z+ . Suppose that g does not divide a. Algorithm 8.13 Testing Whether Two Binary Trees Are Isomorphic Input: The roots r1 and r2 of two binary trees. 1/8 P(x). need 20. What is a face? i=1 9. Example 8.5 The rooted trees T1 and T2 in Figure 8.7 are isomorphic. Input: A connected, weighted graph in which all weights are positive; vertices a and z Output: L(z), the length of a shortest path from a to z 1. Thus dec val remains 1. 3 ➜ Conditional Propositions and Logical Equivalence The dean has announced that If the Mathematics Department gets an additional $60,000, then it will hire one new faculty member. Letting U denote the universal set, we obtain X ∪ (Y − X) = X ∪ (Y ∩ X) [Y − X = Y ∩ X] = (X ∪ Y) ∩ (X ∪ X) [Distributive law] = (X ∪ Y) ∩ U [Complement law] = X∪Y [Identity law]. Which of Exercises 68–70 must also be true? What is the height of the resulting tree? Finally, again recursively invoke this algorithm to move the n − kn disks on peg 2 to peg 4. The domain of discourse is R × R. For the threequeens problem, by symmetry, the only possible first column positions are upper left and second from top. For example, task 1 must be done before task 2. Exercise 53, Section 1 25. For example, if v, w ∈ {1, 3, 5}, there is a directed edge from v to w. (b) Show that Ln+2 = fn+1 + fn+3 , where f1 , f2 , . As shown, a cycle results, which gives a contradiction. Use breadth-first search (Algorithm 3.6) with the vertex ordering fdehagbci to find a spanning tree for the graph of Exercise 9. Show that the cycle (e, b, a, c, d, e) provides a solution to the traveling salesperson problem for the graph shown. The graph would then show which teams have played each other. In how many ways can we select a set of four compact discs that contains more defective than nondefective discs? Persons 2, 3, . Let fk1 be the largest Fibonacci number satisfying n ≥ fk1 . Clearly, D > 0. How many strings can be formed by ordering the letters SCHOOL using some or all of the letters? Show that the derived tree T ∗, as described in the text, has height lg(n + 1) . 33 Sets and Logic If we let p : 2 = 3, q : I ate my hat, the argument may be written p→q q ∴p If the argument is valid, then whenever p → q and q are both true, p must also be true. Thus, we must show that 1 ≤1 x2 + 1 is true for every real number x. Functions, Sequences, and Relations Richard Johnsonbaugh 135 4. We let an denote the worst-case time. We can make such assignments by the following n-step process: Assign the first element of X to one of A, B − A, X − B; assign the second element of X to one of A, B − A, X − B; . A tromino is a polyomino of order 3. Show that the Catalan numbers are given by the recurrence relation 12. Give an example of a graph that has an Euler cycle that is also a Hamiltonian cycle. The vertices under the root are said to be on level 1, and so on. Therefore mn is odd. Let P(x) denote the statement "x is an environmental problem," and Q(x) denote the statement "x is a tragedy." Symbolically, the clarified statement is ∃x(P(x) ∧ ¬Q(x)). Everyone is a professional athlete and plays soccer. The statement is false if, for every x ∈ X and for every y ∈ Y , P(x, y) is false. , si 217 Algorithms was sorted. ∀x ¬P(x) 23. = 6. 607 Network Models Theorem 3.7 Let F be a flow in G and let (P, P) be a cut in G. Give an example (different from those of Example 1.5) of a theorem in the system of real numbers. Then f(G) = 0 = n. Edge e1 is denoted (v2 , v1), and edge e7 is denoted (v6 , v6). (b) If points in the same equivalence class are glued together, how would you describe the figure formed? If T is a full binary tree with i internal vertices, how many total vertices does T have? This contradiction shows that G does not have a Hamiltonian cycle. The movie won the Academy Award as the best picture of 1955. Thus, p → (q → r) is true (regardless of whether p is true or false). (For example, "gladiator" precedes "gladiolus" in the dictionary. k + 2 < x + 3 ≤ k + 3. The domain of discourse is R. ((1, 0), (0, 1)), ((1, 1), (0, 0))}. If X has n members, how many proper subsets does X have? Trace Algorithm 2.1 for the input t = "balalaika" and p = "bala". ■ Try to use some of the ideas in a similar problem. What advantages do prefix and postfix forms of expressions have over the infix form? Prove that for all integers m and n, if m and m − n are odd, then n is even. Suppose that there are n frequencies. b c 4, 3 5, 6, d 1, a 4, 2 4, e 5, z 4, 2 2, 3, 1 f 4, g 5, 2 4. What is Pascal's triangle? The number n2 of ×'s between the second and third | represents n2 such strings. Corollary 2.22 In any graph, the number of vertices of odd degree is even. Are the sets {3, 2, 2}, 5. Nothing is better than a hard disk drive. if (sn > sm) 18. By the first form of the Pigeonhole Principle, at least two pigeons, x1 , x2 ∈ X, are assigned to the same pigeonhole; that is, f(x1) = f(x2) for some x1 , x2 ∈ X, x1 = x2 . Let G be the complete, weighted graph having vertices V and weights w(vi , vj) = s(vi , vj). 4 · 3 50. After the contestant chooses a door, the host opens one of the other three doors that hides a goat. Assume that the universe for Cartesian products is U × U. while (s(i, j) < 10 ∧ not valid(i, j)) s(i, j) = s(i, j) + 1 // if no value found, backtrack if (s(i, j) == 10) { s(i, j) = 0 // retreat moves i and j to the previous position in // which a value is not specified. The following formula may be of use: an−2 an−1 √ with initial conditions a0 = 8, a1 = 1/(2 2) by taking the logarithm of both sides and making the substitution bn = lg an . , wj }, then R(S) receives at most jm edges from S. In how many ways can 10 balls be selected? 13. (A double occurs when both dice show the same number.) 22. At line 4, i is reset to 1 and we return to line 2. We note that if we eliminate one more edge, we will obtain two vertices of degree 2 and we can then carry out two series reductions. C(52, 13) 21. Introduction to Number Theory 37. A string of length n that contains an even number of 1's that begins with 2 can be followed by any string of length n − 1 that contains an even number of 1's, and there are cn−1 such strings. Applying (3.1) to expressions 3 and 6, we derive 7. Suppose that su = 0. We prove only A ∪ ∅ = A. A probability given that some event occurred is called a conditional probability. If n > 1, let kn be the largest integer satisfying If f(n) = C in (2.20), it can be shown that g(n) = C in (2.21) if 1 is not a root of t 2 − c1 t − c2 = 0, 41. 2 lg n + 4n + 3n lg n 4. If n = 4, person 1 gets either order 2, 3, or 4, so the possibilities are 2,?,?,?; 3,?,?,?; and 4,?,?,?. ∃x∀y T3 (x, y) 22. f 6 1000 vehicles A to C 3000 vehicles B to D 4000 vehicles C to D 2000 vehicles. An edge e in a graph (undirected or directed) that is associated with the pair of vertices v and w is said to be incident on v and w, and v and w are said to be incident on e and to be adjacent vertices. Example 5.17 Write the statement Some birds cannot fly, symbolically. Write a program that prints all the ways to spend n dollars under the conditions of Exercise 53, Section 1. Suppose that K3,3 is planar. , Z; and r(x) = −1, for x = +, −, ∗, /. ∀x(x > 1 → 50. We denote the square in row i, column j by (i, j). We conclude this section by giving additional examples. , pk ; otherwise, we could eliminate p and obtain a smaller integer that is divisible by both m and n. Therefore (A − B) ∪ (B − A) ⊆ A B. Since there are C(m + n, m) strings altogether, the formula follows. Input: a, b, c Output: large (the largest of a, b, and c) 1. No heads were obtained. The while loop tests whether p occurs at index i in t. As a special case, we give a linear-time algorithm to determine whether two binary trees T1 and T2 are isomorphic. You must now give an argument that begins with the hypotheses and ends with the conclusion. The relation R on X given by Theorem 4.1 contains the ordered pairs (1, 1), (1, 3), and (1, 5) because {1, 3, 5} is in S. Think about this question now, then analyze it (see Exercise 74, Section 2) after reading Section 2. 1/4 52. c 5, 2 b 3, 2 6, 3 z 4. The professor's algorithm repeatedly compares two elements and, based on the result of the comparison, modifies the original list. Example 8.9 556 The binary trees T1 and T2 in Figure 8.10 are isomorphic. Therefore, |x| = −x = −1 · x = sgn(x)x. In Exercises 19 and 20, write the statement in words and tell whether it is true or false. Six microprocessors are randomly selected from a lot of 100 microprocessors among which 10 are defective. c2 = 4, c3 = 9, c4 = 20, c5 = 29 11. ∀x∃y P(x, y) 85. , sn) } } 17. At lines 11 and 12 we update each of the b 2 a c 4 2 d 1 2 3 3 7 f 5 g Figure 4.1 The graph for Example 4.2. 466 1 2 z e 4 b 6 a 2 0 4 d 1 c 2 4 3 3 f e 7 5 g 1 z 6 Figure 4.2 Initialization in Dijkstra's shortest-path algorithm. It proceeds down a column first. , yj } E1 ∩ E2 = {z1 , . G1 has seven edges and G2 has six edges. Thus m2 ≤ 20 and m ≤ 4. The largest primes known are Mersenne primes. We continue in this way until the entire tree is drawn. Denote the pictures P1 , P2 , . Find all teams. Show that G cannot have a vertex of degree 3. Consider specifically complexity, empirical studies, and special features of the algorithms (see [Knuth, 1998b]). (x, y) ∈ R if 3 divides x + y 11. If it is correct, what is the worst-case time? (b) If x is a real number, then exactly one of the following statements is true: x is in P, x = 0, −x is in P. Consider the four-coins puzzle (all the rules are the same as for the five-coins puzzle except that the number of coins is reduced by one). The first bit is 0 (to identify it as a Class A address). 66 53. (c) R is not antisymmetric. The value 11 has been converted to the binary number c3 c2 c1 c0 = 1011. It is reflexive, symmetric, and transitive. We have our proof. Write an algorithm that finds a maximal spanning tree in a connected, weighted graph. Example 1.6 Computer File Systems Modern computer operating systems organize folders and files using a tree structure. Exercises Find solutions to the following Instant Insanity puzzles. A folder contains other folders and files. Evaluate children from left to right. By Exercise 53, τ(R) is transitive. Define x R y to mean that 3 divides x − y. 200 − 5 + 1 55. If the sentence is a proposition, write its negation. If we then set f(n) = f0 (n) + n and g(n) = g0 (n) + n, we obtain increasing positive functions for which f(n) = O(g(n)) and g(n) = O(f(n)). Find the parents of c and of h. Lancaster. Prove that when the pair fn+2 , fn+1 is input to the Euclidean algorithm, n ≥ 1, exactly n modulus operations are required. Since d is positive and d = 1, d > 1. In a balanced binary tree of height 2, there must be a path from the root to a terminal vertex of length 2. Suppose that algorithm A requires n lg n comparisons to sort n items and algorithm B requires n2 /4 comparisons to sort n items. [Recall that p → q and ¬q → ¬p are equivalent.] This special case of proof by contradiction is called proof by contrapositive. Prove that the number of solutions to the equation x1 + x2 + x3 = n, n ≥ 3, where x1 , x2 , and x3 are positive integers, is (n − 1)(n − 2)/2. Then xy > 0 and |xy| = xy = (−x)(−y) = |x||y|. At line 2, since i > j (4 > 5) is false, we proceed to line 4, where we set k to 4. Since the first die can show any one of six numbers and the second die can show any one of six numbers, by the Multiplication Principle there are 6 · 6 = 36 possible sums; that is, the size of the sample space is 36. , sn) while (true) { m=r max val = n while (sm == max val) { m=m−1 max val = max val − 1 } if (m == 0) return sm = sm + 1 for j = m + 1 to r sj = sj−1 + 1 println(s1 , . Not bipartite 23. We begin at the lower left by evaluating the vertices E, F , and G. Example 3.6 The capacity of the cut of Figure 3.2 is Cbc + Cdc + Cde = 6. , Z and the operators +, −, ∗, /. Then CX (x) = 1. 2an = 7an−1 − 3an−2 ; a1 = 16 a0 = a1 = 1 with initial conditions a0 = a1 = 1 by making the substitution √ bn = an . The relation {(1, 2), (2, 1)} is a symmetric relation on {1, 2}. Show that aaabbb is in L. Finally, the relation is transitive because whenever (x, y) and (y, z) are in R, (x, z) is also in R. This statement is true because, for every real number x, there is at least one y (namely y = −x) for which x + y = 0 is true. Thereafter, given a word in the list (5.5), we add a vertex v and an edge to the tree and place the word in the vertex v. (7.3) k=0 The sum is the same as the sum in the Binomial Theorem, n C(n, k)an−k bk , k=0 n−k k except that the expression a b is missing. Exercises In Exercises 1–5, list the order in which the vertices are processed using preorder, inorder, and postorder traversal. ¬(p ∧ q) ∨ (¬q ∨ r) In Exercises 30–32, represent the given proposition symbolically by letting p : 5 < 9, q : 9 < 7, r : 5 < 7. Show that adding or deleting loops, parallel edges, or edges in series does not affect the planarity of a graph. 11. Prove that Algorithm 3.6 is correct. The capacities of the intermediate systems are shown on the edges. Once the positions for the S's have been selected, there are C(5, 4) ways to choose positions for the four I's. If the path begins with the loop, the remainder of the path must be a path of length n − 1 from v1 to v1 . A function f from X to Y is said to be one-to-one if for each y ∈ Y , there is at most one x ∈ X with f(x) = y. (A terminal vertex will add both a left and right child.) Show that this mapping is one-toone and onto from the set of all nonisomorphic (n − 1)-vertex binary trees to X1 . f is neither one-to-one nor onto. c b j g d k h e f 49. n! = number of permutations of an n-element set 6. The variable n is incremented and becomes 0. In symbols, if n = p 1 p 2 · · · pi , where the pk are primes and p1 ≤ p2 ≤ · · · ≤ pi , and n = p1 p2 · · · pj , where the pk are primes and p1 ≤ p2 ≤ · · · ≤ pj , then i = j and pk = pk for all k = 1, . We will be concerned primarily with algorithms that can be executed by a "traditional" computer, that is, a computer, such as a personal computer, with a single processor that executes instructions step-by-step. In postfix, the operator follows its operands. 340 Example 4.2 Let α = 132 and β = 1324 be strings over {1, 2, 3, 4}. Prove the idempotent laws for sets. If a5 < a2 , we next compare a5 a5 < a1 < a2 < a4 a1 < a5 < a2 < a4 a1 < a2 < a5 < a4 If a3 is less than the second item, only one additional comparison is needed (with the first item) to locate the correct position for a3 . The Basis Step of a proof by mathematical induction corresponds to the base cases of a recursive function, and the Inductive Step of a proof by mathematical induction corresponds to the part of a recursive function where the function calls itself. 203 Functions, Sequences, and Relations 48. In the Inductive Step, we must have k = n/2 ≥ 3. 1 2 n q (The previous Problem-Solving Corner shows that the sum 2. But the subsets of Y are precisely the subsets of X that do not contain x. After returning from the Orient in 1202, Fibonacci wrote his most famous work, Liber Abaci (available in an English translation by [Sigler]), which, in addition to containing what we now call the Fibonacci sequence, advocated the use of Hindu-Arabic numerals. Example 4.5 Use induction to show that 5n − 1 is divisible by 4 for all n ≥ 1. The board is an eight-pointed star with a circular area in the center known as the putahi. q: I'll eat my hat. (h) The subtree rooted at Kronos is shown in Figure 2.2. Kronos Zeus v2 Poseidon Hades Apollo Athena Hermes vi⩾1 v1 Ares Heracles Figure 2.2 The subtree rooted at Kronos of the tree of Figure 2.1. vi v 0v n vi+1 vn⩾1 vj+1 vj vj⩾1 The remainder of this section is devoted to providing alternative characterizations of trees. Is the relation of Exercise 21 reflexive, symmetric, antisymmetric, transitive, and/or a partial order? , Xt } is a pairwise disjoint family, the number of possible elements that can be selected from X1 or X2 or . Specify the disjoint vertex sets. When can we increase the flow in a path from the source to the sink? You heard the "Flying Pigs" rock concert, but you do not have sore eardrums. In Exercises 27–33, C1 , C2 , . For example, if 442 Graph Theory several teams play soccer games, we could let the vertices represent the teams. Example 1.20 Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology; 7 Sets and Logic U A B Figure 1.7 The shaded region depicts both (A ∪ B) and A ∩ B; thus these sets are equal. 12 − 22 + 32 − · · · + (−1)n+1 n2 = (−1)n+1 n(n + 1) 2 2 n(n + 1) 2 1 1 1 1 7. Since the values of the outcomes in F originally summed to P(F), we change the value of each outcome in F by dividing it by P(F) so that the reassigned values sum to 1. Does 2010 represent a number in binary? How many reflexive and antisymmetric relations are there on an n-element set? Example 3.5 In this book, we let lg n denote log2 n (the logarithm of n to the base 2). How many are even? 36 6 Since the sum of 5 can be obtained in four ways [(1, 4), (2, 3), (3, 2), (4, 1)], P(E2) = 4 1 = . If the coins do balance, we disregard these coins; our problem then is to find the bad coin from among the remaining six coins. Properties of an algorithm: Input, output, precision, determinism, finiteness, correctness, generality 3. Since n/2 satisfies 2m−1 < n/2 ≤ 2m , by the inductive assumption the loop iterates at most m − 1 more times. Since T is a simple graph, C cannot be a loop; so C contains at least two distinct vertices vi and vj , i < j. We leave it to the reader to check that this mapping is a bijection. x ∈ /X 7. We must be craftier in deriving a lower bound. P(A) = 0.55, P(D) = 0.10, P(N) = 0.35 49. Relabel the vertices by following the path from w to the root, as shown. This result is important enough to highlight as a theorem. 2 n Therefore, 64. 16 = 79 − 21 · 3 5 = 21 − 16 · 1 1 = 16 − 5 · 3. Apply Helly's Theorem to X1 , . We then invoke this algorithm with i = 4, j = 5 to search for key in s4 = 'M', s5 = 'X'. It is true that for any real number x, either x ≤ 1 or x > 1. We call 3 a divisor or factor of 21. Correctness—the output produced by the algorithm is correct; that is, the algorithm correctly solves the problem. We establish this last equation by induction on y. If the first move is first column, upper left, the second move must be to the bottom of the second column. We say that P and Q are logically equivalent and write P ≡ Q, provided that, given any truth values of p1 , . What is the greatest common divisor? , pn /∴ q, the hypotheses are p1 , p2 , . We conclude by proving a rather special result. We conclude by showing how matrix multiplication relates to composition of relations and how we can use the matrix of a relation to test for transitivity. (For Exercise 19) False; Erin and Garth are distinct persons, but Erin is not taller than Garth. n+1 n n(n + 1) Let bn = an . We illustrate these concepts with an example. xn = 3 + xn−1 , 68. How many comparisons does tournament sort require to find the second-largest element? Show that the sequence b1 , b2 , . It is always present on the bar code, but it may not appear in the printed version.) For example, the UPC for a package of 10 Ortega taco shells is 0-54400-00800-5. , 9}, lexicographic order is the same as numerical order on the positive integers if we interpret the strings as decimal numbers (see Examples 4.4 and 4.5). The probability of getting a sum of 10 is 1/12, the sum of the values of the outcomes that add up to 10. Output: The set of edges E in a minimal spanning tree (mst) 1. The argument in Example 4.2 is used extensively and is known as the modus ponens rule of inference or law of detachment. We draw five balls, then replace the balls, and then draw five more balls. Now suppose that each person gets along with exactly five others. Thus, w = xy mod z, which translates to ab mod z = [(a mod z)(b mod z)] mod z. The sets [xi] are called the equivalence classes. Then f(x) = (g ◦ h)(x). den 19. 12567 14. 7d. Proof We first find all nonisomorphic rooted trees with four vertices in which the root has degree 3; we then find all nonisomorphic rooted trees with four vertices in which the root has degree 2; and so on. Add parallel edges until the resulting graph G∗ has n − 1 edges. By the inductive assumption, 8 divides 3n + 7n − 2. Exercises 7 and 8 refer to nim and nim . Step 4: We open on Broadway and before you can say Step 5: We close on Broadway. It is denoted x. Then k ≤ n − 1 for every k in X. Show that the number of ways to tile a 2 × n board with 1 × 2 rectangular pieces is fn+1 , the (n + 1)st Fibonacci number. Show that any simple, planar graph can be colored using five colors. 230 linear search(s, n, key) { for i = 1 to n if (key == si) return i // successful search return 0 // unsuccessful search } Algorithms Example 3.18 Matrix Multiplication and Transitive Relations If A is a matrix, we let Aij denote the entry in row i, column j. Show that in the best case, merge sort requires (n lg n) comparisons. We use induction on k to show that in this case the statement x = x + 1 is executed k times. The graph in Figure 1.13 is bipartite since if we let V1 = {v1 , v2 , v3 } v1 v2 A graph G = (V, E) is bipartite if there exist subsets V1 and V2 (either possibly empty) of V such that V1 ∩ V2 = ∅, V1 ∪ V2 = V , and each edge in E is incident on one vertex in V1 and one vertex in V2 . Assume that the values of a, b, and c are distinct. Binary operator on X: function from X × X to X 14. We write an algorithm to calculate the number of ways the robot can walk n meters. k=2 27. , vn−1 , vn) be the portion of P1 from d = v0 to e = vn . Let x be any real number whatsoever. 2 48. 553 Trees v a b c T1 d w e x y z T2 Figure 8.3 Nonisomorphic trees. Therefore, equation (1.3) holds. ⩾4 ⩾3 ⩾2 ⩾1 0 1 ⩾2 4 15 1.766 2 3 4 … For example, Ellen Barkin has Bacon number one because she appeared with Bacon in Diner. , where an is the time (best-case, average-case, or worst-case time) required for an algorithm to execute an input of size n. But now T would be a tree with n vertices and fewer than n − 1 edges—an impossibility. Regardless of what value you chose, Farley can always choose a value for y, which makes the statement 0, x + y = 0 false. A graph that can be drawn in the plane without its edges crossing 2. If x∈ / X, then x ∈ Y . Since the security of the RSA system rests primarily on the inability of anyone knowing the value of z to discover the numbers p and q, p and q are typically chosen so that each has 100 or more digits. The worst-case time of Algorithm 3.17 is analyzed as follows. Let T consist of this single vertex and no edges. Since (x, x) is in the relation for all x ∈ X, the main diagonal of the matrix must consist of 1's. 0 22. Summing over all j gives the desired recurrence relation. For example, for our problem, to store or retrieve the number n, we might take as the first choice for a location, n mod 11. Definition 3.2 Let f and g be functions with domain {1, 2, 3, . On what does the security of the RSA encryption system rest? An inventory consists of a list of 80 items, each marked "available" or "unavailable." There are 50 available items. Look at F(n + 1) so you know where you're headed. 42 Sets and Logic Example 5.12 Suppose that P is a propositional function whose domain of discourse is the set {d1 , . Find a recurrence relation for the sequence A0 , A1 , . Summary of Problem-Solving Techniques ■ List elements that are related. If n is an odd integer, n = 2k + 1 for some integer k. He was the author or co-author of nearly 1500 papers. Let α and β be the angles shown in Figure 1.6. The geometry of the situation shows that the price tends to stabilize if and only if α + β > 180◦ . Encrypt the message COOL BEAVIS using the key of Example 4.1. 2. Modify the graph of Exercise 37, Section 2, by inserting an edge between the vertex in row i, column 1, and the vertex in row i, column m, for i = 1, . Subsets of X that contain a {a} {a, b} {a, c} {a, b, c} Subsets of X that do not contain a Thus, (4.13) is true for n = 0. [It suffices to check ordered pairs (x, y) and (y, z) with x = y and y = z.] In words, a relation is transitive if whenever x is related to y and y is related to z, then x is related to z. Base the terminating condition on the fact that the last permutation has the elements si in decreasing order. In our problem, the postage-amount equation np p + nq q was similar in form to the greatest common divisor formula gcd(p, q) = sp + tq. Since p is true and q is false, p ∧ q is false. The Lucas sequence L1 , L2 , . 35 20. For the remainder of the string β we have tm+1 · · · tr = (sm + 2)(sm + 3) · · · . Pseudocode Section 2 5. Write the negation of ∀x∃yP(x, y) using the generalized De Morgan's laws for logic. Which of Exercises 83–85 must also be false? 211 Algorithms 24. A relation is a set of ordered pairs. Since Y con- 134 sists of upper bounds for X, k ≤ n for every k in X. is bounded above and its least upper bound is a + b. Whales Whales Whales Pilots There are three other ways that the tournament can unfold: (a) Whales defeat Scientists; Pilots defeat Whales. Moreover, if P (respectively, P) is the set of labeled (respectively, unlabeled) vertices at the termination of Algorithm 2.4, the cut (P, P) is minimal. If the statement is false, give a counterexample. , an } of Z+ is a ∗ -set of size n if (ai − aj) | ai for all i and j, where i = j, 1 ≤ i ≤ n, and 1 ≤ j ≤ n. [Barker; Copi; Edgar] are introductory logic textbooks. Problem-Solving Tips An Euler cycle starts at a vertex,

traverses each edge exactly one time, and returns to the initial vertex. (We chose a to be a number different from 1, 2, or 3 to avoid a clash with those numbers. According to Definition 5.4, the universally quantified statement ∀x P(x) is false if for at least one x in the domain of discourse, the proposition P(x) is false. What is an Euler cycle? Also c ∈ Y1 and only if s3 = 1 if and only if c ∈ Y2 . If a binary tree of height h has t terminal vertices, then lg t ≤ h. Therefore, lg n ≥ s/2 for all n ≥ 2, which is equivalent to (1/2) lg n ≥ s/4 for all n ≥ 271 Introduction to Number Theory which is in turn equivalent to lg n1/2 ≥ s/4 for all n ≥ . , si , si+1 . Draw all nonisomorphic free trees having three vertices. If we set α1 = b, β1 = aa, α2 = ba, and β2 = a, then f(α1 , β1) = baa = f(α2 , β2). Most general references on computer science contain some discussion of algorithms. Turn off safety lock. We have n n (i − j)2 = i=1 j=1 = n i=1 j=1 n n i=1 j=1 202 Section 3 Review 1. , n (order 2 is missing because person 1 has it). Suppose that G has a Hamiltonian cycle H. ∃x∃y P(x, y) Assume that ∀x∃y P(x, y) is false and that the domain of discourse is nonempty. , n and start vertex s. Height = 4 b e 2. Suppose that condition (4.1) holds. Every even integer greater than 4 is the sum of two primes. We have just observed that the function computes 1 as the value of 1!. Therefore X ∩ Y ⊆ X. 105. (For Exercise 4) ∃x(P(x) ∧ ¬Q(x)). To prove that the existentially quantified statement ∃x P(x) is true, find one element x in the domain of discourse for which P(x) is true. We also show that binary search is Θ(lg n) in the worst case. Therefore, d ≤ dq + r. Whether an integer d > 0 divides an integer or not, we obtain a unique quotient q and remainder r as given by the Quotient-Remainder Theorem. There exist unique integers q (quotient) and r (remainder) satisfying n = dq + r, 0 ≤ r < d. large smalls, i, j, large, small) { if (i == j) { large = si small = si return } m = (i + j)/2 large smalls, i, m, large, small) large smallm+1, j, large, small) if (large > large right) large = large left else large = large right if (small left > small right) small = small right else small = small left else large = large right if (small left > small right) small = small right else small = small left 47. Let T be the decision tree that represents this algorithm for input of size n and let h denote the height of T . A directed edge exists from vertex x1 · · · xn−1 to x2 · · · xn . Again, currently there is no polynomial-time algorithm known for computing nth roots mod z. If person 2 gets order 1, the number of ways for the other persons to get the wrong orders is Dn−2 . 22 + 42 + · · · + (2n)2 = Section 2 5. 2, 12, 6, 14 2 → Examples of Algorithms Algorithms have been devised to solve many problems. x ≤ d1 and x ≤ d2 (Conclusion—uses the conjunction inference rule) At this point, the only part of the proof that is missing is the subproof to show that d ≤ d1 and d ≤ d2 . We now show that X is a proper subset of Z. p 50. The proposition is false. 1001000 26. The value of F is 6. Therefore 1 ≤ n/2 < n and taking k = n/2, we see that k satisfies inequality (5.1). If v is a vertex in G, we say the parity of v is even if the number of edges of the form (v, w) is even; odd parity is defined similarly. 54. Next, insert s3 in s1 , s2 to obtain s1 , s2 , s3 is sorted. Implement some of these algorithms as programs. We then randomly choose one of the edges along which the total search is greater than sum of a, b. A string that is... We will be looking at functions of and of pieces, and z so that rand(1, 5) = 1, rand(2, 5) = 3, rand(4, 5) = 4. 5. 270 Introduction to Number Theory 8's place (2 2 3) 4's place (2 2) 16's place (2 4) 2's place (2 1) 32's place (2 5) 1's place (2 0) 1 0 1 1 0 1 Symbol 5 Symbol 0 Symbol 4 Symbol 1 Symbol 3 Symbol 2 Figure 2.2 The binary number system. An integer greater than 1 that is not prime is called composite. ¬b 5. Find (a) The parent of a. 1 e 7 1 3 3 1 6 c 2 2 0 7 3 b 2 3 d 1 c 2 g z 6 6 Figure 4.4 The second iteration of Dijkstra's shortest-path algorithm. Give an algorithm that uses this number of comparisons to sort six items in the worst case. where v0 = vn , every edge in G occurs exactly one time, and all vertices appear. The initial condition is d0 = 0. This left/right distinction is important in expressions. Following lines 3–7 (root/left/right) of Algorithm 6.1, the traversal proceeds as shown in Figure 6.4. Thus the order of processing is ABCDEFGHIJ. The correct interpretation is: Some object that glitters is not gold. Computer Exercises 1. What is a component of a graph? ¬p ∨ ¬r From 1 and 2 6. f permutation(n) { for i = 1 to n si = i println(s1 , . Such algorithms are impractical to run except for small values of n. The function {(a, 1), (b, 3), (c, 0)} is one-to-one. By Theorem 2.11, the sequence U defined by Un = 8n + dn+ rn is a solution of (2.16). Both kinds appear in literature. Here, for example, numbers in color indicate that an arrow or a solution appears at the end of this chapter. What is the probability that it is a heart? Of these, half, (n2 − n)/2, are above the main diagonal. If g = {(1, 2), (2, 2)} and f = {(2, a)}, f ◦ g = {(1, a), (2, a)}. Therefore, T is a simple graph. Suppose next that there exist x and y such that P(x, y) is true. We have si+1 ≤ large. Why not use (9 + n)! 4.6 Alternate Version of Prim's Algorithm This algorithm finds a minimal spanning tree in a connected, weighted graph G. Is v increasing? [Pearl] also shows that for such game trees, the alpha-beta procedure is optimal. Let X1 denote the set of nonisomorphic full binary trees having n terminal vertices, n ≥ 2, and let X2 denote the set of nonisomorphic full binary trees having n + 1 terminal vertices, n ≥ 1, with one terminal vertex designated as "marked." 30. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is D3 × D3 . Unfortunately, no one has succeeded in finding such a set of invariants. Let G be a network and consider the flow F at the termination of Algorithm 2.4. Some vertices are labeled and some are unlabeled. Notice that 2k−1 ≤ n/2 < 2k . The sequence is unchanged. How many eight-bit strings begin 1100? This book was one of the main influences in bringing the decimal number system to Western Europe. Formally, f ∩ S = {x | x ∈ X for all X ∈ S}. Prove: 3n = n!. If we begin at the root, always go downward, and follow the appropriate edges, we will eventually arrive at a terminal vertex that represents a decision. U A U B CALC 34 12 47 8 25 16 14 U Figure 1.4 A Venn diagram of A ∪ B. h, d 6 4 3 4 3 6 7 g 4 2 6 c 12 30. We call P a propositional function or predicate (with respect to D) if for each x ∈ D, P(x) is a proposition. Since the walk must begin with either a 1-meter or a 2-meter step, all of the ways to walk n meters are accounted for. Given that C0 = C1 = 1 and C2 = 2, compute C3 , C4 , and C5 by using the recurrence relation of Example 1.7. 26. f(n) = n3 Determine whether each function in Exercises 16–21 is one-to-one, onto, or both. Therefore, a = b. (Order is obviously important since, for example, form and from are different words.) Relations generalize the notion of functions. Example 2.12 More Population Growth Assume that the population of Rustic County is 200 at time n = 0 and 220 at time n = 1 and that the increase from time n − 1 to time n is twice the increase from time n − 2 to time n − 1. Finally, we evaluate ¬p ∨ q ∧ r, which is true. Is decreasing? Reflexive, antisymmetric, transitive, partial order 35. We call n the index of the sequence. At line 2, h > large (5 > 1) is true, so we set large to h (5). (An answer is b, with 3 ∈ B, 6 ∈ B.) If v is a vertex in B and there is at least one element in B (namely 4) that is not in A. Suppose that X and Y are sets. Suppose that x ∈ Z. Example 4.6 34 We are given the following hypotheses: If the Chargers get a good linebacker, then the Chargers can beat the Broncos. Executing power = power ∗ b sets power to 8. These observations lead to a loop invariant that can be used to prove that Algorithm 2.3 is correct (see Exercise 12). For example, the number t = 1 is assigned the value 55; the number t = 3.45 is assigned the value 189.75; and so on. The nonzero remainders in the order they are computed by the Euclidean algorithm are 425 mod 196 = 33 196 mod 33 = 31 33 mod 31 = 2 31 mod 2 = 1. Suppose, by way of contradiction, that a ∈ [b]. For every x, there exists y such that P(x, y). |X ∪ Y | = |X| + |Y | − |X ∩ Y |. Prove the Well-Ordering Property. ¬p ∨ q 19. The first bits are 110 (to identify it as a Class C address). The numbers relate to the Catalan numbers. At lines 19 and 20, we label vertex b as (a, 1) and we label vertex d as (a, 1). Assume the form of mathematical induction where the Inductive Step is "If S(n) is true, then S(n + 1) is true." Prove the Well-Ordering Property. →p ∨ q 19. The first bits are 110 (to identify it as a Class C address). Recurrence Relations 1 2 3 Introduction Solving Recurrence Relations Problem-Solving Corner: Recurrence Relations Applications to the Analysis of Algorithms Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises You want to tell me how? (If c = d), {c, d} = {c, c} = {c}.) Since a = b and c = d, we must have {a} = {c} and {a, b} = {c, d} . If p, q, and r are true, find the truth value of the proposition (p ∨ q) ∧ ¬((¬p ∧ r) ∨ q). Using trial and error, we find that if we eliminate edge (g, h) and carry out the series reductions, we obtain an isomorphic copy of K3,3 (see Figure 7.7). Isomorphic free trees 44. , n2 + 1, are distinct. a f a g b f a g Eliminate edge (g, h) b h f h e e e c d b Series reductions c d c d K 3, 3 Figure 7.7 Elimination of an edge to obtain a subgraph, followed by series reductions. Algorithm 4.3 selects an edge (j, k) of minimum weight, where j ∈ V and k ∈ / V , and adds it to Ti to produce Ti+1 . Example 3.24 If we define a relation R from X = {2, 3, 4} to Y = {3, 4, 5, 6, 7} by (x, y) ∈ R if x divides y, we obtain R = {(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}. Since n ≤ 1 + lg n = (lg n), a theta notation for the number of times the statement x ← x + 1 is executed is (lg n). Exercise 48, Section 3, shows another way to prove (7.4). Thus the solution to problem S(n) is 1 , count the number of paths of various lengths. ∃y∀x(x2 < y + 1) 47. In Exercises 28–30, suppose there are 10 roads from Oz to Mid Earth and five roads from Mid Earth to Fantasy Island. By the inductive assumption, the greedy solution GS = to Grs [n − ?]problem is optimal, so |S = | = |GS |. Thus when we arrive at line 7, large will correctly hold the largest of a, b, and c. If you are asked to prove or disprove a universally quantified statement, you can begin by trying to prove it. Also, tell on which vertices edge e1 is incident. Then (A ∩ B) ∪ C = {2, 3, 4, 5} and A ∩ (B ∪ C) = {2, 3} and (A ∩ B) ∪ C = A ∩ (B ∪ C). What does it mean to "decrypt a message"? Example 2.20 Let G be the graph of Figure 2.14. But if x ∈ / X ∪ Y , then x ∈ / X and x ∈ / Y , , v0) is a path from w to v, and thus wRv. Therefore, R is symmetric. Trace Algorithm 2.1 for the input t = "00000000" and p = "001". X − Y = Y − X for all sets X and Y. We prove that Fact = i! is an invariant for the while loop. Thus x ∈ A. ■ Don't be afraid to make assumptions. As we move along P, at some point we must encounter an edge y = (v, w) with v ∈ U, w ∈ V . Prove that for all rational numbers x and y, y = 0, x/y is rational. This algorithm selects the largest item and places it last, then recursively repeats this process. One approach is to use a hash function. a b f c g m 17. Suppose that within a set of strings, we start with a and four other that the game "tournament" appropriate? What is the probability that the student answers every question correctly? Draw the complement of the graph G2 of Exercise 7. What is a sink in a network? P1 (shown dashed) and P2 (shown in color) are distinct simple paths from a to b. Choose y = m to obtain x = ... Thus the equation holds for all x ∈ U. ∀x ¬P(x) 31. (We assume that time is restricted to nonnegative real numbers.) The range is also equal to the set of all nonnegative real numbers. Basis Step (n = 0) If n = 0, the trees input to Algorithm 8.13 are both empty. Now suppose that a = b. Find the names of all employees. Prove one of the following: (a) There exists x such that x ∈ X and x ∈ Y . The vertex v ∈ V is matched with the vertex w ∈ W if and only if the flow in edge (v, w) is 1. Bijection: one-to-one and onto function 11. In this section we give an algorithm for finding a maximal flow. If we next divide 45 by 2, we find 45 = 2 · 22 + 1. Equation (1.3) furnishes an example of a recurrence relation. But the number of subsets of X is 2n ; we have improved equation (7.3). (VLSI is short for Very Large Scale Integration.) The problem is to tile a rectangle of minimum area with the desired components. (p → q) − r 18. (Conclusion) (Notice that we cannot assume that k1 = k2 . Let G be a graph. Combining these inequalities, we have k + 1 ≤ 1 + lg n < k + 2. In Figure 3.1, we have drawn four vectors to represent the elements of the set X of Example 3.4. Next, if the element (x, y) is in the relation, we draw an arrow (called a directed edge) from x to y. Everyone who has a graphing calculator understands the trigonometric functions. {x | x ∈ R and 0 < x ≤ 2} {1, 2} Sets and Logic In Exercises 33–36, show, as in Examples 1.5 and 1.6, that A ⊆ B. The arrow diagram of f is 0 1 2 3 4 13. Since there are 26 spades and hearts, there are C(26, 5) ways to select five cards from among these 26. Replacing n by n − 1 in (2.1), we obtain an−1 = dn−2 + 3. a d b c f e g h i 10. The sum of the degrees of all vertices in a graph is equal to twice the number of edges. At this point, we return to line 8. sequence sat31. See Theorem 2.3. 7. The variable n is repeatedly halved n = n/2 and when n becomes 0, the loop terminates. 32. So we must show that τ(R1) ∪ D = τ(R1 ∪ D). Add an+1 to both sides to get a1 + a2 + · · · + an + an+1 = F(n) + an+1 . Therefore, the graph does not have a Hamiltonian cycle. To disprove a universally quantified statement, find one element in the domain of discourse, called a counterexample, that makes the proposition false. Full binary tree; height = 3; nine terminal vertices 17. Let us begin by trying to prove the statement. Composition of relations generalizes composition of functions. When using a game tree we should use a depth-first search. Substituting this expression for 11 into (2.5), we obtain 1 = 24 · 5 + 23 + 2 + 1. We must be careful not to count an eliminated edge more than once when we apply the Exercise 3.2 to show that a graph does not have a Hamiltonian cycle. If Pete is not taller than Pat. From t100 . The idea is to repeatedly divide the sequence as nearly as possible into two parts and retain the part that might contain the key. Since n < 2k+1 , again taking logs, we obtain lg n < k + 1. Theorem 2.24 If a graph G contains a cycle from v to v, G contains a simple cycle from v to v. When dividing three bags, say B1 , B2 , and B3 , 277 Introduction to Number Theory we obtain a two-bit binary number, say ch. Next, the prospective recipient computes φ = (p − 1)(q − 1) and finds an integer n such that gcd(n, φ) = 1. The project operator chooses specified columns from a relation. , ek incident on v. Which properties of an algorithm—input, output, precision, determinism, finiteness, correctness, generality—does this proposed algorithm have? an = 2(−4)n + 3n(−4)n 24. 47 17. Write the negation of each proposition in Exercises 10–13 in words and symbolically. Two key observations lead to improved time. Let (x, y) ∈ R1 ∩ R2 . x2 y3 z5 ; (x + y + z)10 6. Proof Basis Step (n = 0) We have already observed that if n = 0, Algorithm 4.2 correctly returns the value of 0! (1). A map with distances can be modeled as a weighted graph. We show that bn = n − 1. For example, a "loaded" die is weighted so that certain numbers are more likely to appear than others. At line 2, since i > j (6 > 5) is true, we return to indicate that we failed to find key. For example, x > y, x ≠ y, etc. Subtraction 5.8 Example 5.9 A binary search tree is a binary tree T in which data are associated with the vertices. The proofs in Examples 2.10 and 2.11 are constructive proofs. Given two distinct words, to determine whether one precedes the other in the dictionary, we compare the letters in the words. ■ If none of the conditions of potentially useful definitions and theorems applies, try proof by con- tradiction. The relation {(1, 2)} is not a symmetric relation on {1, 2}. Then f is one-to-one, but not onto. What are the advantages of pseudocode over ordinary text in writing an algorithm? First suppose that a ≤ b. An argument that establishes the truth of a theorem is called a proof. In this section we consider three additional tree-traversal methods. Compute b1 , b2 , and b3 . In Exercises 23–25, tell whether or not each property is an invariant. How many proper subsets does X have? In other words, no proves it to a position if the value is negative. If an argument is in the lead on the same day as the search for the alternative is found... If it's a... value 0. See Theorem 1.22. Definition 4.1 Let α = s1 s2 · · · sp and β = t1 t2 · · · tq be strings over {1, 2, . A 2k × 2k grid is divided into 4k2 squares and four k × k subgrids. Since a = 2, 0, 2a−c 3b is even. If it is a bipartite graph, the program should list the disjoint sets of vertices. Therefore, g is a composition of all buyers. Prove that if a relation R on a set X is reflexive but not symmetric, the collection of pseudo equivalence classes (see Example 4.15) does not partition X. We can use decision trees to show that the algorithm given in Figure 7.3 to solve the five-coins puzzle is optimal, that is, no algorithm that solves the five-coins puzzle has worst-case time less than 3. The nonzero remainders in the order they are computed by the Euclidean algorithm are 480 mod 396 = 84 396 mod 84 = 60 84 mod 60 = 24 60 mod 24 = 12. An integer n is odd if there exists an integer k such that n = 2k + 1. (Any vertex in G is either in V or in W .) A matching for G is a set of edges E with no vertices in common. Then (2.1) and (2.2) are true. List the members of P ({a, b}). Let m, n, d1 , and d2 be integers. In Exercises 14–19, express each decimal number in binary. We are given that for every x, there exists y such that P(x, y) is true. Edges incident on R that cannot be selected are shown dashed in Figure 8.7. This leaves only the edge labeled 4. Prove that if f is a one-to-one, onto function from X to Y , then {(y, x) | (x, y) ∈ f } is a one-to-one, onto function from Y to X. Use proof by cases to prove that min{a, b}, c} = min{a, {b, c}} for all real numbers a, b, and c. Since there are eight possible outcomes, we can solve the problem by comparing two coins against two coins we can solve the problem in two weighings or less in the worst case. Therefore, P(E1 ∩ E2) = 0. 6. We first prove that if A ⊆ B, then B ⊆ A. e5 v1 e6 e1 v2 12. Thus, 110(−67) mod 273 = hs mod φ = 1. ■ When using a divide-and-conquer algorithm, it is not necessary to use a complicated form of recurrence relation. For the recurrence relation may be solved by iteration: dn = 1.1dn−1 = (1.1)2 (dn−2) = · · · = (1.1)n d0 = (1.1)n 1000. ∃x ∃y ¬P(x, y) is not logically equivalent to ¬(∀x ∃y P(x, y)). Letter Frequency Letter Frequency I U B S 7.5 20.0 2.5 27.5 C H M P 5.0 10.0 2.5 25.0 26. , k − 1. The techniques shown in the example is useful in checking solutions of recurrence relations. e3 is incident on v2 and v4 . Hypotheses: Everyone in the class has a graphing calculator. A pattern is developing. Since the domain of discourse is nonempty, we may choose a value for x , k, then the (t1 + t2 + · · · + tk) +able contains a subgraph isomorphic to M. The formulas for the number of nonisomorphic free trees and for the number of nonisomorphic rooted trees with n vertices are quite complicated. How many terminal vertices does it have? What is the maximum number of comparisons required by Algorithm 3.8 to sort an array of size 6? Basis Step (n = 2) f22 = n = 1 · 2 − 1 = f1 f3 + (−1)3 Inductive Step fn fn+2 + (−1)n+2 = fn fn+1 + fn) + (−1)n+2 = fn fn+1 + fn2 + (−1)n+2 = fn fn+1 + fn−1 fn+1 + (−1)n+1 + (−1)n+2 = fn+1 (fn + fn−1) = fn+1 24. Choose one main course. Formal Solution [Problem 1.] Let X = {1, 2, 3, 4}. For example, the equation A = {1, 2, 3, 4} (1.1) describes a set A made up of the four elements 1, 2, 3, and 4. We then compute n mod 2. Compare the number of bits used to encode the text using the Huffman code with the number of bits used to encode the text in ASCII. The reduction to a finite number of cases was helpful in using the computer to help find the cases to be analyzed. Then both x and y belong to some set S ∈ S. This leaves 34 students taking only calculus. It's a good idea to write out exactly what must be proved: If CX∪Y (x) = CX (x) + CY (x) for all x ∈ U, then X ∩ Y = ∅. given in the beginning of this section symbolically and show that it is valid. Show that τ(R) is transitive. Find all labeling corresponds to a matching? Professor E. si , . Compute c = an mod z and send c. These n − 1 possibilities account for the leading n − 1 factor in the recurrence relation). Let's fill in the missing numbers. Now m1 m2 m1 n2 + n2 n1 ¬y = = −1 If you choose x = Farley can choose y = 0). Farley can always win the game, so the statement ∃x∀y(x + y = 0) is false. Three dice are rolled. Suppose that G contains at least one cycle, then i = v . What does it mean for v and w to be adjacent? We conclude the section by looking at Hamiltonian cycles in the n-cube. (9.2) It follows from (9.1) and (9.2) that whatever the value of y is, it will not affect the value of x; thus we need not concern ourselves further with the subtree rooted at vertex C. By the second form of the Pigeonhole Principle, two of the numbers must coincide. Discussion We certainly cannot check 2m2 + 3n2 for all the positive integers m and n, but we can rule out most positive integers because, if 2m2 + 3n2 = 40, the sizes of m and n are restricted. Since z is not circled, we proceed to line 9, where we select vertex f , the uncircled vertex with the smallest label, and circle it (see Figure 4.4). The argument becomes 1. Prove that m m m+n n m+n m+n C(m + n, m) < 1 for all m, n ∈ Z+ . If the domain of discourse is Z × Z, ∃x ∃y ¬P(x, y) is true; however, ¬ (∀x ∃y P(x, y)) is false. b a 3 d Figure 4.9 The moment in the second iteration of Dijkstra's shortest-path algorithm. In words, a function is not one-to-one if there exist x1 and x2 such that f(x1) = f(x2) and x1 ≠ x2 . 84. 32 · 73 · 11 25. What is a binary relation from X to Y? Since n mod 2 = 1, result is set to (result∗x) mod z = 27 mod 11 = 5. Notice that r satisfies 0 ≤ r < d; that is, 0 ≤ 9 < 13. Example 5.10 Consider the existentially quantified statement x ∈ Z | x = 2x . Figure 5.3 A binary tree of height h = 3 with t = 8 terminals. As we have learned and remarked previously, when we are asked to prove a universally quantified statement, a specific example does not prove the statement. 7 The Euclidean algorithm requires at most 40 modulus operations to compute the greatest common divisor of any pair of integers, not both zero, in the range 0 to 1,000,000. Similarly, a path of length 2 from c to c defines an edge incident on c. This initial conditions are S1 = 2, S2 = 3. Is ¬t nonrecreasing? For the BasisStep, consider the set {1, 2}. There is one nonchild—the root. 58 1. Of course, for sufficiently large inputs, algorithm A is considerably faster than algorithm B. Give an example of an Instant Insanity puzzle satisfying: 16. To handle collisions, a collision resolution policy is required. But m(k − 1) < m n = n, m 367 Counting Methods and the Pigeonhole Principle which is a contradiction. The logic game Chapter Self-Test Section 2 1. The number of such selections is (Ck + 1 − 1, k) , an−1 . Each component of a forest is connected and acyclic, and therefore, a tree. Write a program that, given integers n > 0 and φ > 1, gcd(n, φ) = 1, computes the inverse of n mod φ. A = {3, 2, 1}, B = {1, 2, 3} 22. Since mathematical induction to show that every integer n ≥ 2 121. Let each row, column, or diagonal that contains one X and two blanks count −1 . Ck−1 Cn−k . replacing the index k by i, where k = i + 1. Let c and b = d, then {a, b} = {{a}, {a, b}} = {{c}, {c, d}}. 96. Examples: 1 = 0.1666 . , An }, we write S = n Ai , S= i=1 n Ai , i=1 and if S = {A1 , A2 , . If R and S are reflexive, then R ∪ S is reflexive. (e) Buy two tickets to the "Unhinged Universe" rock concert for Friday. In our game, the initial position is shown as 3 . If there is a complete matching, show one. If G is a connected graph and every vertex has even degree and G has only a few edges, we can usually find an Euler cycle by inspection. We define a function g on the edges of M by g ((v, w)) = (vx vy , wx wy). Suppose that there exists i such that si < A. (n2) 10) Using the hint in Exercise 35, construct ∗ -sets of sizes 3 and 4. If we let P (P) denote the set of labeled (unlabeled) vertices, we obtain the cut shown in Figure 3.2. b c 2 2, 3 2, 5, 4 a z 2 2, 4 5, 4 d 2, 2 e 2 Figure 3.2 A network at termination of Algorithm 2.4. The cut (P, P) = {(b, d), (a, d)}, is obtained by letting P be the set of labeled vertices. Consider, for example, the statement p : n is an odd integer. Since Fdc = Cdc and Fde = Cde , we do not label either vertex c or vertex e. The Basis Step proves that the invariant is true before the condition that controls looping is tested for the first time. Again, CX∩Y (x) = CX (x) · CY (x). Thus, the maximum time to search for an item in a binary search tree is approximately proportional to the height of the tree. Figure 2.12 guarantees that there is such an i. For example, in Figure 3.10 a knight on the square marked K can move to any of the squares marked X. In Exercises 11–26, solve the given recurrence relation for the initial conditions given. , whose values begin 1, 2, 3, 5, 8, 13, . , is related to the Fibonacci sequence. We have shown how mathematical induction may be used to prove that a recursive algorithm computes the value it claims to compute. From p = s and p, we may use modus ponens to conclude s. , an2 +1 of n2 + 1 distinct numbers contains either an increasing subsequence of length n + 1 or a decreasing subsequence of length n + 1. The preceding argument holds for arbitrary finite sets X and Y ; it is always true that |X × Y | = |X| · |Y |. 30. File loading please wait... For example, if n = 5, n and n + 2 are prime. It follows that there are n − |V × | edges in E of type I. Using Example 1.11, we find that A ∩ (B C) = A ∩ [(B ∪ C) − (B ∩ C)] = [A ∩ (B ∪ C)] − [A ∩ (B ∩ C)] = [(A ∩ B) ∪ (A ∩ C)] − [A ∩ (B ∩ C)]. Which, in turn, is approximately equal to 70/r. Then there exist integers m1 , n1 , m2 , n2 such that x = m1 /n1 and y = m2 /n2 . Walter, will you serve the nuts—I mean, would you serve the guests the nuts? Prove your results. As shown, suppose a2 = u1 = 1 2 = (1 + i)n . 197 Functions, Sequences, and Relations 12. Also, Y is not a subset of X since 3 ∈ Y , but 3 ∉ X. An algorithm that contains a recursive function 2. State the conjunction rule of inference. The technique demonstrated in Example 2.18 is formalized as Algorithm 2.19. See Theorem 3.6. Section 3 1. (Note that ∅ is an independent set for any graph.) Prove the following result due to [Prodinger]. How many are divisible by 5? Now a(r n+1 − 1) + ar n+1 r−1 ar(n+1 − 1) ar n+1 (r − 1) + r n+1 −1 = r n+1 r−1 a(r n+1 − 1) = r−1. We proceed to line 9, where we select vertex a, the uncircled vertex with the smallest label, and circle it (see Figure 4.3). p → q 51. Suppose that we replace the line k = (i + j)/2 with k = j − 2 in Algorithm 3.2. Is the resulting algorithm still correct (i.e., does it find key if it is present and return 0 if it is not present)? A 3 × 7 rectangle is divided into 21 squares each of which is colored red or black. The base case is n = 1. Suppose that we have propositional function S(n) whose domain of discourse is the set of positive integers. 2010 could represent a number in either decimal or hexadecimal. The weight of edge (i, j) is w(i, j) > 0 and label the vertex set {1, 2, . n ≥ 1, . In this section we explore nested quantifiers in detail. ∀y∃x(x2 < y + 1) 44. We then call this algorithm to compute tn−1 , tn−2 , and tn−3 . Use the Addition Principle to solve Exercises 10–18. , pn ∧ an = 7an−1 − 10an−2 ; a1 = 0, a0 = 4. 46. Now we can readily verify that the relation R = {(a, a), (b, b), (c, c)} on X = {a, b, c} is reflexive, symmetric, and transitive. Find 8 such that if each piece of the chessboard is divisible by 2 and the floors vanish. This sequence is a finite sequence. Input: n, a power of 2 (the board size); and the location L of the missing square Output: A tiling of an n × n deficient board 1. Show that the value V of any flow satisfies m(P) − C(P, P) ≤ V ≤ C(P, P) − m(P, P) for any cut (P, P). . , are cancel. Does 30470 represent a number in binary? 2n Let B0 , . The next time we execute the for loop in lines 7–14, the edges with one vertex in the tree and one vertex not in the tree are Edge Weight (1, 2) (2, 4) (3, 6) (4, 6) (5, 6) 4 5 3 6 2 The edge (5, 6), with minimum weight is selected. Write a recurrence relation and initial condition that define the sequence c1 , c2 , . 38. We have shown that the argument ∀xP(x) ∴ P(d) if d ∈ D is valid. We conclude the proof by presenting a solution based on a graph model to the Instant Insanity puzzle. Notice that by the nature of the calculator, the domain and range are finite sets. We note that P1 = 1. Write a program to determine whether a relation is reflexive. Taking r = s0 and t = t0 , we have that gcd(r0 , r1) = sa + tb. Write a program that adds binary numbers. technical reasons, if G consists of one vertex v and no edges, we call the path (v) an Euler cycle for G. Definition 2.26 A string β is a substring of α if there are strings γ and δ with α = γβδ. R = {(1, 2), (2, 3), (3, 4), (4, 5)}; ordering of X: 1, 2, 3, 4, 5 11. If there is only one equivalence class, what must the relation look like? Thus there are 96 selections in which either Alice or Dolph or both are officers. Prove that (n + 1)2 /8 < cn for all n ≥ 2. Basis Step In this case, {1} is the only nonempty subset of {1}, so the sum is 1 = 1. Given a line and a point not on the line, there is exactly one line parallel to the given line through the point. (3.10) Taking i = n − 3 in (3.10), we obtain rn−3 = rn−2 qn−1 + rn−1 , which may be rewritten as rn−1 = −qn−1 rn−2 + 1 · rn−3 . The pair of functions f and g is called an isomorphism of G1 onto G2 . Describe a difference operation on a relational database. Write a program that lists all the ways that m distinct Martians and n distinct Jovians can wait in line if no two Jovians stand together. Fix a seat for a Jovian. In general, the outcomes are not equally likely. He was an early proponent of programming as a science. ↑ 18. (b) Define f = f · f ◦ · · · ◦ f to be the n-fold composition of f with itself. Since there are n stations, g1 = Y f ... , P(x), ... In the tree, there are five, fn, ... Show that a... This is a simple cycle of length k 11 8. Find the children of Uranus. 9 35 v = 0 21. Momentarily, consider the edges of G to be undirected and let P = (v0 , v1 , . The number of lists of length k which cannot be ... (1.2) (1.2) we obtain j=1 ... (1.2) ... 10 ... P(x) ... ... ■ List the permutations that begin with s1 . The proof is complete. An actor's Bacon number is the length of a shortest path from that actor to Bacon. Let s1 , . (e) Use the result of part (d) to find the relation R ◦ R1 (as a set of ordered pairs). Red pepper and onions are required to make chili. We show that b is not in any pseudo equivalence class. At line 13, we find that s3 is the rightmost element not at its maximum value. {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)} 2. This graph must have a 3-cycle. The tree T has at least n! terminal vertices, so by Theorem 5.6, lg n! ≤ h. Show that AO(k, 2, 1) = 2 for k ≥ 2. Summing over all j, we will count all paths Graph Theory of length n + 1 to k. Therefore it suffices to check the cases m = 1, 2, 3, 4 and n = 1, 2, 3. f(x) = 48. Initially, there are n piles, each consisting of a number of identical tokens. The Inclusion-Exclusion Principle tells us that |X ∪ Y | = |X| + |Y | − |X ∩ Y| = 60 + 60 − 24 = 96. How many strings begin with A, allowing repetitions? Example 3.3 Figure 2.1 shows the labeling at the termination of Algorithm 2.4 for a particular network. X ∗ ∩ Y = (X − Y) ∪ (X − Y) ∩ Z for all sets X, Y , and Z. Example 4.12 There are two equivalence classes for the equivalence relation R of Example 4.5, namely, [1] = [3] = [5] = {1, 3, 5}, [2] = [4] = {2, 4}. Example 4.13 [2] = [4] = {2, 4}. {1, 3, 5} {1, 2, 3, 4, 5} f(c) = 3 13. Stacks {1} {(1, 2), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)} {(1, 2), (1, 3), (1, 4), (1, 5)} {(1, 2, 3, 4, 5) of Example 4.5, namely, [1] = [3] = [5] = {1, 3, 5}, [2] = [4] = {2, 4}. (c) Find x ∗ x mod z = 256 mod 21 = 4, and so set n = 1. Most people are required to do an n-step of... If there is a winning... 1. ... In the given true statement ∀x∃y P(x, y), given any x, it's possible to find a y, which may depend on x, that makes P(x, y) true. For example, the unique simple path from v2 to v7 is (v2 , v1 , v3 , v7). ("Partitions" means that every x in X belongs to exactly one of the subsets of the partition.) The subsets making up the partition can be described in successive steps, we use the Multiplication Principle. Explain a method of verifying that X is a subset of Y . Indeed, the first text on graph theory ([Kőnig]) appeared in 1936. In particular, if he is a freshman, he may not take calculus. A proposition is a sentence that is either true or false, but not both. Since −4 < x, we may subtract by ¬1 to obtain z ≤ p. ... p → q. Theorem 2.11 or 2.14. Why is it easy to solve for the worst-case time of merge sort if the input size is a power of two? The desired conclusion. Let J denote the event "Joe passes," and let A denote the event "Alicia passes." Then P[Joe fails) = P[J) = 1 − P[J) = 0.25 P[both pass) = P[J ∩ A) = P[J]P[A) = (0.75)(0.80) = 0.6 P[both fail] = P[J ∩ A] = P[J ∪ A] = 1 − P[J ∪ A] = 1 − [P[J] + P[A] − P[J ∩ A]] = 1 − [0.75 + 0.80 − 0.6] = 0.05 P[J ∩ A] + P[J ∩ A] = ... Computer and... 1.1 ... and... In the decimal number system, to represent integers we use the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Cubing 3 2 = p/q gives 2 = p3 /q3 , and multiplying by q3 gives 2q3 = p3 . In this way it is generally possible to represent strings of characters, such as text and programs, in less space than if ASCII were used. Proof x · 0 + 0 = x · 0 = x · (0 + 0) = x · 0 + x · 0; therefore, x · 0 = 0. Since 2n−1 = 2n is true when n = 1. If X and Y are sets, we let X × Y denote the set of all ordered pairs (x, y) where x ∈ X and y ∈ Y . x2 + 1 ≥ 2x. Suppose that statement 1 ≤1 x2 + 1 is true for every real number x. Each Ti is connected and acyclic, so we may use our previously proven result, by implies (c), to conclude that Ti has ni − 1 edges. We assign this maximum value to the box vertex. } 385 Recurrence Relations Algorithm 1.4 is a direct translation of equations (1.4) and (1.5), which define the Catalan numbers. ... (4.1) ... (n − 1) = ... Example 4.10 ... { predecessor(v) = null (sub)// (v) } // label a Network Models 8. Continuing in this way, we can show that every block is marked. Theorem 1.21 24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; 22 are taking computer science. The inductive step in the proof of... When we set ... proposition... ... the truth of prior instances of the statement to prove the truth of the current instance. At this point, all the children of the root will have values. The number of strings with the pattern interests in the program 3. If we are defining a relation, R from the set of identifiers by s1 R s2 , provided that the first 31 characters of s1 and s2 coincide, then R is an equivalence relation. Keeping in mind that we're assuming (the removal of any edge from T disconnects T), try to figure out how to deduce a contradiction from the assumption that T contains a cycle before reading on. An r-element subset of {x1 , . TABLE 6.6 ■ SUPPLIER Dept Part No 04 23 04 66 04 96 96 23 3355B2 2A 8C200 42C 900 20A8 1199C 772 9. Since R is symmetric, xRy and yRz, then yRx, and since R is transitive, yRz and zRx, so xRx. Cycle, simple cycle 7. x becomes a4 , and n becomes 7. A Gray code is a sequence s1 , s2 , . Logic is concerned with the form of propositions and the relation of propositions to each other and not with the subject matter itself. If we can describe the partition as constructed in successive steps, we use the Multiplication Principle. Explain a method of verifying that X is a subset of Y . Indeed, the first text on graph theory ([Kőnig]) appeared in 1936. In particular, if he is a freshman, he may not take calculus. A proposition is a sentence that is either true or false, but not both. Since −4 < x, we may... not lift weights. i=1 ' 66. sn > A. What is a maximal flow? In Figure 1.4, ∞ represents an unlimited capacity. In a base case, a solution is obtained directly, that is, without a recursive call. At one time, the following ordinance was in effect in Naperville, Illinois: "It shall be unlawful for any person to keep more than three [3] dogs and three [3] cats upon his property within the city." Was Charles Marko, who owned five dogs and no cats, in violation of the ordinance? Any subset of X × Y having the property that for every x ∈ X, there is a unique y ∈ Y such that (x, y) ∈ X × Y is a function. It is not the case that (5 < 9 and 9 < 7). The classes are "has the HIV virus," which we denote H, and "does not have the HIV virus" (H). It remains to show that every element in X belongs to exactly one member of S; that is, if x ∈ X and x ∈ [a] ∩ [b], then [a] = [b]. If U, a universal set, is specified as U = {1, 2, 3, 4, 5}, then, e.g., {a, c}. The values of the terminal vertices are given, 6 2 3 We say that a fraction p/q, where p and q are integers, is in Egyptian form if Exercises 9 and 10 refer to the sequence c1 , c2 , . 1 2 3 f X a b c d 1 2 3 Y X Figure 1.9 The function of Example 1.22. log3/2 2m/3 Substituting the third formula for 2 into the previous equation yields 5. Example 1.12 What day of the week will it be 365 days from Wednesday? 1 4 3 1 W B G 3 G 1 28. P(3) 11. Write an algorithm that returns the index of the last occurrence of the value key in the sequence s1 , . Similarly, (x, y), (y, z) ∈ S, and since S is transitive, (x, z) ∈ S. . . . , p → (p ∨ r ¬v q p 7. Which of the following must be true? Thus, X ∩ Y is convex. Hint: If X is a nonempty set of real numbers bounded below, let Y denote the set of lower bounds. 34. Example 5.7 Suppose that P is a propositional function whose domain of discourse is the set {d1 , . We can similarly orient cubes 2, 3, and 4. n = 1 576 j 24. 1 0 a c b d T1 T1 Figure 8.1 A tree. Using the generalized De Morgan's laws for logic, we find that the negation is ¬(∀x∃y(xy < 1)) ≡ ∀x¬(∀y(xy < 1)) ≡ ∀x∃y(xy < 1) ≡ ∀x∃y(xy ≥ 1). At line 2, we set large to 12. f(v1) = w6 , f(v2) = w2 , f(v3) = w5 , f(v4) = w7 , f(v5) = w4 , f(v6) = w1 , f(v7) = w3 , f(v8) = w8 . At line 17 we examine edge (c, z). Show that the number of comparisons required is reduced to 7 and r = 5. It is possible to solve some problems in polynomial time whether two arbitrary trees are isomorphic. Look at the parts that perform key computations to gain insight into how to enhance the algorithm's efficiency. Show a maximal matching. If we substitute "Alex Rodriguez" for "baseball player," statement (c) is Alex Rodriguez hit over .300 in 2003. Find the components f · f , g · g, f · g, and g · f . In P(n) Let P(n) be the proposition "n ≥ x2 ." Tell whether each proposition in Exercises 12–20 is true or false. Describe worst-case time of Algorithm 2.1. 19. 2 4 2 This geometric sum is equal to n 1 − 1− Now t(n) ≤ n 1−n 1−n−1 1−2 1 2k 1 2k 1 2 1 2k = 2n 1 − 2k . (c) 1111111, 011111, 11111111, 00011111, 00011111, 00000011, 00000001, 0000000. Given x∀y P(x, y) ∨ ¬x∀y P(x, y) ≡ 0... 8 2 ... temp1 := player [Age ≥ 30] temp2 := temp1 [ID Number = PID] assignment temp2 [Team] Blue Sox, Mutts Algorithms 1 2 3 4 Introduction Examples of Algorithms Analysis of Algorithms Problem-Solving Corner: Design and Analysis of an Algorithm Recursive Algorithms Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises It's so simple. How many eight-bit strings have exactly two 1's? , xn) (1, (4, 6)) 21. Suppose that we have answered part (a). k=0 But this recurrence relation and initial condition a0 = 1 define the sequence of Catalan numbers. The bit, 0 or 1, tells us whether to move right or left. n = O(2n) 61. Let's check that the recurrence relation holds for n = 3: D3 = 2 = 2(1 + 0) = (3 − 1)(D2 + D1). Yes, (a, c), (f, g, e, b) ? (b, a, f, d, g, b, c, f) ?; (b, c, f) ? We can prove the reverse... {1 2, f} An algorithm can be modeled after Example 4.2. Modify Algorithm 4.1 so that it begins by assigning the weight ∞ to each nonexistent edge. ∀x ∀y P(x, y) is true. How many eight-bit strings have exactly two 1's? If there is a good... (3.9) Equation (3.5) may be rewritten as 53 = 273 − 110 · 2. The truth value of the disjunction p ∨ q is also determined by the truth values of p and q, and the definition is based upon the "inclusive" interpretation of "or." Consider the proposition, p ∨ q: It is raining or it is cold. So the proposition, q: It is raining or it is cold, so the proposition (i.e., p ∨ q). The order of the... x ∈ R... so we would consider the proposition, p ∨ q: It is raining or it is cold, so the proposition p ∨ q... (a) Given that x ∈ Z, we are looking at an Euler cycle. Two such examples are shown in Figure 4.10. The truth value of proposition (5.4) is defined similarly: (5.4) is true if and only if P(x) is true for every x in the domain of discourse. Therefore, every path from a to v has length at least L(v). At line 5, since key ('S') is not equal to s2 ('D') , we proceed to line 7. If integers in the range 0 to m, m ≥ 8, are both zero, an input to the Euclidean algorithm, the at most log3/2 2m 3 modulus operations are required. The game begins with the first (left) quantifier. It is worth noting in passing that this result holds in general; that is, if we start with a set with n elements has 2n elements. There are n! (n − r)! r! r-combinations of an n-element set. In the general case, to the professor correct? Let I be the sum of the lengths of the simple paths from the root to the internal vertices. Phony lists of lockers—one for the position to (A ∩ B) ∪ C. In practice, the implementation of Algorithm 2.4 can be simplified by using the adjacency matrix of the graph (see Exercise 11). Events E1 and E2 are mutually exclusive if E1 ∩ E2 = ∅. using the relations of Tables 6.4–6.7: Find the names of all employees who work in either department 23 or department 96. satisfy the same recurrence relation, s1 = f2 , and s2 = f3 , it follows that sn = fn+1 for n ≥ 1. 1346. Like the algorithm for generating r-combinations, the algorithm that generates the permutations of {1, 2, . (1.5) We have to show that x ∈ A ∩ (B ∪ C), that is, x ∈ A and x ∈ B ∪ C. ¬q → (r ∧ p) 52. 61. The length of a string α is the number of elements in α. Since we assume that X ∩ Y = ∅, there is an element x ∈ X ∩ Y . A function f from X to Y is a relation from X to Y having the properties: (a) The domain of f is equal to X. The sets [a] defined in Theorem 4.8 are called the equivalence classes of X given by the relation R. Prove equation (7.5) by induction. Prove that x − y ≤ 1. Explain how we can select five or five balls if the balls are considered distinct? Any sorting algorithm requires at least Ω (n lg n) comparisons in the worst case to sort n items. Let P(x) denote the statement "x is an accountant." Write each statement in Exercises 44–47 symbolically. Algorithm 1.9 Huffman's Codes The most common way to represent characters internally in a computer is by using fixed-length bit strings. If we remove a 7 from S, we obtain a solution S ∗ to the (n−7)-problem. It doesn't matter what the units of time are—2n grows so fast as n increases (see Table 3.1) that, except for small values of n, it would be impossible to run the program. A binary tree with three vertices in which the root has degree 2 must be isomorphic to the tree in Figure 8.12(a). Algorithm 6.3 Inorder Traversal This recursive algorithm processes the vertices of a binary tree using inorder traversal. If the Council approves the funds, then New Atlantic will get the Olympic Games. Implement tournament sort as a program. For example, in the figure that follows, which shows part of our graph, Jeremy gets along Jeremy Lance Bret Tiffany It follows that the degree of every vertex is 5. Graph Theory Sheridan Greyhull Gillette Buffalo Worland Shoshoni Casper Lander Douglas Muddy Gap Figure 1.1 Part of the Wyoming highway system. Algorithm 3.11 computes an by using the formula an = am · an−m . gcd(a, b) The least common multiple can also be computed by using the prime factorizations of a and b. By Theorem 6.5, the probability of obtaining at least one defective microprocessor is 1 − (0.99)n = 0.00735781 = 0.096264219. Since there are two positions (x ∈ Z+ X Z+ , ∃y P(x, y), ∃y ¬ ∀x ∀y P(x, y) is true, prove your conjecture. Theorem 5.6 If a binary tree of height h has t terminal vertices, then lg t ≤ h. [Cull] gives algorithms for solving certain Tower of Hanoi problems with minimum storage space and time complexity. How can we reach this conclusion? Prove or disprove: A (B ∩ C) = (A B) ∩ (A C) for all sets A, B, and C. P(F) 0.999023437 378 P(E)P(F) = P(E ∩ F), and E and F are independent. f(n) = 2n 13. Since a2 = 8, a3 = a2 + 3 = 8 + 3 = 11. Since n = 16 > 0, the body of the while loop executes. n = 47 3. Suppose that X = {1, 2, 3, 4}. Example 5.7 The binary tree in Figure 5.3 has height h = 3 and the number of terminal vertices t = 8. Since 4 ∈ U, but 5 ∈ U (some 4 ∈ U), and 4 ... reflexive, symmetric, ... (3.7) ... j ... v ... c ... etc. What is a proof by contradiction? f(f(f(x))) = (x) ... in an enumerated example... 101101 + 11011 In the octal (base 8) number system, to represent integers we use the symbols 0, 1, 2, 3, 4, 5, 6, and 7. But T is the sum of n odd numbers, and there is no way to show that GK4 does not have a Hamilton cycle. Given an arrow diagram, the function is transitive if whenever there are directed edges from x to y and from y to z, there is also a directed edge from x to z. A contradiction results from the assumption that n ≥ 7. If v is a vertex in G and w is the number of entries of the set of... Suppose ... k ... that n ≥ 7 ... (see Exercise 11, Section 1). Since x and y are rational numbers, there exist integers m1 , n1 , m2 , n2 such that x = m1 /n1 and y = m2 /n2 . x becomes a8 , and n becomes 0. A graph G has an Euler cycle if and only if G is connected and the degree of every vertex is even. P = p → q, Q = ¬p → ¬q 66. ■ Finiteness The algorithm terminates; that is, it stops after finitely many instructions have been executed. p →p T F F T Section 2 1. Use the matrix of the relation to test for transitivity (see Examples 5.7 and 5.8) for the relations of Exercises 4, 6, and 10. How many eight-bit strings either start with a 1 or end with a 1 or both? k=0 28. In words, a relation R is not transitive if there exist x, y, and z such that (x, y) and (y, z) are in R, but (x, z) is not in R. Give the first seven values of the Fibonacci sequence. 71 Functions, Sequences, and Relations Example 3.21 Since the relation R defined on the positive integers by (x, y) ∈ R if x divides y is reflexive, antisymmetric, and transitive, R is a partial order. ¬q ∨ ¬r ≡ B and D. 3; , ss using insertion sort, first insert s2 in s1 so that s1 , s2 is sorted. The link between mathematical induction and recursive algorithms runs deep. 1 0 0 1 1 1 1 0 1 1 0 0 (0) , e1 , v1 , v2 , v2 , (a, d, c, b), (a, c, b) 15. A what is a beta cutoff? Let x ∈ X ∩ Y. 2. Is the relation of Exercise 20 reflexive, symmetric, antisymmetric, transitive, and/or a partial order? List the members of the event "the coin shows a head and the die shows an even number." 3. We consider that is a proper tree, ... P(F) We use Definition 6.13 to compute the probability of getting a sum of 10, given that at least one die shows 5, when two fair dice are rolled. Assume that large is the largest value in the subsequence s1 , . A sequence a1 , a2 , . Continue until the first formula for the remainder is substituted into the last Ex equation. 7. Therefore, k + 3 = x + 3 . In one of the oldest and simplest systems, the sender and receiver each have a key that defines a substitute character for each potential character to be sent. 58. Second, the line sumji − sumj−1 + si shows how a consecutive sum that ends at index i − 1 is related to a consecutive sum that ends at index i. n2 n1 n2 = m2 n1 m2 n1 + n2 m1 n1 m2 ∧ ¬a1 Answer 39–50 for the sequence a defined by an = 1 = 3n − 3. a, c, d, c, c, d, c, b, c, (Note that f(m, n) ≥ 6 for all m, n ∈ Z+ .) 23. 6 6 6 6 In Exercises 51–53, decide whether or not the proposition is true or false. Functions, Sequences, and Relations Exercises 4–7 refer to the relation g r-combination. (a : b means "store item a in cell b.") 1 : 1, 784 : 4, 18 : 5, 329 : 6, 43 : 7, 281 : 8, 620 : 9, 1141 : 10, 31 : 11, 684 : 12 5. Input: p (indexed from 1 to m), m, t (indexed from 1 to n), n Output: i text search(p, m, t, n) { for i = 1 to n − m + 1 { j=1 // i is the index in t of the first character of the substring // to compare with p, and j is the index in p // the while loop compares ti · · · ti+m−1 and p1 · · · pm while (ti+j−1 == pj) { j = j+1 if (j > m) return i } } return 0 } 212 Algorithms The variable i marks the index in t of the first character of the substring to compare with p. q3 = 12, qi . What is a conclusion in an argument? Use Exercises 41 and 42. Statement (d) is similar in form to statement (c): Here the variable is "restaurant." Whenever we substitute a restaurant rated in Chicago magazine for the variable 37 Sets and Logic "restaurant," the statement is a proposition. Notes An accessible introduction to elementary number theory is [Niven, 1980].An extended discussion of the greatest common divisor, including historical background, and other elementary number theory topics are in [Knuth, 1998a]. We can repeat the above to substitute a particular baseball player for "baseball player." Using the generalized De Morgan's laws for logic and the fact that ¬(p → q) ≡ p ∧ ¬q, we find that the negation is ¬(∀x1 ∀x2 ((f(x1) = f(x2)) → (x1 = x2))) ≡ ∃x1 ¬(∀x2 ((f(x1) = f(x2)) → (x1 = x2))) ≡ ∃x1 ∃x2 ¬((f(x1) = f(x2)) → (x1 = x2)) ≡ ∃x1 ∃x2 ((f(x1) = f(x2)) ∧ ¬(x1 = x2)) ≡ ∃x1 ∃x2 ((f(x1) = f(x2)) ∧ (x1 ≠ x2)). Suppose that list 1 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 3 4 6 7 8 0000 32 0 0 0 SOLUTION Figure 5.3 Suppose that we begin by the backtracking algorithm (Algorithm 3.10) in the search for a solution to the four-queens problem. ∀x(x = 1) → P(x)) Let P(x) denote the statement "x is taking a math course." The domain of discourse is the set of all students. B G R B G R W G W B 2 W B W R G 3 R B R W W 4 30. 410 Using iteration, we obtain Cn = (−1)1 Cn−1 = (−1)2 Cn−2 = · · · = (−1)n−2 C2 = (−1)n C2 = (−1)n [D2 − 2D1] = (−1)n . Explain how to construct a Gray code. Section 5 a b a b c d 1 b 1 1 0 [c] 1 1 0 d [1 0 1 0 0 1 1 0 1] 0 1 1 0 0 0. We say our proof was ... Inductive steps... ... antisymmetric. Write a backtracking program that determines whether a graph has a Hamiltonian cycle and, if there is a Hamiltonian cycle, finds it. 273 Introduction to Number Theory 16's place (16 1) 256's place (16 2) 1's place (16 0) B F 4 Symbol 0 Symbol 2 Symbol 1 Figure 2.3 The hexadecimal number system. 12 8 4 9 5 1 10 11 1 4 2 3 13 16 12. If the coin is flipped 10 times, what is the probability of at most five heads? The numbers show how many students belong to the particular region depicted. Use the Addition Principle If you want to count the number of elements in a set and you can divide the set into nonoverlapping subsets. Could the pseudocode of Example 5.7 be written as follows? Use Theorem 3.6 to compute the Gray code C4 . We write 9 and carry 1: + 1 84F 42EA 9 Next, we add 1, 4, and E, obtaining 1316 . A line 7, key < k ('S' < 'D') is false, so at line 10, we set i to 3. This partition together with X−Y partitions 431 Recurrence Relations X. Also write the postfix form and the fully parenthesized infix form of the expression. Deductive reasoning Hypothesis Premises Conclusion Argument Valid argument Invalid argument Rules of inference: modus ponens, modus tollens, addition, simplification, conjunction, hypothetical syllogism, disjunctive syllogism Section 5.3 Theorem [n] = [2n − 1] − · · · The Euclidean algorithm is Algorithm 3.3. Algorithm 3.3 Euclidean Algorithm This algorithm finds the greatest common divisor of the nonnegative integers a and b, where not both a and b are zero. Suppose that an eight-bit string with no two 0's in a row has exactly two 1's. Therefore, Section 5 1. n1 · · · nk 1 1 1 1 + = . (The trees are isomorphic as free trees.) 555 Trees Arguing as in the proof of Theorem 8.3, we can show that there are four nonisomorphic rooted trees with four vertices. , n, n + 1) and edge set 32. Theorem 6.20 Bayes' Theorem Suppose that the possible classes are C1 , . To see this, notice that each subset S of X that contains n can be paired uniquely with the subset obtained by removing n from S (see Figure 3.5). Suppose that the first i-value that is zero so that gcd(a, b) = rn−1 , , 19]] and we cannot immediately use the second form of the Pigeonhole Principle. Suppose that the method of Algorithm 1.1 can be used to find the largest value in a sequence. Therefore, S is a partition of X. 4! 2! 2! We can give an alternate proof of Theorem 3.2 by using relations. Functions, Sequences, and Relations If the condition given in Definition 1.21 for a function f from X to Y to be one-to-one is satisfied. For all x1 , x2 ∈ X, if f(x1) = f(x2), then x1 = x2 . Write an algorithm to carry out the procedure described before Exercise 23. Since ∀x ∈ A∪B, ¬x ∈ A ∪ B. Given X ∗ is the concatenation of two strings, each in X ∗. The complete relation is R = {(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}. Since n mod 2 is equal to 1, result is set to (result∗x) mod z = 3 mod 11 = 3. The probability of getting a sum of 10 given that at least one 5 occurs is 1/11. In symbols, a relation R is antisymmetric if ∀x∀y[(x, y) ∈ R ∧ (y, x) ∈ R] → [x = y]. The number of paths of length k, 1 ≤ k ≤ n, is n k=1 (n − 1)k − 1 (n − 1) = n(n − 1) (n − 1) − 1 k n(n − 1)[(n − 1)k − 1] . 1 0 1 1 0 0 1 0 27. If the graph is planar, redraw it so that no edges cross; otherwise, find a subgraph homeomorphic to either K5 or K3,3 . By the inductive assumption, the sequence x = x + 1 is executed k more times, for a total of total of k + 1 times. Example 5.11 To verify that statement is true, suppose that x ∈ R and x2 > 1. Let x be any real number x. Thus, there is a total of Fn+1 + Fn = Fn+2 ways. In the next section, we derive formulas for this... Inductive Step is complete. Give an example of a graph... nonnegative integer n... ...such that... ... from... Section ... Thus we can choose one of ... two-level, depth-first search. Show that postage of six cents or more can be achieved by using only 2-cent and 7-cent stamps. Theorem 3.5 Suppose that the pair a, b, a > b, requires n ≥ 1 modulus operations when input to the Euclidean algorithm. v2 v4 v3 v5 w2 v7 w3 w6 w5 v6 w7 v8 T2 T1 8. One edge can be chosen in C(2+4−1, 2) = 10 ways. Thus x ∈ B and A ∪ B ⊆ B. Definition 2.6 These classes need not be disjoint. If Y is nonempty, then Y has a least element. Let a = 504 and b = 396. ■ To build a graph model, determine what to represent and what the edges represent. Linear-time algorithm (Algorithm 8.13) to test whether two binary trees are isomorphic Section 9 48. Example 1.14 The statement ∀n ∈ Z+ (2n + 1 is prime) is false. Our results on subgraphs of the hypercube are from [Saad]. 515 Trees (g) The internal vertices are Uranus, Aphrodite, Kronos, and Zeus. The equivalence classes turn out to be X1 , . In this situation the user is in all intervals and is mapped to the problem of the one 1... 3 3 3 3... the subtrees... the i... that... k ... In the following subsections, in the... Use Exercises 17–23 ... the following query, using the relations of Tables 6.4–6.7: List the names of all employees who do not work in department 04. , n + 1} contains an even number of partitions. How many contain the digit 7? Theorem 1.5 Let U be a universal set and let A, B, and C be subsets of U. , an integer, n ≥ 2, are there 2 n−1 such ... Since there are 2n subsets of an n-element set, the algorithm that examines all subsets is Ω (2n). Exercise 3, Section 1 17. We must prove two things: T is connected, and the removal of any edge (but no vertices) from T disconnects T. Analysis of the Euclidean Algorithm We analyze the worst-case performance of Algorithm 3.3. We define the time required to be the number of modulus operations executed at line 6. We have proved that A ⊆ B. Suppose that we can represent who gets along with whom in the graph, leaving just a moment to go. We let vertices represent the actors, and we place an edge between two distinct actors if they appeared in at least one film together (see Figure 1.8). an = (n + 1)n . If d divides n, we say that d is a divisor of n . prv /∴ q, the algorithm is... It is an easy to ... number i... in... and factorial(n) { if (n == 0) return 1 return n ∗ factorial(n − 1) } We show that Algorithm 4.2 computes n! for several values of n. Find the length of a shortest path from a to z that passes through v. Our proof using mathematical induction is... If T is an isomorphism of T1 and T2 at those two vertices. Therefore T and T′ are isomorphic. Suppose that β ∈ X∗ . Johnsonbaugh co-authored a paper with Tadao Murata, Murata co-authored a paper with A. Notice that we are not saying that the conclusion is true; we are only saying that if you grant the hypotheses, you must also grant the conclusion. To send a message, all one needs to do is look up the recipient's encryption key in a publicly distributed table. Example 1.6 Let X = {x | x2 + x − 2 = 0}. A randomized algorithm does not require that the intermediate results of each step of execution be uniquely defined and depend only on the inputs and results of the preceding steps. A value x in the domain of discourse that makes P(x) false is called a counterexample to the statement ∀x P(x). Use proof by cases to prove that |xy| = |x||y| for all real numbers x and y. T is obtained by designating e as the root. n 1. In the conditional proposition p → q, p is the hypothesis. ∀x∃y(x2 + y2 ≥ 0) 54. It is possible to interconnect five processors so that exactly two processors are directly connected to an identical number of processors? In the following algorithm, the values of i, j, and k are as shown in Figure 2.2. Here we have one number and the example... A ≠ B... 1. Since... , r... r ∈ R... Section 1 ... at the termination... a triangle... Use... find a ... a triangle... Suppose that a... B S V P... Since... (see Figure 1.2).† To denote the negative numbers that belong to one of Z, Q, or R, we use the superscript minus. In the terminology of relations, we would say that Bill is related to Computer Science and Art, and that Mary is related to Mathematics. Find the names of all pitcher. , n} such that the sum k tij (4.1) j=1 339 Counting Methods and the Pigeonhole Principle does not exceed C and is as large as possible. q : We will buy more memory. What is an invariant in a graph? Write the contrapositive of each proposition in Exercises 1–10. 106. Computer programs are given. , pn ∴ c 24. Describe Dijkstra's shortest-path algorithm. In words, a relation R is not symmetric if there exist x and y such that (x, y) is in R and (y, x) is not in R. f = e − v + 2. Therefore [b] ∩ [c] = ∅, but [b] ≠ [c]. This collection of pseudo equivalence classes does not partition X. In Section 3.6.1 Find a... Mathematics, Hamilton is most famous for inventing the quaternions, a generalization of the complex number system. In either case, the total number of edges not in T is the total of all n ≥ 2 on 1 ... xn ≥ 0 ... 7. Graphs G1 and G2 are homeomorphic if G1 and G2 can be reduced to isomorphic graphs by performing a sequence of series reductions. By expansion, an operator is an invention... mathematical ... and ... For the following... is ... 1 ... (x) ... if ... Assume that... rule of inference. Step 2—move disk 2 from peg 1 to peg 2. List only the possibilities when person 1 gets order 2. b, 3, 2 c 4, 4 (a, 1), a (−1, 4) 4 (a, 2) 3, d 4, 2 (b, 3), d 4, (b, 3) 2 z 4, 5 (d, 5) b, 2 (d, 2) z (2, 2) 4, 0 5, 4 (f, 1) z 0, 4, 1 of Figure 2.11 After labeling, large = a3. (Consider the elements of the sets to be coordinates.) 65. Suppose that 6 has an inverse modulo 15; that is, suppose that there exists s such that 6s mod 15 = 1. Given a Hamiltonian cycle, do we have... Since 0.98% = 1, it contains the path a b c... d... we... There is a... 6 8 5 H K 3 E 6 8 R 9 0 3 6 8 1 8 3 9 1 3 8 A E C D G J M N P Q Figure 9.10 Evaluating the root of the game tree of Figure 9.9 using depth-first search with alpha-beta pruning. If p is true and q and r are false, find the truth value of the proposition (p ∨ q) → ¬r. Since p1 , p2 , . , it follows that a = c and b = d. Example 6.7 The graphs G1 and G2 in Figure 6.3 are not isomorphic, since G1 has seven edges and G2 has six edges and "has seven edges" is an invariant. Because vertices v and w are adjacent in G2 , it follows that an ∀x ∀Z . Reprove Exercise 11 using induction and proof by contradiction. 6. Within a mathematical system, we can derive theorems. By Theorem 2.17, every vertex has even degree. b b c 4, 4 3, 3 c 4, 3 4, 3 5, 4 4, 3 5, 4 4, 4, 4 (d, 4), 2 (d, 2) (d, 3) (d, 4) 3, d d z 4, 4 (a, 2) (d, 2) 2 Figure 2.13 After increasing the flow on path (a, d, e, f, z) by 1. To compute a29 mod z, we successively compute a mod z, a5 mod z, a13 mod z, a29 mod z. We now assign positions to the n1 items of type 1 in C(n, n1) ways. Therefore, nonisomorphic rooted trees with four vertices are shown in Figure 8.9. Binary trees are special kinds of rooted trees; thus isomorphism of binary trees must preserve the relation and must preserve the roots. If Sn denotes the elements of the sequence are a1, ..., there... the a... Since ... nq + 2t q-cent stamps. As an example of the use of the geometric sum, if we take a = 1 and r = 2 in (4.12), we obtain the formula 1 + 2 + 22 + 23 + · · · + 2n = 2n+1 − 1 2n+1 − 1. To compute a29 mod z, we successively compute a mod z, a5 mod z, a13 mod z, a29 mod z. We now assign positions to the n1 items of type 1 in C(n, n1) ways. These n − 1 possibilities account for the leading n − 1 factor in the recurrence relation. In either case, the ... mathematical... The sum is the sum... of ... (1.1) ... First, Algorithm... n − 1 ... 4.6 is correct; that is, at the termination of Algorithm 4.6, T is a minimal spanning tree. The function f is neither one-to-one nor onto. What is the probability that no two lockers are consecutive? The entries in the table give the value of 2m2 + 3n2 for the indicated values of m and n. Section 7 1. For example, in the C programming language, only the first 31 characters of identifiers are significant. 246 Algorithms Chapter Review Section 1 Section 3 1. At Wimbledon, when a player loses, she is out of the tournament. Find two integers a and b, each less than 100, that maximize the number of iterations of the while loop of Algorithm 3.3. 23. Composition of relations R2 ◦ R1 : {(x, z) | (x, y) ∈ R1 and (y, z) ∈ R2 } Section 5 144. algor2(i, j) { if (i == j) return i m = (i + j)/2 algor2(i, j), algor2(k, j)} } Exercises 59–63 refer to Algorithm 3.16. The basis steps are n = 1, . 3 2 − 1, for h = 3. If m = 2, the number of edges... Throughout this section, G denotes a network with source a, sink z, and capacity C. Thus δ(f(v)) = k), so G2 has a vertex, namely f(v), of degree k. How many truth tables are there for an n-variable function? Show how to find the input in Figure 1.11 and stop. Throughout this section, G denotes a network with source a, sink z, and capacity C. Thus δ(f(v)) = k), so G2 has a vertex, namely f(v), of degree k. If a person has made each way to the number of a... in order... 2 2... n ≥ 1... Section 1.9... b... 1 ... Is the formula for ... r ≥ 1 ... Euler cycle 5 24. We can then write f(x) = g(h(w(x))). Permutation of x1 , . (0.95)P(H) + (0.02)(1 − P(H)) Solving for P(H) gives P(H) = .0206 . c = an mod z = 14419 mod 221 = 53 12 = 60 − 24 · 2. At line 2, the condition i ≥ 1 is true. We may assume that p/q > 1. A flow in a network assigns a flow in each directed edge that does not exceed the capacity of that edge. Now x ∈ A or x ∈ B. Addition 21. Subclasses of paths are obtained by prohibiting duplicate vertices or edges or by making the vertices v0 and vn of Definition 2.1 identical. , xn is the middle value of x1 , . Thus the worstcase run time is (n2). A simple path is a path with no repeated vertices. The quotient 14 becomes the new

value of n. In the first case, if the student stays with A, the answer will be correct; but if the student switches to B, the answer will be incorrect. Justify your answers. The value of each vertex, as given by the evaluation function, is given under the vertex. (a) b5 = 35, b10 = 120 (b) (n + 1)2 − 1 (c) Yes (d) No 8. The Tower of Hanoi puzzle consists of three pegs mounted on a board and disks of various sizes with holes in their centers. A1 = 2280, A2 = 2599.20, A3 = 2963.088 3. Let P(x) denote the statement "x is sweetness and light." Symbolically, the clarified statement is ∃x ¬P(x). The check digit is computed as follows. A robot can take steps of 1 meter or 2 meters. Thus we assume s1 > A s2 > A .. To prove p → q, proof by contrapositive proves the equivalent statement ¬q → ¬p. If the input sequence is 2, 12, 6, 14 and x = 4, the algorithm returns false because no distinct pair in the sequence sums to 4. The variables A, B, C, D, and E are referred to as operands. Either of the Huffman trees gives an optimal code; that is, either will encode text having the frequencies of Table 1.2 in exactly the same (optimal) space. Since X and Y are not disjoint (both Alice and Dolph could be officers), we cannot use the Addition Principle. Show that g is a common divisor of a and b. What is the probability that the sum of the numbers on the dice is 10? The coins are labeled C1 , C2 , C3 , C4 , C5 . These 90 numbers range in value from 1 to 86. Prove that 14. (For Exercise 10) The nonzero remainders in the order they are computed by the Euclidean algorithm are 490256 mod 337 = 258 337 mod 258 = 79 258 mod 79 = 21 79 mod 21 = 16 21 mod 16 = 5 16 mod 5 = 1. The first time we arrive at the body of the while loop, the statement x = x + 1 is executed n times. 132 0 1 102 15 5 257 3 4 5 6 2 558 7 8 32 9 10 Figure 1.6 Cells in a computer memory. Proof Since S and T are solutions of (2.13), Sn = c1 Sn−1 + c2 Sn−2 , 400 Tn = c1 Tn−1 + c2 Tn−2 . An example of a lemma about real numbers is ■ If n is a positive integer, then either n − 1 is a positive integer or n − 1 = 0. Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that 5n − 1 is divisible by 4 for all n ≥ 1. (d) Replacing n by n − 1 in definition 2.1, we obtain sn−1 = 2n−1 + 4 · 3n−1 . For each graph G = (V, E) in Exercises 11–13, find V , E, all parallel edges, all loops, all isolated vertices, and tell whether G is a simple graph. List the members of the event "the sum of the numbers on the dice is even." 6. The general discrete mathematical references [Liu, 1985; and Tucker] devote several sections to the topics of this chapter. Find 3 i=1 4 ai . Write an algorithm based on breadth-first search that finds the minimum length of each path in an unweighted graph from a fixed vertex v to all other vertices. Theorem 7.9 Euler's Formula for Graphs If G is a connected, planar graph with e edges, v vertices, and f faces, then f = e − v + 2. Thus a theta notation for the number of times we execute x = x + 1 is (n). P(d) for every d ∈ D ∴ P(d) if d ∈ D ∴ ∀x P(x) 13. Is the graph model for Bacon numbers (see Example 1.6) a simple graph? {a, b, c, d} In Exercises 77–82, answer true or false. , ym }. Similarly, Tournament Sort requires at most k comparisons to 583 Trees find the third-largest, at most k comparisons to find the fourthlargest, and so on. Suppose we have an algorithm that finds the largest value among x1 , . c2 = 2, c3 = 3, c4 = 12, c5 = 13 14. Show that the number of stackings in which the cubes are stacked 1, 2, 3, and 4, reading from bottom to top, is 331,776. Example 4.6 The matching of Figure 4.1 is represented as a flow in Figure 4.3. Since the flow is maximal, the matching is maximal. How many eight-bit strings have at least one 1? At line 4, we then return sum(n − 1) + 2n = 2 + · · · + 2(n − 1) + 2n, which is the correct value. If pi appears in a and pj appears in b, include pmin(i,j) in the prime factorization of the greatest common divisor. We could then put an edge between two vertices (teams) if the two teams represented by the two vertices played at least one game. All of the bad coins also have the same weight, but they are lighter than the good coins. For example, either can be used to determine whether a graph is connected: If we can visit every vertex in a graph from an initial vertex, the graph is connected; otherwise, it is not connected (see Exercises 30 and 31). Now z ∈ / X, because if it were, we would have z ∈ Y . Since the vertices at the lowest level may not be terminal vertices, some method must be found to assign them a value. A proof of ∃x P(x) that does not exhibit an element a of the domain of discourse that makes P(a) true, but rather proves ∃x P(x) some other way (e.g., using proof by contradiction), is called a nonconstructive proof. No Euler cycle 34. Notice that when m is divided by pi , the remainder is 1: m = pi q + 1, q = p1 p2 · · · pi−1 pi+1 · · · pn . No, assuming the interpretation: It shall be unlawful for any person to keep more than three [3] dogs and more than three [3] cats upon his property within the city. A vertex cover of G is a subset V of V such that for each edge (v, w) ∈ E, either v ∈ V or w ∈ V . 6!/(3! 3!) = 20 6. 2n + 1 ≤ 2n , n = 3, 4, . (Here the height of a "missing subtree" is defined to be −1.) State whether each tree in Exercises 18–21 is balanced or not. To decrypt a message is to transform an encrypted message so that it can be read. Find all part numbers supplied by department 96. 6 12 2 8 6 10 16 1 7 20 10. Therefore, if the Skyscrapers win, I'll be quite full. If an n-bit string begins with 10 and does not contain the pattern 111, then the (n − 2)-bit string following the initial 10 cannot contain the pattern 111; therefore, there are Sn−2 strings of type (b). If a blank space is represented as 1, A as 2, B as 3, and so on, the message SEND MONEY would be represented as 20, 6, 15, 5, 1, 14, 16, 15, 6, 26. We show that if there are n contestants in a single-elimination tournament, a total of n − 1 matches are played. Thus the total number of vertices of T is 2i + 1, and the number of terminal vertices is (2i + 1) − i = i + 1. This statement is false because there is at least one x, namely x = 1, such that x > y is false for every positive integer y. Find all substrings of the string babc. Let U be a universal set. Similar computations show that the sequence continues: x3 = 5, x4 = 7, x5 = 10, x6 = 9, x7 = 2, x8 = 8, x9 = 6, x10 = 3. (This time we cannot conclude that at least three weighings are required in the worst case.) However, closer inspection shows that, in fact, three weighings are required. This last inequality may be written |f(n)| d d −d< , − < 2 |g(n)| 2 for all n ≥ N, or |f(n)| 3d d < < , 2 |g(n)| 2 for all n ≥ N, or 3d d |g(n)| < |f(n)| < |g(n)|, 2 2 Therefore, f(n) = (g(n)). Therefore x ∈ Z, and again Y ⊆ Z. If false, give a counterexample and if true, prove it. We can do much better. Is associative? Example 1.6 For the function f of Example 1.3, we may write f(1) = a, f(2) = b, f(3) = a. The same method that we use to add decimal numbers can be used to add binary numbers; however, we must replace the decimal addition table with the binary addition table + 0 1 0 1 0 1 1 10 (In decimal, 1 + 1 = 2, and 210 = 102 ; thus, in binary, 1 + 1 = 10.) Example 2.10 Binary Addition Add the binary numbers 10011011 and 1011011. The formula can be directly verified for n = 1. [Benjamin] contains an outstanding collection of combinatorial proofs. Suppose that n = dq1 + r1 0 ≤ r1 < d n = dq2 + r2 0 ≤ r2 < d. Suppose that G has two components [see Figure 2.9(c)]. Thus the entry in row i, column j in A2 , which counts the number of edges incident on f(vi) and f(vj), is also equal to k. If they find a pair di , dj for which P(di , dj) is false, the condition ¬P(di , dj) in the if statement is true; so the code returns false [to indicate that ∀x∀y P(x, y) is false] and terminates. It follows that 1, 2, . Example 1.5 A rooted tree is often used to specify hierarchical relationships. Give the truth table for the disjunction of p and q. The last n − 2 positions can be filled in En−2 ways since any rise/fall permutation of the remaining n − 2 numbers gives a rise/fall permutation of 1, . 387 Recurrence Relations 2 1 3 Figure 1.3 After recursively moving the top n − 1 disks from peg 1 to peg 2 in the Tower of Hanoi. 1. w2 v4 v3 w8 17. Presumably, we can just use the definition of CX CX∪Y (x) = 1. At line 4, the function correctly computes the value (n − 1)! · n = n!. Basis Step (n = 1) We need 0 multiplications to compute a1 . Suppose that we have n dollars. Show that in a rise/fall permutation of 1, 2, . Since there are 2n subsets of {1, 2, . Let us examine the situation more closely. Define inverse relation and give an example of an inverse relation. Since x ∈ A, x ∈ A ∪ B by the definition of union. We next define the floor and ceiling of a real number. i=3 13. Therefore, we are correct in reasoning that we must eliminate two edges from the graph of Figure 3.5 to produce a Hamiltonian cycle. Since c | a and c | bq, by Theorem 1.3(b), c | a − bq (= r). The sender is said to encrypt the message, and the recipient is said to decrypt the message. (b) Each solution of (3.1) satisfying the given conditions is equivalent to selecting 29 items, xi of type i, i = 1, 2, 3, 4, where, in addition, we must have at least one item of type 1, at least two items of type 2, and at least three items of type 3. Example 2.7 Let G be the graph whose vertex set consists of the 50 states of the United States. We next show that Algorithm 4.1 is correct. Canceling Cn , we obtain (n + 2)k = (4n + 2). Yes, (a, b, d, g, m, l, h, i, j, e, f, k, c) 31. Attacking the Problem First, let's be clear what we must do. Initially the pointers reference the first elements in the sequences. The alpha value of a vertex is dependent on the current state of the search and changes as the search progresses. We show that the distinct dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers. Unlike Theorems 2.17 and 2.18, no easily verified necessary and sufficient condition is known for a graph to have a Hamiltonian cycle. Formal Solution Part (a): Suppose that n persons all have the wrong orders. In each case the problem is to find a maximal flow. , sj , arranged in nondecreasing order merge sort(s, i, j) { // base case: i == j if (i == j) return // divide sequence and sort m = (i + j)/2 merge sort(s, i, m) merge sort(s, m + 1, j) Recurrence Relations 9. Find the center(s) of the tree of Figure 1.5. † 36. Part (b): Making the given substitution, we obtain Cn = −Cn−1 . 2 n−1 k=0 In Exercises 34 and 35, let Sn denote the number of routes from the lower-left corner of an n×n grid to the upper-right corner in which we are restricted to traveling to the right, upward, or diagonally northeast [i.e., from (i, j) to (i + 1, j + 1)] and in which we are allowed to touch but not go above a diagonal line from the lowerleft corner to the upper-right corner. (c) Is b increasing? We first count the number of selections in which Alice is an officer. for all y ≥ 0. Inductive Step Suppose that 6 divides 7n − 1. The least common multiple, written lcm(m, n), is the smallest positive common multiple of m and n. Good references on game trees are [Nievergelt; Nilsson; and Slagle]. Write the truth table of the proposition ¬(p ∧ q)∨(p∨¬r). In algebra, operator precedence tells us to evaluate · and / before + and −. We have found at least one value for x and at least one value for y that make P(x, y) true. But we already solved this problem in part (a)! The number of routes from the lower-left corner to the upper-right corner of a 2 × 1 grid is equal to C(2 + 1, 1) = 3. What is the relationship between the sum of the degrees of the vertices in a graph and the number of edges in the graph? If we reach an empty cell or return to our original position, we conclude that n is not present; otherwise, we obtain the position of n. An example is s = 01111000. If a and b are real numbers and n is a positive integer, then (a + b)n = n = C(n, 0)an+1 b0 + k=1 C(n, k)an−k bk . However, if it is not raining (i.e., p is false) or it is not cold (i.e., q is false) or both, then we would consider the proposition, p ∧ q: It is raining and it is cold, to be false. Each node has communications facilities and local storage. In the hint to Exercise 17, we showed that if C = (v0 , v1 , . Therefore, the for loop runs to completion, and the algorithm correctly returns 0 to indicate that p was not found in t. Calculus students will recognize this exercise as a discrete version of the calculus theorem: If f is a continuous function on [a, b] and f(a) > 0 and f(b) < 0, then f(c) = 0 for some c in (a, b). k Also, C(n, k) = n(n − 1) · · · (n − k + 1) nn · · · n nk ≤ = . Write the decimal number 430 in binary and hexadecimal. 1 12 0 20 1 12 0 1 1 0 1 8 12 5 0 0 7 1 0 8 12 1 0 2 3 Figure 1.13 Constructing an optimal Huffman code. , a10 denote the ages. Algorithm to construct a binary search tree Section 6 33. [If (c), then (d).] Suppose that T is connected and has n − 1 edges. Since G is cycle-free, every time we trace an edge, we arrive at a new vertex. If the coin is flipped 10 times, what is the probability of "approximately" five heads, that is, exactly four or five or six heads? Let G1 and G2 be simple graphs. If x ≤ d, then from x ≤ d and d ≤ d1 , we may deduce x ≤ d1 . Under basement water, which is highlighted, are the files 11-18-03, DSC01007 11-18-03, and others, which appear on the right of Figure 1.7. Example 1.7 Hierarchical Definition Trees Figure 1.9 is an example of a hierarchical definition tree. How is this Cartesian product denoted? a g (a) Find a self-complementary graph having five vertices. Let p: The Council approves the funds. 12,340 In Exercises 20–25, add the binary numbers. S1 = 1(2) 2 2(3) S2 = 2 .. A vertex will represent a city at a particular time (see Figure 1.5). (a) At line 2, since 4 = 0, we proceed to line 4. (a) Show that R is an equivalence relation on S. If, at any point, we cannot add an edge incident on a vertex w, we backtrack to the parent p of w and try to add an edge incident on p. Therefore R is symmetric. Find i=1 10 i . and the Inductive Step is complete. Write an algorithm that evaluates vertices of a game tree to level n using depth-first search. ¬(a ∨ d) ∴ ¬b We use (3.4) to replace hypothesis 1 with the two hypotheses a ∨ ¬b a ∨ c. for all n ≥ N. pn = n/(S + T) 36. 67. Shirley will either take the bus or catch a ride to school. Since i is labeled, we must have Fij = Cij ; otherwise, we would have labeled j at lines 19 and 20. Solving gives an = n! bn = (n!/4)[5 − (−3)n]. Then Gn is a Gray code for every positive integer n. Use resolution and proof by contradiction to reprove Example 3.6. 1. In Algorithm 3.10, backtracking is used to search for solutions to the 4-queens problem. The idea of the backtracking algorithm is to scan the grid (we chose to scan top to bottom, left to right), skipping positions where numbers were preassigned, and, at the next available position, we try 1, then 2, then 3, and so on, until we find a legal value (i.e., a value that does not conflict within its 3 × 3 subsquare, within its row, or within its column). In 1997 the IBM chess program, Deep Blue, defeated Garry Kasparov, who had been world champion since 1985, in a six-game match. Show that if a graph G is partitioned into connected subgraphs so that each edge and each vertex in G belong to one of the subgraphs, the subgraphs are components. The set X ∩ Y = {x | x ∈ X and x ∈ Y } is called the intersection of X and Y . C(90, 4)/C(100, 4) 23. Is the statement Section 3 9. Because of the symmetry of the graph, if v and w are vertices in K5 , there is the same number of paths of length n from v to v as there is from w to w. In the Maryland Cash In Hand game, the contestant chooses seven distinct numbers among the numbers 1 through 31. Proof Let P (P) be the set of labeled (unlabeled) vertices of G at the termination of Algorithm 2.4. Consider an edge (i, j), where i ∈ P, j ∈ P. [(c) → (a)] We prove the contrapositive: If n is even, then n2 + 1 is odd. Using a search engine on the web is another example of a searching problem. n = 2, φ = 3 34. At line 17 we examine edges (d, c) and (d, e). (a) (b) (c) (d) Mary will be a good student if she studies hard. Similarly, Y ⊆ X. t1 = t2 = t3 = 1, tn = tn−1 + tn−2 + tn−3 for all n ≥ 4. However, when d becomes 7, n mod d = 637 mod 7 equals 0, so the algorithm returns d = 7 to indicate that n = 637 is composite and 7 is a divisor of 637. By letting Y = range of f , we obtain a bijection from X to Y . Since 77 > 64, it is immediately inserted to 64's right 64 2 51 i j 5 2 Since 15 < 64, 64 must move one position to the right 64 5 8. Use breadth-first search (Algorithm 3.6) with the vertex ordering chbgadfe to find a spanning tree for graph G of Figure 3.1. 4. By the Multiplication Principle, there are 3 · 5 · 4 = 60 possibilities. 54. Sum the values obtained. v = a, w = b v3 e2 e3 v3 444 v1 e1 c d v3 e6 v5 30. 519 Trees Problem-Solving Corner Trees Problem Let T be a simple graph. Traditionally, number theory was a pure branch of mathematics—known for its abstract nature rather than its applications. , xn 2. On the other hand, ordered pairs and n-tuples take order into account. Summing the 353 Counting Methods and the Pigeonhole Principle 1, 1 1, 2 1, 3 1, 4 1, 5 1, 6 2, 1 2, 2 2, 3 2, 4 2, 5 2, 6 3, 1 3, 2 3, 3 3, 4 3, 5 3, 6 4, 1 4, 2 4, 3 4, 4 4, 5 4, 6 5, 1 5, 2 5, 3 5, 4 5, 5 5, 6 6, 1 6, 2 6, 3 6, 4 6, 5 6, 6 Figure 6.2 Rolling two fair dice. Implement Algorithms 3.1 (selection sort) and 3.8 (merge sort) and other sorting algorithms as programs and compare the times needed to sort n items. We next give an algorithm to compute s and t satisfying gcd(a, b) = sa + tb, where a and b are nonnegative integers not both zero. We consider the question of when an n-cube can simulate a ring model with 2n processors. Example 9.2 Evaluate the root of the tree of Figure 9.9 using depth-first search with alpha-beta pruning. Basis Steps (n = 6; 7) f6 = 8 > 7.59 = (3/2)5 . How many edges are there? Similarly, to denote the positive numbers that belong to one of the three sets, we use the superscript plus. Show that if n is a power of 2, bn = 4n − 2 lg n − 4. Find a formula for the sequence s defined by n ai . 11. We can then appeal to (5) to conclude that T is a tree. † Exercise 28. Suppose that a particular wine has feature set F and P(R | F) = 0.2, P(T | F) = 0.5, P(S | F) = 0.3. Since class table wine has the greatest probability, this wine would be classified as table wine. Can you find another isomorphism of the rooted trees of Figure 8.7? Example 8.6 The rooted trees T1 and T2 of Figure 8.8 are not isomorphic since the root of T1 has degree 3 but the root of T2 has degree 2. , sj , is divided into two nearly equal sequences, si , . It does however prove indirectly using proof by contradiction that there is such an i. By using (1.2) and (1.3), we can compute any term in the sequence just as we did using instructions 1 and 2. Since there are now eight outcomes rather than 10, we can conclude that any algorithm to solve the four-coins puzzle requires at least two weighings in the worst case. w6 w1 v2 v7 w5 In Exercises 10–12, determine whether each pair of binary trees is isomorphic. Then x R a and x R b. 33. If we can tile an n × n deficient board, where n is not necessarily a power of 2, then the number of squares, n2 − 1, must be divisible by 3. Define null string. Then s1 and s3 have the same number of zeros. The vertices represent offices. return 1000 4. Furthermore, the graph is either not connected or contains a cycle. i=0 The idea is that averaging smooths noisy data. (1.4) To apply this recurrence relation for n = 1, we need to know the value of A0 . Modify the usual method of multiplying base 10 integers for base 2 to produce an algorithm to multiply binary numbers bm bm−1 · · · b1 b0 and bn bn−1 · · · b1 b0 . 604 Network Models We then return to the top of the while loop (line 11). But if x ∈ B, x ∈ A ∪ B by the definition of union. "Resolution is refutation complete" means that resolution will be able to derive a contradiction from a set of inconsistent clauses. f −1 = {(1, a), (3, b), (0, c)}. Since f is onto, there exists y ∈ Y such that f(y) = z. In words, for every real number x and for every real number y, if x and y are positive, their sum is positive. In case (d), we decrease the flow in both edges by . Thus, the given statement may be written symbolically as ∀x∃y L(x, y). How many are taking music or French (or both) but not business? ¬(p ∨ q) ↔ r 54. Among the solutions are 1, 2, 3, 4, 5 and 3, 4, 1, 2, 5. We would have to eliminate two edges incident at c and one edge incident at each of a, b, d, and e. Use complete sentences, which may include symbols. Give examples of searching problems. Also, find the equivalence classes [1], [2], [3], and [4]. , or n; so there are n−1 possible ways for person 1 to get the wrong order. Assume that a person invests $4000 at 17 percent interest compounded annually. Therefore there are 2n reflexive relations on an n-element set. Actually, it suffices to check only √ 2, 3, . How do we denote X is a subset of Y ? The inductive step is complete. Thus the permutation following the given permutation is not of the form 163−−− . large = a. Exercise 9 is to give an algorithm that sorts four items using five comparisons in the worst case. Find Answer 17–24 for the sequence v defined by 17. P = (p → q) → r, Q = p → (q → r) 76. 433 Graph Theory 1 2 3 4 5 6 7 8 434 Introduction Paths and Cycles Problem-Solving Corner: Graphs Hamiltonian Cycles and the Traveling Salesperson Problem A Shortest-Path Algorithm Representations of Graphs Isomorphisms of Graphs Planar Graphs Instant Insanity Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises Well, I got on the road, and I went north to Providence. 3 − 2 = 1 18. 437 Graph Theory press between the corresponding holes. 13. Let X = {San Francisco, Pittsburgh, Chicago, San Diego, Philadelphia, Los Angeles}. Assume that the algorithm computes the maximum for input of size n − 1, and suppose that the algorithm receives input of size n. If a and b are nonnegative integers, not both zero, there exist integers s and t such that gcd(a, b) = sa + tb. 91 Proofs A prime number of the form 2p − 1, where p is prime, is called a Mersenne prime [named for Marin Mersenne (1588–1648)]. A simple cycle is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices. Show that disjunctive syllogism (see Figure 4.1) is valid. By Theorem 5.14, part (b), the negation ¬(∃x ¬P(x)) of the preceding proposition is equivalent to ∀x ¬¬P(x) or, equivalently, ∀x P(x). , positions. {(x, y) | x divides 2 − y} In Exercises 9–14, determine whether the given relation is an equivalence relation on the set of all people. Since edge (e, b) is improperly oriented, its flow is decreased by 1. Suppose that six distinct integers are selected from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. By the Pigeonhole Principle, some name is assigned to at least two persons. .it's a prime number. Example 6.2 Write the assertion Everybody loves somebody, symbolically, letting L(x, y) be the statement "x loves y." "Everybody" requires universal quantification and "somebody" requires existential quantification. Show how the algorithm constructs the 4-cube from the 3-cube. Then successively use the recurrence relation to replace each of the resulting terms by certain of their predecessors. We must show that x ∈ A. Let S = δ(x1) + δ(x2) + · · · + δ(xm), T = δ(y1) + δ(y2) + · · · + δ(yn). An Euler cycle consists of the single vertex and no edges. Find all connected subgraphs of the following graph containing all of the vertices of the original graph and having as few edges as possible. gcd recurs(a, b) { make a largest if (a < b) swap(a, b) return gcd recurs1(a, b) } 7. These exercises are due to Martin Gilchrist. How many functions are there from {1, 2} to {a, b}? If we begin at the root, answer each question, and follow the appropriate edge, we will eventually arrive at a terminal vertex that chooses a restaurant. dn = 24. The road inspector's problem can be rephrased for the graph model G in the following way: Is there a path from vertex Gre to vertex Gre that traverses every edge exactly once? We prove the formula by using mathematical induction. Assuming that cm,k = m + k and b1 = 0, show that bn ≤ where d is a positive real number and m is an integer satisfying m > 1. The player who removes the last token loses. Example 2.1 Consider the path from a to z in Figure 2.2. All the edges in P are properly oriented. Next, we label the vertices b and d adjacent to a. Basis Step (n = 1) Since the sequence 0, 1 is a Gray code, the theorem is true when n is 1. We show that X is not a subset of Z. ■ It is sometimes possible to convert a recurrence relation that is not a linear homogeneous equation with constant coefficients into a linear homogeneous equation with constant coefficients. The digraph of an antisymmetric relation has the property that between any two distinct vertices there is at most one directed edge. We call such a route a good route. If I study hard, then I get A's. Edges of infinite capacity connect A, t1 , to A, t2 , and B, t1 to B, t2 to indicate that any number of cars can wait at city A or city B. Then there exists an integer k1 such that n2 = 2k1 . 1 1 4 2 R 2 B 15. We next prove equation (3.1). (1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6) 8. The eccentricity of a vertex v in a tree T is the maximum length of a simple path that begins at v. Many algorithms currently in use are not general, deterministic, or even finite. Show that T ∈X1 XT = X2 . 110110 + 101101 23. If x ∈ X, from the definition of "intersection," we conclude that x ∈ X ∩ Y . Show that an = (lg n). A planar map is a planar graph where the faces are interpreted as countries, the edges are interpreted as borders between countries, and the vertices represent the intersections of borders. There are no isolated vertices. Show that T(n) = O(4 n). n = 9 2. Our functions also satisfy f0 (n) = n and g0 (n) = n2 for infinitely many n [obviously different n than those for which f0 (n) = n2 and g0 (n) = n]. Sets and Logic For statement (c), ∃x∃y P(x, y), to be true, for some x and for some y, P(x, y) must be true. By a tiling of a figure by trominoes, we mean an exact covering of the figure by trominoes without having any of the trominoes overlap 107 Proofs 2 k1 2 k 2k 2 k1 T 2 k 2k Figure 4.5 Tiling a 4 × 4 deficient board with trominoes. Find a formula for zi . (For Exercise 13) 25 28. A sequence is a function in which the domain consists of a set of consecutive integers. Notes Virtually any reference on discrete mathematics contains one or more chapters on graph theory. Thus the board contains seven colorful pairs, one in each column. Example 1.13 International Standard Book Numbers As of January 2007, an International Standard Book Number (ISBN) is a code of 13 characters separated by dashes, such as 978-1-59448-950-1. How many times is the print statement executed? Suppose that n is composite. Since 1 ∈ A, but 1 ∈ B, A = B. Section Review Exercises 1. Since (x, y), (y, x) ∈ R, by transitivity we have (x, x) ∈ R. Let m and n be integers satisfying 1 ≤ m ≤ 2n . Implement Algorithm 2.1, text search, as a program. In Exercises 21–24, show, as in Examples 1.2 and 1.3, that A = B. Example 6.10 Two fair dice are rolled. If the statement is existentially quantified (i.e., there exists x . If R is transitive, then R−1 is transitive. At termination, L(z) is the length of a shortest path from a to z. , b − 1. To prove that R ◦ S is transitive, we must show that if (x, y), (y, z) ∈ R ◦ S, then (x, z) ∈ R ◦ S. Since n + 2 ≥ 6, n+1 3 < fn+2 . Show that there are 24 orientations of a cube. The {1} is a vertex cover of G of size 1. Each equivalence class consists of a set of mutually isomorphic graphs. 9an = 6an−1 − an−2 + 5n2 53. a0 = C0 , a1 = C1 . Case 1 is x ≥ 0 and case 2 is x < 0. b a 3 5 4 c 2 7 5 e 7 h 4 5 2 f i 2 1. We can use Theorem 3.5 to solve this problem if we consider the problem to be that of labeling each book with the name of the student who receives it. What is a lemma? Thus we obtain the recurrence relation dn − dn−1 = 2 (dn−1 − dn−2), which may be rewritten dn = 3dn−1 − 2dn−2 . Since a and b are also vertices in T and T is a tree, there is a path P from a to b. Since four outcomes are possible after identifying two good coins, no algorithm that begins by comparing one coin against one coin can solve the problem in two weighings or less in the worst case. X − Y, difference of X and Y , relative complement: set of elements in X but not in Y 19. Formula for the sum of the first n positive integers: n(n + 1) 1 + 2 + · · · + n = 2 32. 10/C(12, 3) 41. If x ∈ X ∩ Y, CX ∩ Y (x) = 1 = 1 · 1 = CX (x)CY (x). Comment The technical name of a permutation in which no element is in its original position is a derangement. The range of f is the set of all nonnegative real numbers. Combining these equations was crucial to proving the base cases. Also, 2n + 3 lg n ≥ 2n for all n ≥ 1. A shortest path is given by (a, b, c, z). Line 1 is not precisely stated since how to list the subsets of S and their sums is not specified; thus the algorithm lacks the precision property. a b c e d 27. x + y = x + y 79. For the Inductive Step, assume that the greedy algorithm is optimal for all amounts of postage n ≤ 6i. Prove that ta is the least upper bound of the set {tx | x ∈ X}. A free tree T is a simple graph satisfying the following: If v and w are vertices in T , there is a unique simple path from v to w. Consider, for example, the following argument: All mathematicians wear sandals. Example 2.9 The sequence 100, 90, 90, 74, 74, 74, 30 is nonincreasing, but it is not decreasing. The resulting graph will have nine edges; since K3,3 has nine edges, this approach looks promising. Let's check that the recurrence relation also holds for n = 4: D4 = 9 = 3(2 + 1) = (4 − 1)(D3 + D2). un = u3n+1 un/2 n ≥ 1, where f1 , f2 , . Discussion We let x be an arbitrary real number and d be an arbitrary positive real number. For example, if we substitute "Yugo Inn" for "restaurant," statement (d) is Yugo Inn rated over two stars in Chicago magazine, which is false. Therefore, for all a, b, c, min{min{a, b}, c} = min{a, min{b, c}}. Then x ∈ A∪B and x ∈ / A∩B. Define a relation R on S by (x, y) R p(x , y) if 44. Since n > 2, the algorithm executes the return statement return walk(n − 1) + walk(n − 2) By the inductive assumption, the values of walk(n − 1) and walk(n − 2) are correctly computed by the algorithm. , s(i+j)/2 and s(i+j)/2+1 , . 22 + 1 5 It is not the case that every value of x results in a true proposition. What is an invalid argument? P(x) = 1/n, where n is the size of the sample space. 185 Functions, Sequences, and Relations Comments In programming languages, usually only some specified number of characters of the names of variables and special terms (technically these are called identifiers) are significant. In Example 2.6, we saw that the remainders when n is successively divided by 2 give the binary expansion of n. In how many ways can this be done if the balls are considered distinct? , Cn , each pair of these classes is mutually exclusive, and each item to be classified belongs to one of these classes, for a feature set F we have P(F | Cj) . Write a recursive algorithm to find the maximum of a finite sequence of numbers. [The statement could also be written ∃x Q(x), where Q(x) is the propositional function "x cannot fly." As in algebra, there are many ways to represent text symbolically.] The domain of discourse is the set of birds. We call the eight squares across the top and bottom outside squares, and we call the other eight squares inside squares. To represent three sets, we use three overlapping circles (see Figure 1.6). c x3 x1 d x2 x4 b ⎛a b c d e f⎞ 16. Example 3.10 In Figure 3.2, the value of the flow and the capacity of the cut are both 6; therefore, the flow is maximal and the cut is minimal. The netid is given by the next 21 bits, which identifies the network. The greatest common divisor of m and n, not both zero, is the largest common divisor of m and n. c d 5. 9 25. The sum of the degrees of the vertices in a graph equals twice the number of edges in the graph. Definition 3.1 If p and q are propositions, the proposition if p then q (3.2) is called a conditional proposition and is denoted p → q. a 0⎜0⎟ ⎜1⎟0⎟ ⎝0 1 0 2 0 1 2 0 1 0 0 0 0 1 0 1 0 0 1 0 0 2 0 1 0 0 1 0⎟ ⎜1⎟ ⎜0⎟0⎠ 0 a b c d e f ⎛a b c d e f⎞ 4 1 0 1 1 1 2 0⎟1⎟ ⎜1⎟1⎟ ⎝0 1 1 0 1 1 3 1 1 1 0 1 1 0 1 1 1 0 1 2 0⎟ ⎜3⎟ ⎜1⎟1⎠ 0 17. 1 2 3 X Example 1.2 The domain and codomain of the function defined by (1.1) are both equal to the set of all nonnegative real numbers. Write a program to display the cobweb of economics. (b) In how many ways can this be done if either Alice or Ben must be chairperson? The computation shows the successive divisions by 2 with the remainders recorded at the right. How many different dinners consist of one main course and one beverage? 375 Counting Methods and the Pigeonhole Principle Section 4 1. e1 and e2 are parallel edges. Therefore, both x and z belong to S and x R z. 2 14 3 6 5 7 10 8 12 6 4 9 14. Rational numbers are quotients of integers, thus Q for quotient. Enclose the missing square in a corner (n − 6) × (n − 6) subboard. Ceiling of x, x : least integer greater than or equal to x 8. By inspection, a1 = 1 and a2 = 2. Binary tree Left child in a binary tree Right child in a binary tree Full binary tree If T is full binary tree with i internal vertices, then T has i + 1 terminal vertices and 2i + 1 total vertices. Run shuffle (Algorithm 2.4) many times for the same input sequence. The graph of a function f whose domain and codomain are subsets of the real numbers is obtained by plotting points in the plane that correspond to the elements in f . The root is called Desktop. Write a program that tests whether A is nonincreasing. Thus n − d(q + 1) is in X. 11000 23. , sn be a sequence† satisfying (a) s1 is a positive integer and sn is a negative integer, (b) for all i, 1 ≤ i < n, si+1 = si + 1 or si+1 = si − 1. Addition Principle Suppose that X1 , . Since R1 is antisymmetric, x1 = x1 . Then S(n) is true for every integer n ≥ n0 . Explain how a linear congruential random number generator works, and give an example of a linear congruential random number generator. Solving a recurrence relation by iteration 8. We ultimately label z (see Figure 2.11), and at line 36 we find the path P = (a, b, e, f, z). Conclusion: There is not gas in the car, or the car transmission is defective. We turn the method of Example 2.6 into an algorithm. x=1 24. Hypotheses: Everyone in the discrete mathematics class loves proofs. As one constructs the proof, it is essential to keep the goal in mind: x ∈ (X ∩ Y) − (X ∩ Z). Prove C(k, m)Hk = C(n + 1, m + 1) Hn+1 − k=m n(1 + x)n−1 = n C(n, k)xk−1 . Now consider orderings of four B's, two S's, and two M's. Thus s and t satisfy g = s b + t r, where g = gcd(b, r). {un } is not a recurrence relation because, if n is odd and greater than 1, un is defined in terms of the successor u3n+1 . Z × Z 70. Pigeonhole Principle (Third Form) Let f be a function from a finite set X into a finite set Y . If the number of cases to prove is finite and not too large, the cases can be directly checked one by one. (Because there are two goats, the host can open a door that hides a goat no matter which door the contestant first chooses.) The host then gives the contestant the option of abandoning the chosen door in favor of the still-closed, unchosen door. Case (c) is similar to case (b), except that we increase the flow in e1 by and decrease the flow in e2 by . Then d = min{d1 , d2 } = 2. Suppose, in particular, that (b, b) ∈ R. sort(a, b, c, x, y, z) { x=a y=b z=c if (y < x) swap(x, y) if (z < x) swap(x, z) if (z < y) swap(y, z) } 3. 358 10. What factors in addition to the amount of memory used should be considered when choosing a code, such as ASCII or a Huffman code, to represent characters in a computer? Similarly, from x ≤ d and d ≤ d2 , we may deduce x ≤ d2 . A student chooses A. , sj , i ≥ 1, sorted in nondecreasing order, a value key, i, and j Output: The output is an index k for which sk = key, or if key is not in the sequence, the output is the value 0. Suppose that S(n0) is true; and for all n > n0 , if S(k) is true for all k, n0 ≤ k < n, then S(n) is true. Draw a precedence graph for each computer program in Exercises 50–52. It is denoted X1 × X2 × · · · × Xn . Therefore ∅ = U. We can depict the situation as shown in Figure 1.1, where an arrow from j to x means that we assign the letter x to the integer j. The equivalence class [1] containing 1 consists of all x such that (x, 1) ∈ R. Choose x1 ∈ X. Since 60n2 + 5n + 1 ≥ 60n2 for all n ≥ 1, we may take C2 = 60 in Definition 3.2 to obtain 60n2 + 5n + 1 = (n2). Formally, f is a subset of X × Y having the property that for every x ∈ X, there is a unique y ∈ Y such that (x, y) ∈ X × Y. 11 9 13 3 6 1 9 7 16 6 4 4 8 16 9 14 6 7 8 10 8 16 12 13 20 24. If x < 0, then −d < 0 < −x = |x| < d; that is, −d < −x < d. What is the probability that the wheel lands on 0? (Each bi is 0 or 1.) 6. The value obtained is the same as if we had evaluated all the vertices. For example, the binary number 101101 can be written 1011012 . See if you can decide before reading on. Find the probability that a patient has the HIV virus if the ELISA test is positive. Example 5.15 Let P(x) be the statement x2 1 > 1. 16 17. We would then multiply a1 a2 by a3 a4 to obtain (a1 a2)(a3 a4). b; d 7. P(E1) + P(E2) equals P(x) for all x ∈ E1 plus P(x) for all x ∈ E2 , which is equal to P(E1) + P(E2), except that P(x), for x ∈ E1 ∩ E2 , is counted twice. Prove that if T has n internal vertices, then E = I + 2n. Example 3.9 Using an argument similar to that in Example 3.7, we show that lg n! = (n lg n). 143 Functions, Sequences, and Relations If the range of a function f is equal to its codomain Y , the function is said to be onto Y . If the property is an invariant, prove that it is; otherwise, give a counterexample. We call the values for which a recursive function does not invoke itself the base cases. Since P(H) = 1/2 = P(T), we have 1 1 1 P(H ∩ T) = = = P(H)P(T). , i − 1, is or ln 2 . Searching A large amount of computer time is devoted to searching. First, s1 , . The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v. Thus for any x, P(x, y) is true for any particular y. replacing the index i by k, where i = k + 1. Therefore, Adding these inequalities yields s1 + s2 + · · · + si + · · · + sn > nA since si > A. Then set s = s mod φ. is connected and has n − 1 edges. False a 4 b 2 1 5 d 6 c 16. Whales Pilots 13. In the second proof-by-contradiction technique shown in Section 3, we again assume that the graph with n vertices has a Hamiltonian cycle. n+1 n = (n + 1)[lg(n + 1) − lg n]. Each computer is compatible with m > 0 disk drives and each disk drive is compatible with m computers. Show that b1 = 0 and b2 = 2. 497 Graph Theory 5. Adleman, that is believed to be secure. 1 k Therefore, 1 ,b1) k ,bk) gcd(m, n) · lcm(m, n) = [pmin(a · · · pmin(a]· 1 k 1 ,b1) k ,bk) · · · pmax(a] [pmax(a 1 k 1 ,b1)+max(a1 ,b1) k ,bk)+max(ak ,bk) = pmin(a · · · pmin(a 1 k = pa11 +b1 · · · pakk +bk = [pa11 · · · pakk][pb11 · · · pbkk] = mn. 0= In Exercises 3–9, find the coefficient of the term when the expression is expanded. Later in this section we explain carefully why it cannot be done. We may let V1 be the set containing the one vertex and V2 be the empty set. Find the probability of obtaining at most one defective microprocessor. T is connected and acyclic. The prefix form of (6.1), obtained by applying preorder traversal to the tree of Figure 6.7, is − ∗ +ABC/DE. a b e v5 f e3 v6 e4 v3 e5 v2 e2 v2 e2 e3 v5 v3 e4 e5 a b c e d g f h i v4 e6 e7 v7 e9 10. For example, the worst-case time for the algorithm given by the decision tree of Figure 7.7 is equal to 3. Show all solutions to the five-queens problem in which one queen is in the first column, second row. The feature is "tests positive," which we denote Pos. (1.4) Combining (1.3) and (1.4), we have f −1 (y) = x = log2 y. What is the probability that the student answers every question incorrectly? Definition 1.3 Let G be a transport network. i! j! (n − i − j)! 0≤i+j≤n 32. Therefore f is one-to-one. Thus we may compute the maximum consecutive sum that ends at index i by executing 238 if (sum + si > 0) sum = sum + si else sum = 0 Formal Solution Input: Output: s1 , . Tile the two 6 × (n − 7) subboards using the result of Exercise 34, Section 4. If the trees are isomorphic, give an isomorphism. Proof of Theorem 3.7 Given a > b ≥ 0, let r0 = a, r1 = b, and ri equal the value of r after the (i − 1)st time the while loop is executed in Algorithm 3.3 (e.g., r2 = a mod b). Exercises Formulate the arguments of Exercises 1–5 symbolically and determine whether each is valid. Section Review Exercises † 1. If edge e is associated with vertices v and w, v and w are said to be incident on e. Prove that the product of two consecutive integers is even. ≤ 2n for all n ≥ 1, so t(n) = O(n). (c) How many equivalence classes are there? Therefore, T is acyclic. Compute b2 , b3 , and b4 . The adjacency matrix of the simple graph of Figure 5.2 is a b A= c d e b c d e Figure 5.2 The graph for Example 5.2. ⎛a b c d e ⎞ 0 1 0 0 1 ⎜1 0 1 0 1⎟ ⎜ ⎟0 1 2 0 1⎟. Because the amount of potential data is usually so much larger than the available memory, hash functions are usually not one-to-one (see Example 1.14). The indegree of v, in(v), is the number of edges of the form (w, v). Prove that at least two of the six have a sum equal to 11. If xy is rational, the proof is complete. 40 40 30 30 35 40 1 12 40 3 9 18. Theorem 8.12 Figure 8.13 The two nonisomorphic binary trees with two vertices. If the item is not found, because the sequence is sorted, an additional comparison (line 7) will locate the half of the sequence in which the item appears if it is present. We first proved the base cases (k = 4, 5). The worst-case time is (log n). Does the graph of Figure 3.7 have a Hamiltonian path? 534 Definition 5.1 A binary tree is a rooted tree in which each vertex has either no children, one child, or two children. Eighteen persons have first names Alfie, Ben, and Cissi and last names Dumont and Elm. By the first form of the Pigeonhole Principle, some diagonal contains two marked squares. |x − xi |, i=1 |x − xi | > n−1 |x − xi | + (xn − x1), i=2 for all n ≥ 3. 1 → Introduction Figure 1.1 shows the results of the semifinals and finals of a classic tennis competition at Wimbledon, which featured four of the greatest players in the history of tennis. q : It is raining, Exercises 74 and 75 provide further motivation for defining p → q to be true when p is false. 214 (216 − 2) 77. 5 · 4 · 3 41. Functions to compute both sides of the equation for all x ∈ U and verify that the two sides are equal. Suppose that the worst-case time of an algorithm is (n). How many times are the print statements executed? 2 Using induction or otherwise, prove that 42. Example 2.2 Figure 2.1 shows a trace of Algorithm 2.1 where we are searching for the pattern "001" in the text "010001". Given an arrow diagram, the relation is not reflexive if some vertex has no loop. The relation {(1, 1)} is not a reflexive relation on {1, 2}. P = {a, d} for Exercise 1, Section 1 2. A − B 43. By the definition of set difference, since x ∈ Y − Z, x ∈ Y and x∈ / Z. Give an example of a graph with six vertices that has exactly two articulation points. What is the maximum number of communication links that can be broken with communication among all offices still possible? Given an n-element set X, C(n, k) counts the number of k-element subsets. Write a program that finds an Euler cycle in a connected graph in which all vertices have even degree. This function is not onto Y because there is no arrow pointing to d. Let p : 4 < 6 and q : 9 > 12. Since b is a least upper bound for X and a is an upper bound for X, b ≤ a. 35. a b c d e 17. R = {(1, δ), (2, α), (2,), (3, β), (3,)}; ordering of X: 1, 2, 3; ordering of Y : α, β, , δ 2. There are n ways to choose the first item, n − 1 ways to choose the second item, . By the Multiplication Principle, the number of orderings is C(n, n1)C(n − n1 , n2)C(n − n1 − n2 , n3) · · · C(n − n1 − · · · − nt−1 , nt) (n − n1)! (n − n1 − · · · − nt−1)! n! ··· = n1 ! (n − n1)! n2 ! (n − n1 − n2)! nt ! 0! n! = . Problem-Solving Tips The key to solving problems involving functions is clearly understanding the definition of function. It then follows that at termination, Tn−1 is a minimal spanning tree. In Exercises 33–39, show that gcd(n, φ) = 1, and find the inverse s of n modulo φ satisfying 0 < s < φ. We note that the length of a shortest path from v to a vertex on level i of the spanning tree is i. In the RSA public-key cryptosystem, how does one decrypt c? A face is characterized by the cycle that forms its boundary. Thus dn is given by dn = 180 + 20 · 2n . Assign probabilities to the outcomes that accurately model the likelihood of the various numbers to appear. If R1 is a relation from X to Y and R2 is a relation from Y to Z, the composition of R1 and R2 , denoted R2 ◦ R1 , is the relation from X to Z defined by R2 ◦ R1 = {(x, z) | (x, y) ∈ R1 and (y, z) ∈ R2 for some y ∈ Y }. Let ai denote the position of the ith available item. What is the ring model for parallel computation? If there is no maximum-sum subsequence (which would happen, for example, if all of the values of the sequence were negative), the algorithm should set the first and last indexes to zero. Pseudorandom numbers are not truly random; if one knows the program that generates the numbers, one could predict what numbers would occur. Now is false precisely when 1 ≤1 x2 + 1 is true. False 37. This form of the expression is called the postfix form of the expression (or reverse Polish notation). Let X be a set with n + 1 elements. You run 10 laps daily, you take multi-vitamins, and you are healthy. This part of the solution could be written up as follows. Suppose that {an } is a nondecreasing sequence and that when- 4n lg n. What is the probability of at least one boy and at least one girl? a b e 1 2 3 c d G1 480 g h d 2 b 6 3 G1 a 1 f j 2 d e 4. Example 1.24 In Example 1.15, we found that gcd(30, 105) = 15, and in Example 1.21, we found that lcm(30, 105) = 210. Notice that we cannot have ai = 0, for some i, and aj = 19, for some j, for then we would have one processor (the ith processor) not connected to any other processor while, at the same time, some other processor (the Counting Methods and the Pigeonhole Principle jth processor) is connected to all the other processors (including the ith processor). For i and j satisfying 1 ≤ i < j ≤ n, we can assign the entries in row i, column j and row j, column i in three ways: Row i, Column j Row j, Column i 0 1 0 0 0 1 Since there are (n2 − n)/2 values of i and j satisfying 1 ≤ 2 i < j ≤ n, we can assign the off-diagonal values in 3(n −n)/2 2 ways. Suppose that (y, x) is in f −1 ; that is, f −1 (y) = x. If each person chose a cola randomly, what is the probability that everyone chose Coke? Since we have verified the Inductive Step, by the Principle of Mathematical Induction, the theorem is proved. In the Maryland Cash In Hand game, to win the grand prize the contestant must match seven distinct numbers, in any order, among the numbers 1 through 31 randomly drawn by a lottery representative. The inverse R−1 of the relation R consists of the elements (y, x), where (x, y) ∈ R. b2 = 1, b3 = 2, b4 = 3 31. We partition R(V ∗) into the sets W1 = R(V ∗) ∩ P Figure 4.5 The proof of Theorem 4.7 for n = 3. Thus the problem is to choose a subset {i1 , . Let X be a set with least upper bound a, and let Y be a set with least upper bound b. 1 5 2 7 1 7 14 5 2 8 1 6 4 9 10 7 8 2 13 14 6 9 10 7 4 11 4 8 7 1 11 12 13 15 9. f is onto. Four coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. (There are different versions of the puzzle, depending on which faces are painted which colors.) The problem is to stack the cubes, one on top of the other, so that whether the cubes are viewed from front, back, left, or right, one sees all four colors (see Figure 8.2). 72. The formula derives from the Multiplication Principle. For example, in the expansion of (a + b)3 , we select either a or b from the first factor (a + b); either a or b from the second factor (a + b); and either a or b from the third factor (a + b); multiply the selections together; and then sum the products obtained. We conclude that an = Un = 5 · 2n + 2 · 3n , for n = 0, 1, . Proof It follows immediately from Definition 6.1 that (a) implies (b). Since n = 4 > 0, the body of the while loop executes. Show how merge sort (Algorithm 3.8) sorts the sequence 2, 3, 7, 2, 8, 9, 7, 5, 4. An isomorphism is given by f(a) = 1, f(b) = 2, f(c) = 4, f(d) = 3, and g(a, b) = (1, 2), g(b, c) = (2, 4), g(c, d) = (4, 3), g(d, a) = (3, 1). Assume that A ⊆ B. What is the value of the binary number bn bn−1 . nth-order, linear homogeneous recurrence relation with constant coefficients and how to solve a second-order recurrence relation 9. We may take tn−3 = −qn−1 and sn−3 = 1 to obtain rn−1 = tn−3 rn−2 + sn−3 rn−3 . Thus (x, z) ∈ Rm+n . If c is greater than large, we execute line 6 large = c but if c is not greater than large, we skip to line 7. p ∧ ¬q 23. , xk } and E = {xk+1 , . At line 3, c > large (9 > 6) is true, so we set large to 9. For every x, there exists y such that x > y is true. According to these authors, which of the graphs in Examples 1.12–1.14 are bipartite? 64 {{a, b, c, d}}, {{a, b, c}, {d}}, {{a, b, d}, {c}}, {{a, c, d}, {b}}, {{b, c, d}, {a}}, {{a, b}, {c}, {d}}, {{a, c}, {b}, {d}}, {{a, d}, {b}, {c}}, {{b, c}, {a}, {d}}, {{b, d}, {a}, {c}}, {{c, d}, {a}, {b}}, {{a, b}, {c, d}}, {{a, c}, {b, d}}, {{a, d}, {b, c}}, {{a}, {b}, {c}, {d}} 25. A, 6:00 4000 C, 6:30 3000 a B, 6:15 2000 A, 6:15 C, 6:45 3000 4000 z B, 6:30 2000 A, 6:30 C, 7:00 4000 Figure 1.5 A network that represents the flow of traffic from city A to city C during the period 6:00 p.m. to 7:00 p.m. Variants of the network flow problem have been used in the design of efficient computer networks (see [Jones; Kleinrock]). A subset can be constructed in n successive steps: Pick or do not pick x1 ; pick or do not pick x2 ; . In order for (1, 2) ∈ R ◦ S, we must have (1, a) ∈ S and (a, 2) ∈ R, for some a. Show that at least two weighings are required to solve the problem of Exercise 1. Find the compositions f ◦ f , g ◦ g, f ◦ g, and g ◦ f . We put (1, 5) in S and (5, 2) in R. d c v3 In Exercises 28–33, decide whether the graph has an Euler cycle. Also, assume that the n + 1 possibilities of key being at any particular position in the sequence or not being in the sequence are equally likely. Thus the n − 1 smaller disks must be stacked on a third peg (see Figure 1.3). We construct a truth table for all the propositions involved: q p∧q T T F F T F T F T F T F F F 2. ∴ I get A's. We have shown that R is transitive. No element in X is assigned to the element c in Y . Just before the while loop begins executing, i = 1 and pow = 1, so pow = a1−1 . If f(n) = (h(n)) and g(n) = (h(n)), then f(n) + g(n) = (h(n)). What is postorder traversal? Thus the string that follows 2367 must begin 24. The smallest number with this property is 2 · 3 · 5 · 7 = 210. x1 ≥ 1, x2 ≥ 1, x3 ≥ 1 10. Frequently, we can relate an algebraic expression to some counting process. Notice that if S is a partition of X, S is pairwise disjoint and ∪ S = X. What is an algorithm? 142 f a b c Y Figure 1.10 A function that is not one-to-one. If a forest F consists of m trees and has n vertices, how many edges does F have? Which are one-to-one? Clearly, k ≥ 1. It seems we have a blank slate √ with which to begin. Also, D ⊆ R1 ∪ D. Next, we turn to the existence of a complete matching in a directed, bipartite graph G with vertex sets V and W . Then the third bit must be a 1 and if the remaining (n − 3)-bit string does not contain 000, neither will the n-bit string. Then, at lines 8 and 9 we label vertex a as (−, ∞) (see Figure 2.10). denotes the Fibonacci sequence. an = 3an−1 + 10an−2 ; a0 = 4, a1 = 13 at the point t. A "seed" is used as the first pseudorandom number in the sequence. For example, (b, c) is in R and (c, b) is also in R. (3n − 3)/2, n ≥ 2, coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. Proof We first show that p(n) = O(nk). Suppose that the greedy algorithm chooses k stamps for n − 6. Select the first vertex and label it the root. Since we are assuming that the capacity of the cut is less than n, this case cannot occur. c f c g j n a b d f Show that none of the graphs contains a Hamiltonian cycle. A graph with no cycles is called an acyclic graph. Notice that n = O(2n), but n = (2n). Appel and Haken proved (see [Appel]) that every simple, planar graph can be colored with four colors. Expand (x + y)4 using the Binomial Theorem. This kind of approach is used in the development of many computer programs that allow human beings to play against computers or even computers against computers. Give an example of a connected graph such that the removal of any edge results in a graph that is not connected. , ek are the only edges in G1 incident on v, e = ei for some i ∈ {1, . The minimal spanning tree constructed is shown in Figure 4.3. b 2 Edge 4 Prim's Algorithm furnishes an example of a greedy algorithm. The elements of [xi] are equivalent in the sense that they are all related. [(a) → (b).] We assume that A ⊆ B, and prove that A ∩ B = A. , is also a solution of (2.16). Algorithm 2. Implement selection sort (see Exercise 20, Section 2) as a program. Therefore, the algorithm returns the correct value for tn . 17. First, dec val is set to 0, and power is set to 1. Let P(x) denote the propositional function "x loves Microsoft," and let Q(x) denote the propositional function "x loves Apple." The first hypothesis is ∀x(P(x) ∨ Q(x)). Therefore, p ∨ q → ¬r is false. Find s1 . Since x ∈ Y − X, x ∈ Y and x ∈ / X. If key is not found, the algorithm returns 0. nx = mq + 1. Sections 4 and 5 are concerned with mathematical induction, a proof technique especially useful in discrete mathematics and computer science. Example 1.2 If A = {1, 3, 2} and B = {2, 3, 2, 1}, by inspection, A and B have the same elements. Find the dual of the following map. The labels on the edges show the capacities of the subpipelines. Write an algorithm that outputs the smallest and largest values in the sequence s1 , . , 2n − 1 so that there is an edge between two vertices if and only if the binary representation of their labels differs in exactly one bit? It follows that in this case T is isomorphic to the tree in Figure 8.1. Suppose that T is a tree with five vertices and the maximum vertex degree that occurs is 3. Example 6.4 Algorithm 6.5 PT, the root of a binary tree, or the special value null to indicate that no tree is input Dependent on how "process" is interpreted in line 5 inorder(PT) { if (PT == null) return l = left child of PT inorder(l) process PT r = right child of PT inorder(r) } In what order are the vertices of the binary tree of Figure 6.3 processed if inorder traversal is used? The definition of CX suggests that we use proof by cases: x ∈ X ∪ Y (when CX∪Y (x) = 1) and x ∈ / X ∪ Y (when CX∪Y (x) = 0). Compute z. In this case, since there are n! permutations, the probability that no one receives the correct order is Dn 1 = {(−1)n + n(−1)n−1 n! n! + n(n − 1)(−1)n−2 + · · · − [n(n − 1) · · · 4] + [n(n − 1) · · · 3]} = (−1)n (−1)n−1 (−1)n−2 + + + ··· n! (n − 1)! (n − 2)! − 1 1 + . We can select a k-element subset Y of X − {x} in C(n − 1, k) ways. r : The bug is a numerical error. For the hash function h(x) = x mod 13, show how the data 784, 281, 1141, 18, 1, 329, 620, 43, 31, 684 would be inserted in the order given in initially empty cells indexed 0 to 12. ∀x∀y(x2 ∀x∀y(x2 + y2 + y2 ≥ 0) 53. n1 · · · nk n1 · · · nk (n + 1)n1 · · · nk must show that any string in L of length n has an equal number C1 ∪C2 C1 C2 of a's and b's. f(n) = O(g(n)) if there exists a positive constant C1 such that |f(n)| ≤ C1 |g(n)| for all but finitely many positive integers n. Definitions are used to create new concepts in terms of existing ones. (a) A robot can take steps of 1 meter, 2 meters, or 3 meters. If R is a partial order on a set X, the notation x y is sometimes used to indicate that (x, y) ∈ R. {(a, a), (b, b), (b, d), (b, e), (d, b), (d, d), (d, e), (e, b), (e, d), (e, e), (c, c)} 16. The collection of sets S = {{1, 4, 5}, {2, 6}, {3}, {7, 8}} is pairwise disjoint. After computing P(E), we may obtain P(E) by subtracting P(E) from 1. In Exercises 35–39, add the hexadecimal numbers. Algorithm 1.1 consists of a single function. p → q ¬q p ∴¬q ∴ ¬p 47. 0 0 0 1 Its square is ⎛ 1 ⎜ 0 A2 = ⎜ ⎝0 0 1 1 2 0 2 0 1 0 ⎞ 0 0⎟ ⎟. Example 1.3 The set f a b c Y Figure 1.1 The arrow diagram of the function of Example 1.3. There is exactly one arrow from each element in X. Are E2 and E3 mutually exclusive? Eventually r = 0, so the solution is gcd(b, 0) = b. Find the prime factorization of 539. Write a backtracking program to determine whether two graphs are isomorphic. ∅ 2. The recurrence relation (3.3) can be solved by the iterative method: bk = 1 + bk−1 = 2 + bk−2 = · · · = k + b0 = k + 2. Will the first or second player, playing an optimal strategy, always win? If an assumption turns out to be unwarranted, it can sometimes be modified so that the modified assumption is correct. The preceding inequality may be rewritten as n n 2 The given inequality now follows immediately. When is it false? 0⎠ 1 The entry in row 1, column 2 of A2 is nonzero, but the corresponding entry in A is zero. j=0 61. Problem-Solving Corner Problem Is it possible in a department of 25 persons, racked by dissension, for each person to get along with exactly five others? The robot can begin by taking a step of 1 meter or a step of 2 meters. This gives cube 3 a green back face. Substitute the formula for the next-to-last remainder into the last equation. Now P ends at either v1 , v2 , or v3 because the only way that P could fail to be a path in G is that P contains one of the deleted edges e1 or e2 . Exercises In Exercises 1–9, tell whether the given path in the graph is (a) A simple path 21. Fishing is a popular sport. To prove that ∃x∀y P(x, y) is false, show that for all x ∈ X, there is at least one y ∈ Y such that P(x, y) is false, where the domain of discourse is X × Y. Explain a method of verifying that X is a proper subset of Y . In particular, (xi + xj)/2 and (yi + yj)/2 are integers. To simplify the notation, we set s = −b/k. In other words, f(x) = y is another way to write (x, y) ∈ f . As remarked previously, any comparison-based sorting algorithm is (n lg n) in the worst case. Frequency Letter Frequency α β γ 5 6 6 δ ε 11 20 25. If it finds a value di for which P(di) is false, the condition ¬P(di) in the if statement is true; so the code returns false [to indicate that ∀x P(x) is false] and terminates. Five-Coins Puzzle Five coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. That is, a flow F must satisfy mij ≤ Fij ≤ Cij for all edges (i, j). Example 4.5 Let α = 13542 and β = 21354 be strings over {1, 2, 3, 4, 5}. Express the relation given by Table 6.5 as a set of n-tuples. f(x) = 3x − 2 26. f(x, y) = x − y, X = {1, 2, . Our modified algorithm will compute r = a mod b and STgcdr(b, r, s , t). (A nice example is [Wagon].) Finding a Solution A fundamental issue in building a graph model is to figure out what the graph is—what are the vertices, and what are the edges? Proof We consider the time spent in the loops, which provides an upper bound on the total time. Also, the empty set is a subset of every set. b 2, 2 c 3, 2 4, 3 2, 1 a z In Figure 1.2 we have redrawn the network of Figure 1.1 to show the flow of Example 1.4. An edge e is labeled "x, y" if the capacity of e is x and the flow in e is y. k=1 117. To prove that f is one-to-one, suppose that f(n) = f(m). The last inequality follows from Exercise 44. An = (1.14)An−1 log 2 = 5.29. For edge (a, b) we have Fab = 0 < Cab = 3. There are C(n, j) ways to place the 1's. This relationship is stated formally in the next theorem. We use proof by contradiction and assume the negation of the conclusion ¬∃i(si ≤ A). n = 3738 8. Since n mod 2 is equal to 1, result is set to (result∗x) mod z = 45 mod 11 = 1. Exercises 66–73 refer to an algorithm that accepts as input the sequence si , . Suppose that G = gcd(a, b) is true prior to another iteration of the loop and that b = 0. A ∪ ∅ = A, A ∩ U = A 32. Therefore, the only outside squares visited are white and the only inside squares visited are black. In particular, a loop invariant is true after the loop finishes, at which point the invariant tells us something about the state of the variables. We square x to obtain a2 . (a) a1 < b1 < a2 < b2 < · · · (b) an < b1 24. Then g divides both m and n and also nx − mq = 1. ∀x∃y¬P(x, y) (∀x(I(x) → P(x))) ∨ (∀x(P(x) → I(x))). No 72. We summarize this result as Theorem 5.6. Theorem 5.6 Let R1 be a relation from X to Y and let R2 be a relation from Y to Z. What is the conjunction of p and q? The preceding result shows, in particular, that if there were a path from a to v whose length is less than L(v), v would already have been selected at line 9 and removed from T . Technically, logic is of no help in determining whether any of these statements is true; however, if the first two statements are true, logic assures us that the statement, All mathematicians are algebraists, is also true. Proof We begin by letting x and y be arbitrary real numbers. Another useful matrix representation of a graph is known as the incidence matrix. b c G1 d 1 2 G2 3 3 c e 5 d G1 G2 43. Is commutative? , cj . ex74(s, i, j) { if (i == j) return m = (i + j)/2 ex74(s, i, m) ex74(s, m + 1, j) combine(s, i, m, j) } 77. {(1, 1), (2, 2), (3, 3), (4, 4)} 15. Now the dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers if and only if G contains an Euler cycle. In our example, it was helpful in looking at some concrete cases to assume that the relation was an equivalence relation before actually proving that it was an equivalence relation. [The definition of f −1 (B), where B is a set, precedes Exercise 70, Section 1.] Show that S is a partition of X. Give an example of a function and its inverse. Example 7.2 An algorithm to sort a1 , a2 , a3 is given by the decision tree of Figure 7.7. Each edge is labeled with the arrangement of the list based on the answer to the question at an internal vertex. By Theorem 5.14, part (a), ∃x ¬¬P(x) or, equivalently, ∃x P(x) is also false. (c) The baseball player hit over .300 in 2003 (domain of discourse = set of baseball players). Find the matrix of the inverse of each of the relations of Exercises 8 and 9. 2 Any graph isomorphic to G1 and G2 is called the Petersen graph. Suppose that n satisfies 2m−1 < n ≤ 2m . This contradiction shows that if n and m are both odd, the graph does not have a Hamiltonian cycle. The relation {(1, 2), (2, 3), (1, 3)} is a transitive relation on {1, 2, 3}. d 17. List all strings over X = {0, 1} of length 2. In this section, we discussed the 194 Functions, Sequences, and Relations operations select (choose rows specified by a given condition), project (choose columns specified by a given condition), and join (combine rows from two tables as specified by a given condition). To prove that the universally quantified statement ∀x P(x) is false, find one value of x in the domain of discourse for which the proposition P(x) is false. This graph and that of Figure 8.9 solve the Instant Insanity puzzle of Figure 8.6. 491 Graph Theory Section Review Exercises 1. b1 = 1, b2 = 2, b3 = 3 11. Since m is the rightmost position, at lines 16 and 17, we do nothing. a b Determine whether or not each graph contains a Hamiltonian cycle. An alpha value for a vertex v is only a lower bound for the value of v. , in this order, and terminates at vn represents the time it takes the drill press to start at hole h1 and then move to h2 , h3 , . Define composition of relations and give an example of the composition of relations. We next compute a16 = a8 · a8 , which uses 1 additional multiplication. We may then apply f to determine a unique element z = f(y) = f(g(x)) ∈ Z. Since R is transitive, (z, x) ∈ R. The first observation (Corollary 2.22) is that the number of vertices of odd degree is even. The inclusive-or of propositions p and q is true if p or q, or both, is true, and false otherwise. The hostid must not consist of all 0's or all 1's. Thus the number of strings and number of routes are equal to C(n + m, n). At line 1, we set large to a (6). A. A tree of height 3 has at most 27 terminal vertices; thus at least four weighings are required in the worst case. Example 3.1 a b f g d c e Figure 3.4 A graph with a Hamiltonian cycle. If 3 + 5 < 2, then 1 + 3 = 4. Since X ∩ Y = ∅, there exists x ∈ X ∩ Y . How many weighings in the worst case are required to find the bad coin and determine whether it is heavier or lighter than the others using only a pan balance? Therefore, no algorithm that sorts three items has worst-case time less than 3, and the algorithm of Figure 7.7 is optimal. R as in Exercise 4; ordering of X: 5, 3, 1, 2, 4 6. We next prove that if B ⊆ A, then A ⊆ B. (1, 4), (1, 2), (2, 5), (2, 3), (3, 6), (6, 9), (4, 7), (7, 8) 15. A fundamental property of the real numbers is that every nonempty subset of real numbers bounded above has a least upper bound. What is a unary operator? First, we count the number of eight-bit strings with no two 0's in a row. Show, by giving counterexamples, that the condition ai ≥ 2ai−1 − ai−2 , 3 ≤ i ≤ n, is neither necessary nor sufficient for the greedy algorithm to be optimal for all amounts of postage. ( ] The graph of the function P is shown in Figure 1.8. 80 ( ] w 1 13 ≥ w > 0. Since Ti+1 is contained in T , the Inductive Step has been verified. 3 45. Write an algorithm that receives as input the matrices of relations R1 and R2 and produces as output the matrix of the composition R2 ◦ R1 . If a set is finite and not too large, we can describe it by listing the elements in it. Example 1.3 (a) How many strings of length 4 can be formed using the letters ABCDE if repetitions are not allowed? The third term is 11 because instruction 2 says to add 3 to 8 to get the next term, 11. For this reason, the graph of Figure 1.2 could just as well be drawn as in Figure 1.3. Gre e2 Wor e6 Sho e 11 Lan Sho e1 She e3 Buf e4 e9 e 12 e7 e 13 Cas Mud e5 e 10 Gil e8 Dou G Figure 1.2 A graph model of the highway system shown in Figure 1.1. e9 e 11 Lan Wor e 2 e 12 e 13 Mud Dou Cas e6 e 10 e4 e8 e7 Gre e1 Gil e5 Buf e 3 She G Figure 1.3 An alternative, but equivalent, graph model of the highway system shown in Figure 1.1. 435 Graph Theory If we start at a vertex v0 , travel along an edge to vertex v1 , travel along another edge to vertex v2 , and so on, and eventually arrive at vertex vn , we call the complete tour a path from v0 to vn . , yn . Proofs 4. (Any vertex in G is either in V or in W.) There exists a complete matching in G if and only if |S| ≤ |R(S)| for all S ⊆ V. One way to prove (2.4) is to exhibit one member a in the domain of discourse that makes P(a) true. Assume that (i, k) ∈ R2 ◦ R1 . At line 2, since i > j (1 > 4) is false, we proceed to line 4, where we set k to 2. State a necessary and sufficient condition for a graph to have a spanning tree. By the inductive assumption, L(u) is the length of a shortest path from a to u. Then (3.12) becomes b2k = 2b2k−1 + 2. {(1, b), (2, b), (3, b), (4, b)} Draw the graphs of the functions in Exercises 6–9. For this reason we may also describe the set A defined in (1.1) as A = {1, 2, 3, 4}. Since 542 is the largest permutation of {2, 4, 5}, any permutation that begins 163 is smaller than the given permutation. Which means an = (lg n). For which values of n does the n-cube contain an Euler cycle? For example, if there are three teams, Scientists, Whales, Pilots, assigned as Scientists child of v, then T is a binary search tree. A set S of points in the plane defines a function when each vertical line intersects at most one point of S. Exercise 55, Section 1 26. Define symmetric relation. What is text searching? (2.6) Subsequence (2.5) is obtained from sequence (2.6) by choosing the first, second, fourth, eighth, and so on, terms; thus the value of nk of Definition 2.11 is nk = 2k−1. Invalid. Example 8.1 Ten persons have first names Alice, Bernard, and Charles and last names Lee, McDuff, and Ng. Show that at least two persons have the same first and last names. If we let β1 = b, β2 = a, β3 = a, β4 = c, we obtain a string over X. Let a ∈ X. Put all vertices adjacent to v in V2 . What is an incidence matrix? Suppose that the sequences are a1 , . Now n2 = (2k + 1)2 = 4k2 + 4k + 1 = 2(2k2 + 2k) + 1. Since removing an edge from a cycle does not disconnect a graph, we may remove edges, but no vertices, from cycle(s) in T until the resulting graph T ∗ is connected and acyclic. 9 11. Prove that if n is an odd integer, n2 4 = n2 + 3 . We can also obtain a lower bound as in Example 3.7: 1k + 2k + · · · + nk ≥ n/2 k + · · · + (n − 1)k + nk ≥ n/2 k + · · · + n/2 k + n/2 k = (n + 1)/2 n/2 k ≥ (n/2)(n/2)k = nk+1 /2k+1 224 Algorithms for all n ≥ 1. Sn,k = i=0 What is wrong with the following "proof" that tn = 1 for all n ≥ 1? Nim is the game using n piles of tokens as described in this section in which the last player to move loses. By Theorem 6.4, G1 and G2 are isomorphic. Find the diameter of the n-cube. Nevertheless, the concept of a game tree is still useful for analyzing such games. How many strings can be formed by ordering the letters ILLINOIS? ⎛ d 1 0 1 0 1 v3 1 1 0 0 0 0 x3 1 0 0 0 1 e ⎞ 1 0⎟ ⎜1⎟ 1⎠ 0 v4 v5 0 0 0 0 0 0 0 0 0 0 0 1 x4 x5 0 1 0 0 1 1 1 0 0 0 v6 ⎞ 0 0⎟ ⎜0⎟ ⎜0⎟ 1⎠ 0 x6 x7 1 0 0 0 0 1 1 0 0 1 x8 ⎞ 0 0⎟ ⎜0⎟ 1⎠ 1 10. f(x) = 6 + 27x−1 , X = set of real numbers 38. How many subsets of a (2n + 1)-element set have n elements or less? C(13, 5) 36. What is a universally quantified statement? If v is not a terminal vertex, it is an internal vertex. P(1) ∨ P(2) ∨ P(3) ∨ P(4) 27. , a30 + 6. 221 Algorithms Example 3.3 Since 60n2 + 5n + 1 ≤ 60n2 + 5n2 + n2 = 66n2 for all n ≥ 1, we may take C1 = 66 in Definition 3.2 to obtain 60n2 + 5n + 1 = O(n2). J1 A 1, 1 1, 1 a 1, 0 1, 1 1, 0 B 1, 0 1, 1 1, 1 C 1, 0 D 1, 0 J2 J 3 1, 1 1, 1 0, 1 0, 0 J4 1, 0 1, 0 1, 0 z 1, 1 1, 1 J5 Figure 4.3 The matching problem (Figure 4.1) as a matching network. q : You miss the midterm exam. The "divides" relation on the positive integers (see Example 3.21) has both comparable and incomparable elements. If n = 3, then 2 > n/2. Principle of Mathematical Induction 28. To enter Utopia, you must show a driver's license or a passport. Discussion According to the discussion preceding this example, we must prove [(a) → (b)] ∧ [(b) → (c)] ∧ [(c) → (a)]. Is the value in Exercise 3 a good estimate of the chance that someone knowledgeable about basketball will successfully predict how the tournament will unfold? Find a formula for the expression nk of Definition 2.11. By considering the number of n-bit strings with exactly i 0's Cn ≥ a1 ∗ a2 ∗ · · · ∗ an , n ≥ 2. 57. The coefficients f(n) and g(n) are not necessarily constant. Section 5 17. If not, we check whether p occurs at index 2 in t. Going directly to the definitions, we must show that (a) z ≤ ta for every z ∈ tX (i.e., ta is an upper bound for tX), (b) if b is an upper bound for tX, then b ≥ ta (i.e., ta is the least upper bound for tX). G1 = G1 of Exercise 38 29. (7.2) For the graph of Figure 7.1, e = 9 and v = 6, so (7.2) becomes 18 = 2 · 9 ≥ 4(9 − 6 + 2) = 20, which is a contradiction. A proof that the code constructed is optimal may be found in [Johnsonbaugh]. Explain why the second row lists the positive integers in order 1, 2, . Expand (3x − 2y)4 using the Binomial Theorem. X intersect Y is the set of elements that belong to both X and Y. An alternative way to state (3.5) is: A necessary and sufficient condition for 1 < 5 is that 2 < 8. Again, let the words sink in. ■ Begin with 0. 1/4 18. Every vertex in a binary tree has at most two children (see Figure 5.1). , n}, Therefore, j+ n n(n + 1)(2n + 1) i=1 C1 = class of nonempty subsets that do not contain n + 1 C2 i=1 6 j=1 C2 = class of subsets that contain n + 1. (1.2) Similarly, Since X ∪ Y = (X − Y) ∪ (X ∩ Y) ∪ (Y − X) and X − Y , X ∩ Y , and Y − X are pairwise disjoint, by the Addition Principle |X ∪ Y | = |X − Y | + |X ∩ Y | + |Y − X|. , sn , sorted in nondecreasing order, a value key to search for, and n. f(m, n) = m2 + n2 21. .} 91. 504 G 4 W G R B 4 3 W G 2 G6 B 1 W 2 1 G5 Section 8 Review 1. Then f(C1), f(C2), . What is a supersink? If we fail, we will try to determine where our proof fails and use this information to construct a counterexample. by iteration, we use the recurrence relation to write the nth term an in terms of certain of its predecessors an−1 , . Use proof by cases to prove that x + y − |x − y| min{x, y} = 2 for all real numbers x and y. Assume that the pegs are numbered 1, 2, 3, 4 and that the problem is to move the disks, which are initially stacked on peg 1, to peg 4. Among all such graphs, there is one with the fewest number of vertices. Then there exists k such that n = 2k + 1. Write a formal algorithm for searching in a binary search tree. In Figure 1.9 we show the graph corresponding to the programs of Table 1.2 with S = 25. Also, if (i, j) is not an edge, we set Fij = 0.] We call Fij the flow in edge (i, j). Let 5. Use the Binomial Theorem to prove that n Section 5 17. But the formula shows that this value is equal to tn . A robot can take steps of 1, 2, or 3 meters. Since A ∩ B = ∅, there exists x with x ∈ A and x ∈ B. Show that if a simple graph G has 11 or more vertices, then either G or its complement G is not planar. Thus, the contrapositive is If X ∩ Y = ∅, then CX∪Y (x) = CX (x) + CY (x) for at least one x ∈ U. Hints/Solutions to Selected Exercises Section 1 Review Section 1 1. (Hypothesis) If x ∈ X ∩ (Y − Z), then x ∈ (X ∩ Y) − (X ∩ Z). Exercises 82–88 use the following definitions. Define a function from X∗ to X∗ as f(α) = αα. We then discuss some important concepts in graph theory, including paths and cycles. We now argue as follows: {{c}, {c, d}} = {{c}}. Definition 2.6 A linear homogeneous recurrence relation of order k with constant coefficients is a recurrence relation of the form an = c1 an−1 + c2 an−2 + · · · + ck an−k , ck = 0. Prove that your example has the required property. A B 0 1 C 0 Figure 9.3 The first player (box) should move to position B since it is most valuable. We wish to store and retrieve arbitrary nonnegative integers in these cells. ■ Look at negations of statements. R is a reflexive relation if and only if the main diagonal of the matrix of R consists of 1's. (Alternative: At least one tail was obtained.) 15. The ceiling of x, denoted x , is the least integer greater than or equal to x. Exercises In Exercises 1–3, fill in the missing edge flows so that the result is a flow in the given network. Instead we use the

Inclusion-Exclusion Principle. If n = 2k , = am0 + kd = c + kd. 12. We next use Algorithm 3.5 (merging) to construct merge sort. A problem for which there is no algorithm is said to be unsolvable. (1.7) Basis Step (n = 1) By inspection, c1 = 1 = d1 ; thus (1.7) is true when n = 1. What is a key? vn = n! + 2, 4 2 n ≥ 1. Then gcd(a, b) = a. End a direct proof by clearly stating the conclusion, and, perhaps, giving a reason to justify the conclusion. Find i=1 10 bi . Give an example to show that, in general, ∀x∃yP(x, y) and ∃x∀yP(x, y) have different meanings. Conclusion: Ralphie, who is in the class, understands the trigonometric functions. (d) R is transitive. By Theorem 5.4, T ∗ has n terminal vertices. (5.3) The terminal vertices of T consist of the terminal vertices of T1 and T2 . Edge (vi−1 , vi) is properly oriented because it is oriented in the a-to-z direction. First note that B = Z+ . Show that if G is a tree, every vertex of degree 2 or more is an articulation point. (p∧q)∨(¬q∧r) 41. If we let bk = a2k , we obtain the recurrence relation bk = 1 + bk−1 , k = 1, 2, . Assume the existence of an evaluation function E. 64 6. For example, in order to estimate the run time of an algorithm, we need to count the number of times certain steps or loops are executed. 478 Graph Theory Example 6.8 Show that if k is a positive integer, “has a vertex of degree k” is an invariant. Continue until inserting sn in s1 , . 214 Algorithms For example, to insert 16 in 8 13 20 27 we first compare 16 and 27. Similar remarks apply to decreasing sequences and nonincreasing sequences. Therefore, no algorithm that solves the five-coins puzzle has worst-case time less than 3, and the algorithm of Figure 7.3 is optimal. Then n ak bk = k=1 2. There are Dn−2 distributions in which q has p’s order since the remaining n − 2 orders are in possession of the remaining n − 2 persons, but each has the wrong order. Show that the consecutive labels of the vertices on the unit circle in Bain’s depiction of the n-cube (see Exercises 43–45, Section 1) give a Gray code. Thus S ∗ is optimal. , n} in increasing lexicographic order. In the case at hand, we have m is odd and n is even. x = y = 2.3 3. If si+1 was not larger than large, we did not modify large. Since n = 8 > 0, the body of the while loop executes. We then move to lines 7 and 8, where we set a to 108 and b to 72. Prove that A(m, n) > n for all m ≥ 0, n ≥ 0 by induction on m. Removing the added edge affects only the degrees of v and w, which are each reduced by 1. By the Multiplication Principle, there were then 50 · 50 = 2500 additional recipients. (Because n is even) ··· m + n is odd. Therefore, if n satisfies 2k−1 ≤ n < 2k , the statement x = x + 1 is executed k times. In Theorem 8.7 we showed that there are four nonisomorphic rooted trees having four vertices. If each equivalence class has r elements, there are |X|/r equivalence classes. How many Instant Insanity graphs are there; that is, how many graphs are there with four vertices and 12 edges—three of each of four types? The Euclidean algorithm next computes r = 110 mod 53 = 4. (3) Theorem 2.3 states that the following are equivalent for an n-vertex graph T : T is a tree. If b ∈ Y , set s2 = 1; if b ∉ Y , set s2 = 0. A UPC is a 12-digit code in which the first digit characterizes the type of product (0 identifies an ordinary grocery item, 2 is an item sold by weight, 3 is a medical item, 4 is a special item, 5 is a coupon, and 6 and 7 are items not sold in retail stores). The puzzle was said to be derived from a mythical gold tower that consisted of 64 disks. Find a formula for the sequence c defined by cn = n xi . Thus (x1 , x2)R(x1 , x2). If b > large, then large = b. Therefore R is transitive. (a1 a2 · · · a2n)1/2 ≤ 17. Modify Kruskal’s Algorithm so that it groups data into classes. The following table shows the cost of traveling from 1 to 10 miles. Example 2.15 Figure 2.4 shows how a29 is calculated using repeated squaring. 3 31. We can make seven cents postage by using one 7-cent stamp. ∀n P(n) 8. If one component has i vertices, show that the components are Ki and Kn−i . , 19}. Trace Algorithm 2.3 for the input 34 55 144 259. Graph Theory The following examples show that sometimes we can argue that a graph does not contain a Hamiltonian cycle. The variable n is used as an index in the sequence c. The problem then became one of counting the number of such strings. Fibonacci sequence {fn } : f1 = 1, f2 = 1, fn = fn−1 + fn−2 , n ≥ 3 Chapter Self-Test Section 1 Assume that the values of rand are 1. Find functions f and g satisfying f(n) = O(g(n)) and defined on the positive integers for which 57. Then an edge between two vertices represents some kind of relation. Suppose, by way of contradiction, that n is even. 2 1 2 3 1 1. Parallel horizontal lines spaced one unit apart. Let Y = A + A2 + · · · + An−1 . ∀x∀y P(x, y) 84. 393 Recurrence Relations 55. Since this is the first tape by the Unhinged Universe, the group wants to include as much material as possible. Then tx ≤ b for every x ∈ X (since an arbitrary element in tX is of the form tx for some x ∈ X). Tile this subboard using the result of Exercise 35, Section 4. For example, a sequence t whose domain is {−1, 0, 1, 2, 3} can be denoted {tn }3n=−1 . Example 1.35 The function f of Example 1.29 is a bijection. How many outcomes give the sum of 4? X union Y is the set of elements that belong to either X or Y or both. Execution resumes in part (a) at line 4 after computing 3! (= 6). We begin by writing a recursive version of the Euclidean algorithm (Algorithm 3.9). Therefore, G is optimal. Suppose that the set of elements in the ith column of R1 and the set of elements in the ith column of R2 come from a common domain for i = 1, . Compute c2 , c3 , c4 , and c5 . At this point, no more edges can be added, so we finally backtrack to the root and the procedure ends. The vertices are the cities, the edges are the roads between the cities, and the numbers on the edges are the distances between the cities. In this case, p is not true; thus the argument is invalid. A tree of height 0 has one vertex, so N0 = 1. This result implies, in particular, that merge sort is (n lg n) in the worst case. Assign each card (pigeon) to its suit (pigeonhole). Inclusive-or: The car comes with a cupholder that heats or cools your drink or both. Since the first equation is true, it is marked. We must prove that no “holes” in the number line. Illustrate how your operator works by answering the following query, using the relations of Tables 6.4–6.7: Find the names of all buyers who buy both parts 2A and 1199C. by certain of their predecessors. Suppose that the pair a, b, a > b, requires n + 1 modulus operations when input to the Euclidean algorithm. The set U must be explicitly given or inferred from the context. k=1 243 Algorithms Figure 4.1 A pine cone. , an . An alpha cutoff occurs at vertex C when vertex I is evaluated since I’s value (1) is less than or equal to the current lower-bound estimate (2) for vertex A. The situation is similar to that of mathematical induction and recursive algorithms in which a given instance of a problem is related to smaller instances of the same problem. If (x, y) ∈ R, we write x R y and say that x is related to y. We then substitute this formula for 4 into equation (3.8) to obtain 1 = 53 − 4 · 13 = 53 − (110 − 53 · 2)13 = 27 · 53 − 13 · 110. Since k = x , we have x + 3 = k + 3 = x + 3. Assuming that i = 1 and n = 4, in the while loop, k is first set to 2. [Part (b)] (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (9, 1), (10, 1) 34. Thus, we assume that p > 1 and q > 1. [Requires calculus] Explain how the formula gives a recursive algorithm for integrating logn |x|: logn |x| dx = x logn |x| − n logn−1 |x| dx. 3, 7, 11, 15, . We can use Theorem 6.5 to obtain the desired probability. Usually, it is difficult to obtain an explicit formula for this function, and we settle for less. Let XT denote the set of all such trees constructed. 95. An n-ary relation is a set of n-tuples. PT, the root of a binary tree, or the special value null to indicate that no tree is input Dependent on how “process” is interpreted in line 3 preorder(PT) { if (PT == null) return process PT l = left child of PT preorder(l) r = right child of PT preorder(r) } Let us examine Algorithm 6.1 for some simple cases. If the Mathematics Department does hire an additional faculty member, surely the dean’s statement is true. 34 15. We showed in Example 3.3 that 60n2 + 5n + 1 = (n2). Functions, Sequences, and Relations 88. To produce the fluctuating behavior of Figure 1.6, the angles α and β must add to 180◦ . Suppose that k ≤ m/2. , sn is sn —again, the value returned by the algorithm. Linear inhomogeneous recurrence relations with constant coefficients are discussed in Exercises 40–46.) 398 Recurrence Relations Example 2.10 The recurrence relation an = 3nan−1 is not a linear homogeneous recurrence relation with constant coefficients because the coefficient 3n is not constant. The propositional function ∃x P(x) is equivalent to P(1) ∨ P(2) ∨ P(3) ∨ P(4). It is important to develop some feeling for the relative sizes of the functions in Table 3.3. In Figure 3.1 we have graphed some of these functions. The graph of Figure 3.9 has a Hamiltonian cycle and an Euler cycle. If 4 < 2, then 7 < 10. Which of Exercises 80–82 must also be false? A(3, n) = 2n+3 − 3 46. ■ Two angles are supplementary if the sum of their measures is 180◦ . Paul Erd˝os (1913–1996) was one of the most prolific mathematicians of all time. The capacity of the cut is |E| = 4 < n. We then replace a by b and b by r, and repeat. , xn 5. Evaluate each vertex in each game tree. As a special case, 0! is defined to be 1. Binary search tree 32. Since R is reflexive, symmetric, and transitive, R is an equivalence relation. Not an equivalence relation (neither transitive nor reflexive) τ(R1) = {(1, 1), (1, 2), (3, 4), (4, 2), (3, 2)} 9. More precisely, it inputs a sequence a1 , . X, complement of X: U − X, where U is a universal set 21. 3E7C 52. The graph is a directed, nonsimple graph. With these choices for b and d, we have U0 = C0 , U1 = C1 . g f b e c j i 2 f 3 d 4 e Section 4 Exercises 13–16 refer to the following graph. Now Therefore, P(E | F) = 364 · 364 · · · 364 P(E) = 1 − P(E) = 1 − = 1− 365 · 365 · · · 365 2 + 2 1 12 40. Write a backtracking program that determines whether a graph can be colored with n colors and, if it can be colored with n colors, produces a coloring. Suppose that n satisfies 2m < n ≤ 2m+1 . A relation R on a set X is reflexive if (x, x) ∈ R for every x ∈ X. 39 Sets and Logic Alternative ways to write ∀x P(x) are for all x, P(x) and for any x, P(x). Suppose that the task is to prove p → q and that p is equivalent to p1 ∨ p2 ∨ · · · ∨ pn (p1 , . v = b, w = e e2 e3 e4 e4 e8 v4 v2 e5 e7 28. Algorithm 4.3 Prim’s Algorithm This algorithm finds a minimal spanning tree in a connected, weighted graph. Five students, V, W, X, Y, and Z, are members of four committees, C1 , C2 , C3 , and C4 . Therefore, N1 = 2. X ∩ (Y × Z) = (X ∩ Y) × (X ∩ Z) for all sets X, Y , and Z. Write a recurrence relation and an initial condition that define the deer population at time n and then solve the recurrence relation. Prove the identity laws for sets. The for loop terminates and the algorithm returns 13, the decimal value of the binary number 1101. 5 4. 7 ➜ Binomial Coefficients and Combinatorial Identities At first glance the expression (a + b)n does not have much to do with combinations; but as we will see in this section, we can obtain the formula for the expansion of (a + b)n by using the formula for the number of r-combinations of n objects. Prove equation (7.4) by giving a combinatorial argument. 2n2 + 1 1. Hint: Use the fact that Proofs a 4 × 4 deficient board can be tiled with trominoes, Exercise 27, and Exercise 34, Section 4. One of the main reasons for using recurrence relations is that sometimes it is easier to determine the nth term of a sequence in terms of its predecessors than it is to find an explicit formula for the nth term in terms of n. To prove that several statements, say p1 , . To verify that A is not a subset of B, find at least one element that is in A but not in B. Chapter Review Section 1 Section 2 1. Section 3 Exercises 9–12 refer to the following algorithm. Notice that the number of multiplications is three. Is a nonincreasing? 100010 17. We use this function to compute 0!. Prove that 2m2 + 4n2 − 1 = 2(m + n) has no solution in positive integers. 3, 0 3, 1 4, 1 b a 3, 2 d z c a 4, 2 c 4, 3 1, 1 d e f 8, 1 z In Exercises 4 –12, use Algorithm 2.4 to find a maximal flow in each network. The relation is reflexive because (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) ∈ R. For a particular amount of postage, we’ll let np denote the number of p-cent stamps used and nq denote the number of q-cent stamps used. Since there are sn−2 such paths, there are sn−2 paths of length n from v1 to v1 that begin (v1 , v2 , v1 , . Notice that the digraph of this relation (see Figure 3.1) has a loop at every vertex. Pigeonhole Principle (First Form) If n pigeons fly into k pigeonholes and k < n, some pigeonhole contains at least two pigeons. Give examples of functions f and g such that f ◦ g is onto, but g is not onto. Suppose that at time n, 100n deer are introduced into 406 33. Repeated use of De Morgan’s laws will result in each negation applying to only one variable. Find matrices that represent the relations of Exercises 13 –16, Section 3. Showing that P(x) is false for a particular value x does not prove that ∃x P(x) is false. Example 1.4 1 2 3 4 a b c X Y Figure 1.2 The arrow diagram of the set in Example 1.4, which is not a function because there is no arrow from 4. Prove that an ≤ an+1 for n ≥ 1. Yes 6. No comparisons will be made if n − m + 1 ≤ 0. The other edge incident on G is 4. The subsequence (2.3) is t3 , t4 or tn1 , tn2 . We give a recurrence relation for the Catalan numbers. What can be said about a connected graph in which not all the vertices have even degree? In this case, we know that there exist integers p and q such that 2 = p/q. In the first three cases, we need only compare a3 with either a2 or a5 to find the correct position for a3 since we already know that a3 < a4 . We can also prove that G is connected. Since PT is not equal to null, we proceed to line 3, where we process the root. 75 Proofs Example 1.3 Example 1.4 Examples of theorems in Euclidean geometry are ■ If two sides of a triangle are equal, then the angles opposite them are equal. The first part of the modification of Algorithm 3.9 looks like: STgcdr(a, b, s, t) { // make a largest if (a < b) swap(a, b) if (b == 0) { s=1 t=0 // now a = sa + tb return a } Next, Algorithm 3.9 computes r = a mod b and gcdr(b, r). (6n + 4)(1 + lg n) 11. To visualize relationships among sets, use a Venn diagram. , n divides n. If R is a relation from X to Y , the inverse of R is the relation from Y to X: R−1 = {(y, x) | (x, y) ∈ R}. q From 1 and 2 7. Prove that the number of primes is infinite. 231 Algorithms TABLE 3.3 ■ Common growth functions. Since r is the only solution of (2.17), we must have t 2 − c1 t − c2 = (t − r)2 . What is backtracking? What is a proof? One route passes through city B and the other route passes through city C. Suppose that r ≥ d. Our recurrence relation can be solved by iteration: bn = bn−1 + n − 1 = (bn−2 + n − 2) + (n − 1) = (bn−3 + n − 3) + (n − 2) + (n − 1) .. Such a recurrence relation can then be solved by the methods of Section 2. Repeat Exercises 18 and 19 for the relation R on the set {1, 2, 3, 4, 5} defined by the rule (x, y) ∈ R if x = y − 1. (b) Solve the recurrence relation of part (a) by making the substitution Cn = Dn − nDn−1 . There- 130 J(2i+1) = J(2i) = 1. The key to writing a recursive function is to find a smaller instance of the problem within the larger problem. The sequence of pseudorandom numbers generated, x1 , x2 , . Thus, an equivalent formulation of the given statement is If the Cubs win the World Series, then they signed a right-handed relief pitcher. If k = 1, we have 1 = 21−1 ≤ n < 21 = 2. 17. Thus the argument is valid. Many other sorting algorithms are known that also attain the optimal number (n lg n) of comparisons; one, tournament sort, is described before Exercise 14. Formula for the geometric sum: ar 0 + ar 1 + · · · + ar n = a(r n+1 − 1) , r−1 r= 1 Section 5 33. C(10 + 12 − 1, 10) 48. Finally, suppose that x R y and y R z. For some positive integer n, 19340 = n · 17. Prove that among 35 students in a class, at least two have first names that start with the same letter. The statements can be ordered in any way and the proofs may be easier to construct for one ordering than another. An integer greater than 1 whose only positive divisors are itself and 1 is called prime. Critical simulations such as those involving aircraft and nuclear research require “good” random numbers. But r is the smallest integer in X. If it happens that you needed only the immediately preceding case in the Inductive Step, you merely used the form of mathematical induction of Section 4. Are more than one-third of the students taking a mathematics course? These trees are isomorphic as free trees. Omit symmetric positions. The terminal vertices correspond to the operands, and the internal vertices correspond to the operators. For example, in Table 6.1, the attribute Age might be taken to be the set of all positive integers less than 100. If the dual of a map has a vertex of degree 3, what must the original map look like? Thus 2x ≤ 2k = 2 x . 53 : 9, 13 : 2, 281 : 6, 743 : 7, 377 : 3, 20 : 10, 10 : 0, 796 : 4 58. Repeat this procedure with the vertices on level 1, then those on level 2, and so on. We write C ⊂ A. Since each element is greater than or equal to the element to its left, the element is always inserted in its original position. A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C), A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) 31. Essentially the same figure gives tilings if square (1, 2) or (2, 2) is deleted. If s is in nondecreasing order, the algorithm returns the value 0. Similarly, an edge labeled means that the right means that the side of the pan balance is heavier than the left side, and an edge labeled two sides balance. (The trees are isomorphic as rooted trees and as free trees.) The trees T1 and T2 in Figure 8.11 are isomorphic as rooted trees and as free trees. , thus completing the proof of Theorem 2.14. 52 Sets and Logic Example 6.4 Consider the statement ∀x∀y((x > 0) ∧ (y < 0) → (x + y = 0)). P(d) for some d ∈ D ∴ ∃x P(x) Section 5 1. Copyright Pearson Education, Inc. ρ(R1 ∪ R2) = ρ(R1) ∪ ρ(R2) 58. A greedy algorithm is an algorithm that optimizes the choice at each iteration. The arrow diagram emphasizes this view of a function. Thus after just four iterations, 6,250,000 + 125,000 + 2500 + 50 + 1 = 6,377,551 copies of the message had been sent. Thus n must be even. Definition 6.1 Graphs G1 and G2 are isomorphic if there is a one-to-one, onto function f from the vertices of G1 to the vertices of G2 and a one-to-one, onto function g from the edges of G1 to the edges of G2 , so that an edge e is incident on v and w in G1 if and only if the edge g(e) is incident on f(v) and f(w) in G2 . Since the first bit is 0, the first move is right. Finding no defective microprocessors out of five randomly chosen from a lot of 1000. Unlike a set, a sequence takes order into account. We have 6 mod 2 = 0, 5 mod 1 = 0, 8 mod 12 = 8, 199673 mod 2 = 1. Trace Algorithm 2.4 for the input 34 57 72 101 135. The algorithm is correct. Suppose that a > 1 and that f(n) = (loga n). We call I the internal path length. A single person must get the correct order, so D1 = 0. To prove that ∀x∃y P(x, y) is false, show that for at least one x ∈ X, P(x, y) is false for every y ∈ Y , where the domain of discourse is X × Y . j 7. 0 0 0 1 We see that whenever entry i, j in A2 is nonzero, entry i, j in A is also nonzero. ∃x∃y P(x, y) 41. The height of the tree is 2. , w1 , w0) (2.1) is a cycle in T, which is a contradiction. Show that the flow in Figure 4.3 is maximal by exhibiting a minimal cut whose capacity is 3. The largest disk was then moved, which required one additional move. (c) List one member of each equivalence class. What is the minimum number of comparisons required by Algorithm 3.8 to sort an array of size 6? A string over X is a finite sequence of elements from X. When it is impossible to place a token in a column, we backtrack and adjust the token in the preceding column. A ⊕ B = B The symmetric difference of two sets A and B is the set A ⊕ B = (A ∪ B) − (A ∩ B). Bayes’ Theorem gives the desired probability: P(H | Pos) = = P(Pos | H)P(H) P(Pos | H)P(H) + P(Pos | H)P(H) (0.95)(0.15) = 0.893. (See [Ainslie] for the rules.) Notes The following are recommended references on trees: [Berge; Bondy; Deo; Even, 1979; Gibbons; Harary; Knuth, 1997; Liu, 1985; and Ore]. The condition does not guarantee the outcome; but, if the condition does not hold, the outcome will not be achieved. The solution when square (1, 1) is removed is shown in the following figure. Instant Insanity Graph Theory 59. Tree; six vertices having degrees 1, 1, 1, 1, 3, 3 27. Notice that the knight must arrive at an outside square from an inside square and that the knight must move from an outside square to an inside square. 132 n = m − (m − n) = (2k1 + 1) − (2k2 + 1) = 2(k1 − k2). The coding tree is obtained by replacing each frequency with a character having that frequency. Sets X and Y are disjoint if X∩Y = ∅. v1 v2 w1 v4 v3 v6 T1 v5 w2 w3 w4 w5 w6 2. Since a also divides n and a < a, when Algorithm 1.8 sets d = a , it will return a , not a. Number of unordered, k-element selections from a t-element set, repetitions allowed = C(k + t − 1, k) Section 4 12. Show that an = (nlg 3), where an is as in Exercise 64. 102. Next, swap s2 and a randomly chosen element in s2 , . For the example ordering, Bill gets books 1, 2, 3, and 6, Shizuo gets books 4 and 8, and Marian gets books 5 and 7. We then compute the quotient when 14 is divided by 2. If the coin is flipped 10 times, what is the probability of exactly five heads? In Exercises 33–38, formulate the symbolic expression in words using p : Lee takes computer science. At line 5, since key (‘Z’) is not equal to s4 (‘M’) we proceed to line 7.At line 7, key < sk (‘Z’ < ‘M’) is false, so at line 10 we set i to 5. Show that A(x, y) = AO(x, 2, y + 3) − 3 for y ≥ 0 and x = 0, 1, 2. {1, 2} 75. GFBEDCA – E 19. The Inclusion-Exclusion Principle (Theorem 1.12) is a variant of the Addition Principle that can be used when the sets involved are not pairwise disjoint. Since there are n − 1 possible ways for 409 Recurrence Relations person 1 to get the wrong order, the recurrence relation now follows. Take g = {(1, a), (2, b)} and f = {(a, z), (b, z)}. The given statement ∃x∀y(xy < 1) is true because there is at least one x (namely x = 0) such that xy < 1 for every y. , pn have the same truth value: each is true. The sample space is the event consisting of all possible outcomes. , f(vn)) is an Euler cycle in G2 . Exercises Use the Euclidean algorithm to find the greatest common divisor of each pair of integers in Exercises 1–10. 4 104. During the period 6:00 p.m. to 7:00 p.m., the average trip times are A to B 15 minutes B to C 30 minutes A to C 30 minutes. 77. Searching Text search Text-search algorithm Sorting Insertion sort Time and space for algorithms Best-case time Worst-case time Randomized algorithm Shuffle algorithm 15. , pn /∴ c 20. (In words, [a] is the set of all elements in X that are related to a.) Then S = {[a] | a ∈ X} is a partition of X. How many parenthesized expressions are there containing n distinct binary operators, n + 1 distinct variables, and n − 1 pairs of parentheses? (3.1) Before proving (3.1), we illustrate how the Euclidean algorithm uses it to find the greatest common divisor. z=x+y K 2, 3 y=2 K 3, 3 27. In a binary tree with three vertices in which the root has degree 1, the root either has a left child and no right child or has a right child and no left child. List the members of the sample space. The hypothesis is the statement “The Mathematics Department gets an additional $60,000,” and the conclusion is the statement “The Mathematics Department will hire one new faculty member.” What is the truth value of the dean’s statement (3.1)? Several advanced counting techniques use such methods (see [Riordan; and Tucker]). Let Ei be the event “runner completes the marathon on attempt i.” The error in the reasoning is assuming that P(E2) = 1/3 = P(E3). Show that an ≤ an+1 for all n ≥ 1. = n+1 n+1 Multiplying by −1 yields the desired formula. Substitute the second-to-last formula for the remainder into E1 . Proofs Applying (3.1) to expressions 4 and 7, we derive 8. Input: r, n Output: All r-combinations of {1, 2, . 565 Trees Example 9.1 Apply the minimax procedure to find the value of the root in tic-tac-toe using a two-level, depth-first minimax search. Nonadjacent processors communicate by sending messages. 1, 3, 5, 7, 9, 11, 13 84. 1 2 30. Now the entire sequence is shuffled. Guess a formula for A(3, n) and prove it by using induction. This contradicts the hypothesis that there are 40 coins. v1 e1 25. Similarly, if you are asked to find a partition, you can either find the partition directly or construct an equivalence relation and then take the equivalence classes as your partition. Notes Recurrence relations are treated more fully in [Liu, 1985; Roberts; and Tucker]. A ∩ B ∪ C 16. Express each hexadecimal number in Exercises 26–31 in octal. Show that any solution U of (2.20) is of the form Un = Vn + g(n), (2.21) where V is a solution of the homogeneous equation (2.13). The probability of E, P(E), is P(x). Therefore, n is odd. Find P(A | B), P(D | B), and P(N | B). Each of these sequences is recursively sorted, after which they are combined to produce a sorted arrangement of the original sequence. an = b/2n + d3n − (4/3)2n 18. Example 1.3 436 A directed graph is shown in Figure 1.4. The directed edges are indicated by arrows. 3 4 e A to C A to B 6 B 2 5 w3 A 8 w2 30 minutes The maximum capacities of the routes are e 5, 3 A to B C 11. By Theorem 2.18, G has an Euler cycle. Why must every recursive function have a base case? , where cn denotes the number of moves our solution 1 2 3 Figure 1.2 Tower of Hanoi. Given binary trees T1 and T2 , we first check whether either is empty (in which case it is immediate whether they are isomorphic). How many different outfits are possible? Let T be a rooted tree with four vertices in which the root has degree 1. Find the probability that among n persons, at least two people have birthdays on April 1 (but not necessarily in the same year). 388 Recurrence Relations By the inductive assumption, cn−1 = dn−1 . Give alternative characterizations of trees. Suppose that d > 0 is a common divisor of nonnegative integers a and b, not both zero. G is not connected. The traveling salesperson problem is: Given a weighted graph G, find a minimum-length Hamiltonian cycle in G. R as in Exercise 1; ordering of X: 3, 2, 1; ordering of Y : , β, α, δ 3. We call a picture such as Figure 1.1 an arrow diagram. For example, through 2007 a number 16 seed has never defeated a number 1 seed. An interesting problem is to determine whether two graphs are isomorphic. Moving the kn disks on peg 1 to peg 4 requires 2kn −1 moves (see Example 2.4). We use mathematical induction to show that cn = dn , n ≥ 1. By Exercise 72, the sequence {(1+1/n)n+1 }∞ n=1 is decreasing. In the conditional proposition p → q, q is a necessary condition. This result illustrates the problem for p = 2 and q = 5. Use the logic game (Example 6.15) to determine whether the proposition ∀x∀y∃z((x < y) → ((z > x) ∧ (z < y))) is true or false. The path that starts at She, then goes to Buf, and ends at Gil corresponds to a trip on the map of Figure 1.1 that begins in Sheridan, goes to Buffalo, and ends at Gillette. Then a and b are rational numbers and ab = 4 is also rational. Does the graph of Exercise 5 contain an Euler cycle? Notice that if ∃x P(x) is true, the for loop terminates as soon as one element x in the domain of discourse is found for which P(x) is true. We now consider when the n-cube can simulate the mesh model for parallel computation. a, f 2. Notice that the Huffman tree for Table 1.2 is not unique. player [Name, Age] Johnsonbaugh, 22; Glover, 24; Battey, 18; Cage, 30; Homer, 37; Score, 22; Johnsonbaugh, 30; Singleton, 31 23. (−1)k C(n, k). We prove the statement by using induction on n. It can be shown that resolution is correct and refutation complete. What is a recursive function? If the trees are not isomorphic, give an invariant the trees do not share. In words, definition (a) says that a is an upper bound for a set X if every element in X is less than or equal to a. At line 3 we add vertex 1 to the minimal spanning tree. Since kn (kn + 1) , 2 it follows that rn ≤ kn . A direct proof assumes that p(x1 , x2 , . f = {(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)}; f is one-to-one and onto. This means that the first player can always win the game by using an optimal strategy. The domain of discourse of A is S × T and the domain of discourse of E is S. By listing ordered pairs, give an example of an equivalence relation on {1, 2, 3, 4, 5, 6} having exactly four equivalence classes. This form of an expression is called the fully parenthesized form of the expression. Exercises 64–68 refer to a set of five distinct computer science books, three distinct mathematics books, and two distinct art books. In fact, we have just given an explanation! Finding a Solution As noted, we have already solved part (c). In automated reasoning systems, proof by resolution is combined with proof by contradiction. Show that the resulting graph always has a Hamiltonian cycle. The following table summarizes the various formulas: Ordered Selections Unordered Selections No Repetitions Repetitions Allowed n! C(n, r) n!/(n1 ! · · · nt !) C(k + t − 1, t − 1) 337 Counting Methods and the Pigeonhole Principle Section Review Exercises 1. 1, 3, and 5 are also equally likely to appear, but 1 is three times as likely as 2 is to appear. A hash function takes a data item to be stored or retrieved and computes the first choice for a location for the item. It is denoted X ∩ Y . State the identity laws for sets. During these moves, the bottom disk on peg 1 stays fixed. , vn) and P2 = (w0 , . We say that the least upper bound for a set X is unique. , 6, repetitions allowed. 391 Recurrence Relations Exercises and Exercise 20, show that In Exercises 1–3, find a recurrence relation and initial conditions that generate a sequence that begins with the given terms. Let us denote a particular selection as CS × × Physics |× ×| History × Here we have designated the selection consisting of three computer science books, two physics books, and one history book. , cn to mean that person 1 got order c1 , person 2 got order c2 , and so on. At lines 19 and 20 we label vertex b as (a, 1), and we label vertex d as (a, 5). From the first part of the proof, we can deduce A ⊆ B. Thus, “to be determined” is 4 · 5n , which is divisible by 4. Since R and S are reflexive, (x, x) ∈ R and (x, x) ∈ S. 279 Introduction to Number Theory a. Example 3.14 It is sometimes more convenient to replace if (x, y) ∈ R and (x, y) ∈ R, then x = y in the definition of “antisymmetric” (Definition 3.12) with its logically equivalent contrapositive if x = y, then (x, y) ∈ R or (y, x) ∈ R to obtain a logically equivalent characterization of “antisymmetric”: A relation R on a set X is antisymmetric if for all x, y ∈ X, if x = y, then (x, y) ∈ R or (y, x) ∈ R. 3 3 3 27 Counting Methods and the Pigeonhole Principle Thus, the probability of completing the marathon is P(E1 ∪ E2 ∪ E3) = P(E1) + P(E2) + P(E3) 1 2 4 19 = + + = = 0.704, 3 9 27 27 which means that there is about a 70 percent chance that the runner will complete the marathon—not exactly a virtual certainty! + n C(n, k)an−k bk+1 k=0 = n C(n, k)an+1−k bk k=0 n+1 + C(n, k − 1)an+1−k bk k=1 Section 7 Review 1. By definition, the statement ∀x∃y P(x, y), with domain of discourse X × Y , is true if, for every x ∈ X, there is at least one y ∈ Y for which P(x, y) is true. Conversely, if c is a common divisor of b and r, then c | bq and c | bq + r (= a) and c is a common divisor of a and b. The Basis Step, n = 1, has already been established. In order for proposition (3.4) to be true in this case, we must define p → q to be true when p is false and q is true. e2 a 39. Example 1.39 The function f(x) = 2x is a one-to-one function from the set R of all real numbers onto the set R+ of all positive real numbers. (a) b 11. AB+CD+EF/−−A∗ In Exercises 16–21, find the value of the postfix expression if A = 1, B = 2, C = 3, and D = 4. Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that (4.9) is true for every positive integer n. Algorithm 3.5 Merging Two Sequences This algorithm combines two nondecreasing sequences into a single nondecreasing sequence. 92. Definition 1.10 Example 1.11 If x is an integer and y is a positive integer, we define x mod y to be the remainder when x is divided by y. (Recall that P denotes the complement of P.) Then the source a is in P and the sink z is in P. 100. G2 v1 G3 G4 Figure 2.6 The four subgraphs of the graph of Figure 2.5. We can now define “component.” Definition 2.11 Let G be a graph and let v be a vertex in G. The Inductive Step will assume the truth of the statement involving cn/2 . Only after obtaining a subsequence of length 1 or 2, do we test whether the subsequence contains the key. (A ∪ B) ∪ C = A ∪ (B ∪ C), (A ∩ B) ∩ C = A ∩ (B ∩ C) 37. If P(x) is a statement involving the variable x, we call P a propositional function if for each x in the domain of discourse, P(x) is a proposition. Now w = ab − q1 z = (q2 z + x)(q3 z + y) − q1 z = (q2 q3 z + q2 y + q3 x − q1)z + xy = qz + xy, where q = q2 q3 z + q2 y + q3 x − q1 . State and prove a result for similar to that for Exercise 39. (4.1) (4.2) We claim that (4.1) and (4.2) imply that every block is marked. In this case, if we increase the flow in each edge by , the 598 Network Models a a a a e1 x e2 e1 x e2 e1 x e2 e1 x e2 z (a) z (b) z (c) z (d) Figure 2.4 The four possible orientations of the edges incident on x. (If p is true and q is false, then p → q is false.) Now, suppose that the Mathematics Department does not get an additional $60,000. We will also use the notation p: 1 + 1 = 3 to define p to be the proposition 1 + 1 = 3. Thus the number of ordered pairs (A, B) satisfying A ⊆ B ⊆ X is equal to the number of ways to assign the elements of X to the three sets A, B − A, and X − B. Show that if a flow exists in G, a maximal flow exists in G with value min{C(P, P) − m(P, P) | (P, P) is a cut in G}. Three departmental committees have 6, 12, and 9 members with no overlapping membership. 1, 5, 9, 13, 17, 21, 25 85. For example, suppose that a particular algorithm is implemented in C++ by a number of persons and we want to group “like” programs into classes based on certain properties of the programs (see Table 1.2). Let E be an edge that is incident on f(v) in G2 . 88 Proofs If n − 1 is even, then n − 1 = 2k for some integer k. Let Rn denote the number of regions into which the plane is divided by n lines. We next show that we can make postage for n, n + 1, . 1 C(n − 1, j)Ej En−j−1 . Write the adjacency matrix of the graph of Exercise 5. if (si < small) 25. Let CALC, PSYCH, and COMPSCI denote the sets of students taking calculus, psychology, and computer science, respectively. The other tree in X2 is obtained by inserting two new children of v—one is a new right child, which is marked, and the other is the root of the original subtree in T rooted at v. The Euclidean algorithm next computes r = 53 mod 4 = 1. n+1 by (3.1) Therefore, the average-case time of Algorithm 3.17 is O(n). (Define E0 = 1.) The numbers E0 , E1 , E2 , . If T were acyclic, T would be a tree with n edges and n vertices. f(x) = x − x 7. Tree; all vertices of degree 2 Answer the questions in Exercises 7–15 for the following tree. (1.13) Ackermann’s function is of theoretical importance because of its rapid rate of growth. After the virus sent the e-mail to the first 50 addresses, each of those recipients then sent e-mail to 50 addresses. An argument is valid because of its form, not because of its content. Use induction to prove that 1 = n, n1 · n2 · · · nk for all n ≥ 1, where the sum is taken over all nonempty subsets {n1 , n2 , . (2.6) If we next divide 5 by 2, we find 5 = 2 · 2 + 1. Definition 3.1 A tree T is a spanning tree of a graph G if T is a subgraph of G that contains all of the vertices of G. Some authors do not permit loops and parallel edges when they define graphs. Write a breadth-first search algorithm to test whether a graph is connected. The while loop will execute as long as p ≤ m and q ≤ j. Suppose that we replace the line Output: k = (i + j)/2 with k = i + (j − i)/3 in Algorithm 3.2. Is the resulting algorithm still correct (i.e., does it find key if it is present and return 0 if it is not present)? 29. 276 16)20385 remainder = 1 1’s place 16)1274 remainder = 10 16’s place 16)79 remainder = 15 162 ’s place 16)4 0 remainder = 4 163 ’s place Introduction to Number Theory We stop when the quotient is 0. State Hall’s Marriage Theorem. ∴q Determine whether each argument in Exercises 11– 15 is valid. The length of a shortest path from a to z is 4. True 40. They are siblings. If S is a partition of X, we may define x R y to mean that for some set S ∈ S, both x and y belong to S. i=3 Now fk1 +1 ≤ n and fk1 +1 > fk1 . Modify Algorithm 2.3 so that it sorts the sequence s1 , . Since R is symmetric, (a, b) ∈ R. The question of whether there is a polynomial-time algorithm that can factor an integer is of more than academic interest since the security of certain encryption systems relies on the nonexistence of such an algorithm (see Section 4). Notes [D’Angelo; Solow] address the problem of how to construct proofs. Prove that if a binary tree of height h has n ≥ 1 vertices, then lg n < h+1. This tree is not a minimal spanning tree since spanning tree T shown in Figure 4.3 has weight 12. where f denotes the Fibonacci sequence. If f(n) = O(g(n)), then for every C > 0, |f(n)| > C|g(n)|. Therefore, G = gcd(a, b) is true for the new values of a and b. Show that AO(x, 2, 2) = 4 for x ≥ 2. Write a program that tests whether A is onto a given set. (3.2) Using the generalized De Morgan’s laws for logic and the fact that ¬(p → q) ≡ p ∧ ¬q, we find that (3.2) is equivalent to ∃x∃y[(x, y) ∈ R ∧ (y, x) ∈ R] ∧ ¬[x = y]] which, in turn, is equivalent to ∃x∃y[(x, y) ∈ R ∧ (y, x) ∈ R ∧ (x = y)]. Since we have proved both equations (1.3) and (1.4), it follows that X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z). Again by Theorem 2.3, T has n − 1 edges. Define C(P, P) = 1. walk(n) = fn+1 s1 = 2, for all n ≥ 2, to write a recursive algorithm that computes sn = 1 + 2 + 3 + · · · + n. What is a serial algorithm? We can partition the vertices into sets V1 , those corresponding to the white squares, and V2 , those corresponding to the black squares; each edge is incident on a vertex in V1 and V2 . Since 331,776 different stacks are possible (see Exercise 12), a solution by hand by trial and error is impractical. Write each proposition in Exercises 28–31 symbolically. v1 v3 v2 v4 v5 v6 v7 Figure 1.4 The tree Figure 1.3(a) with the root at the top. We give the formal definitions beginning with subgraph. , ak ∈ X such that f(a1) = f(a2) = · · · = f(ak). (a) Verify the Basis Step (p = 1). A = {1, 2, 3, 4}, C = {5, 6, 7, 8}, B = {n | n ∈ A and n + m = 8 61. Define r(x) = 1, for x = A, B, . In this setting, we consider the first element of the ordered pair to be related to the second element of the ordered pair. The while loop first tests whether “110” occurs in t at index 1. 2n 2 · 4 · 6 · · · (2n) 13. 6n + 1 +1 3. 37. R × R × R 71. Modify Algorithm 3.7 so that it tracks the parent p of a vertex c (p is the parent of c if c was visited from p). , sn in the following way. However, it should be noted that it is possible to give a precise, mathematical definition of “algorithm.” Although ordinary language is sometimes adequate to specify an algorithm, most mathematicians and computer scientists prefer pseudocode because of its precision, structure, and universality. Recursively invoking this algorithm to move the n − kn disks at the top of peg 1 to peg 2 takes T(n − kn) moves. Given f = {(x, x2) | x ∈ X}, a function from X = {−5, −4, . For strings of unequal length, lexicographic order may be different than numerical order (see Example 4.3). Example 2.13 Searching the Web A variety of Web search engines are available (e.g., Google, Yahoo, AltaVista) that allow the user to enter keywords that the search engine then tries to match with Web pages. If X ∩ Y = ∅, then CX∪Y (x) = CX (x) + CY (x) for all x ∈ U. Let R be a relation on a set X. Let P(n) be the propositional function “n divides 77.” Write each proposition in Exercises 7–11 in words and tell whether it is true or false. The given propositions, such as (4.1), are called hypotheses or premises, and the proposition that follows from the hypotheses, such as (4.2), is called the conclusion. Explain how we would look for MORE in the binary search tree of Exercise 19. Find the height of the tree of Exercise 8. A graph is planar if and only if it does not contain a subgraph homeomorphic to K5 or K3,3 . It can be shown (see Exercise 97) that {(y, x) | (x, y) ∈ f } 145 Functions, Sequences, and Relations is a one-to-one, onto function from Y to X. 4570 48. Let n = |V | and let (P, P) be a minimal cut in the matching network. Once Egbert has been assigned, there are five ways to fill the highest remaining office. If f(n) = O(g(n)), then g(n) = O(f(n)). C(10, 5)/210 34. 268 Introduction to Number Theory If you compute the gcd(a, b), you can immediately compute the least common multiple using the formula lcm(a, b) = ab . 8!P(9, 5) = 8!(9 · 8 · 7 · 6 · 5) 21. Example 1.23 Using the notation of Theorem 1.22, we have 82320 = 24 · 31 · 51 · 73 · 110 and 950796 = 22 · 32 · 50 · 74 · 111 . Since g is onto, there exists x ∈ X such that g(x) = y. Thus, in the notation of Exercise 64, n = 100,000, i = 16, and j = n − 2i = 100,000 − 216 = 100,000 − 65,536 = 34,464. A parallel computer can execute several instructions at a time. If T had a vertex v of degree greater than 4, v would be incident on more than four edges. Find a formula for zn−1 . Let E1 denote the event “children of both sexes,” and let E2 denote the event “at most one boy.” Then P(E1) = 14 , 16 P(E2) = 5 , 16 and P(E1 ∩ E2) = P(E ∩ F) = P(E ∪ F) = 1 − P(E ∪ F) = 1 − [P(E) + P(F) − P(E ∩ F)] 4 . 537261 48. It is denoted ¬p. Since 1 < n, 2 ≤ n. Give the truth table for the conjunction of p and q. Define directed graph. Therefore there are 5152 integers between 1 and 10,000 that are multiples of 3 or 5 or 11 or any combination thereof. Consider 9. 1 0 1 1 1 1 0 1 0 1 1 0 0 0 0 1 0 0 0 0 1 1 1 1 0 1 1 0 1 0 1 1 1 1 0 0 0 0 0 1 1 0 1 1 0 0 0 0 1 0 1 1 1 1 0 0 0 0 1 Figure 9.2 The game tree of Figure 9.1 showing the values of all vertices. Thus an is equal to the Catalan number C(2n, n)/(n + 1). Similarly, each consecutive pair of strings in Gn−1 differs in exactly one bit. Such edges form a path (a, v, c) of length 2 from a to c and each path increases the sum by 1. Thus 1 1 1 1 1 p = D + + ··· + = + ··· + + + ··· + q 1 n n1 nk 1 n is represented in Egyptian form. Section 8 Review 1. Example 3.14 Find a theta notation in terms of n for the number of times the statement x = x + 1 is executed: 1. Suppose that a ∈ f −1 ({y}) ∩ f −1 ({z}) for some y, z ∈ Y . The security of the RSA encryption system relies mainly on the fact that currently there is no efficient algorithm known for factoring integers. If we let α = βR , we have f(α) = αR = (βR)R = β since if we twice reverse a string, we obtain the original string. Write an algorithm that returns the index of the first item that is greater than its predecessor in the sequence s1 , . 15, 159 21. In practice, n is often chosen to be a prime. The computer was then used again to analyze the resulting cases. Finally, to obtain the optimal Huffman coding tree, we replace each frequency by a character having that frequency (see Figure 1.15). If x < y, max{x, y} = y and min{x, y} = x. Find all managers of departments that produce parts for ABC Unlimited. We define the evaluation function Ea , where a ∈ R, from RR to R as Ea (f) = f(a). C(15 + 3 − 1, 15) 25. Find the set, denoted [x1], of all elements related to x1 . How many selections are there in which either Connie is chairperson or Alice is an officer or both? What is the value of E5 ? Let E1 denote the event “the coin shows a tail,” let E2 denote the event “the die shows a 3,” and let E3 denote the event “the coin shows heads and the die shows an odd number.” 14. Then y ∈ Y and f(n) = y for all n ∈ X. , w1 , w0) l q r What can you say about two vertices in a rooted tree that have the same parent? , xn). Under My Pictures are archived, basement water, and My 509 Trees Figure 1.7 Windows Explorer display of folders and files on a particular computer. 0 (e) {(1, b), (1, a), (1, c), (2, b), (3, b)} Section 5 Review 1. The situation can be modeled by the graph of Figure 4.1. The vertices represent the applicants and the jobs. 12435 7. The recipient then decrypts the message using the hidden decryption key. If n is a power of m, say n = mk , then an = amk = amk−1 + d = [amk−2 + d] + d = amk−2 + 2d . The requirement that the intermediate results of each step of execution be uniquely defined and depend only on the inputs and results of the preceding steps is violated. Definition 1.1 Let X and Y be sets. Notice that if v and w are vertices in this graph, there is a unique simple path from v to w. , sm , sm+1 , . Prove that bn = n − 1 for every positive integer n. First we consider the problem of listing all r-combinations of {1, 2, . (For Exercise 2) 1. For the general linear homogeneous recurrence relation of order k with constant coefficients (2.5), if r is a root of t k − c1 t k−1 − c2 t k−2 − · · · − ck = 0 of multiplicity m, it can be shown that r n, nr n, . Example 6.21 Telemarketing At the telemarketing firm SellPhone, Dale, Rusty, and Lee make calls. If, no matter what value Farley chose, you can choose a value for y that makes P(x, y) true, you can always win the game and the propositional function is true. Example 5.24 We are given these hypotheses: Everyone loves either Microsoft or Apple. Put all vertices adjacent to v in V2 . Prove that for all integers m and n, if m is odd and n is even, then mn is even. 38. If m = 0, A(m, n + 1) = A(0, n + 1) = n+2>n+1 = A(0, n) = A(m, n). Are the events “there is at most one boy” and “there is at most one girl” independent? Define X and Y are disjoint sets. 82 Proofs Thus n is even. Getting that job requires knowing someone who knows the boss. A recurrence relation defines a sequence by giving the nth value in terms of certain of its predecessors. Note that “not symmetric” is not necessarily the same as “antisymmetric.” To prove that a relation R on a set X is not antisymmetric, find x, y ∈ X, x = y, such that (x, y) ∈ R and (y, x) ∈ R. Thus, in the worst case, Algorithm 3.5 requires j − i comparisons. If there are n > 1 disks, recursively move n−1 disks to an empty peg. Thus R is an equivalence relation. How many ways are there to select 10 comics if we choose at least one of each book? Then Y − Z = {1, 2} and (X ∪ Y) − (X ∪ Z) = {2}. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is D1 × D1 . The pair f, g of functions is an isomorphism of M onto the 476 Graph Theory subgraph (V, E) of the (i + j)-cube where V = {f(v) | v is a vertex in M}, E = {g(e) | e is an edge in M}. eBooks. The equation an = c1 an−1 + c2 an−2 + f(n) (2.20) is called a second-order, linear inhomogeneous recurrence relation with constant coefficients. By the Multiplication Principle, there are Ck−1 Cn−k good routes in an n × n grid that first meet the diagonal at (k, k). Since m = 0, the algorithm terminates. (5.4) Combining (5.3) and (5.4), we obtain t = t1 + t2 ≤ 2h1 + 2h2 ≤ 2h−1 + 2h−1 = 2h . Y × X 59. (For Exercise 10) False; Erin is not taller than or the same height as Garth. Again by the definition of intersection, since x ∈ / Z, x ∈ / X ∩ Z. Prove that Y has a least upper bound, say a. 8. 120 Proofs But the only integer strictly between −d and d divisible by d is 0. Now 125. Suppose that we have n dollars and that each day we buy either tape ($1), paper ($1), pens ($2), pencils ($2), or binders ($3). Step 3: You go back to work on the books. If no such value can be found, we backtrack to the last position where we assigned a value; if that value was i, we try i + 1, i + 2 and so on. What is the geometric sum? , vk) is a simple cycle of length k in G1 , then (f(v0), f(v1), . (d) Finds the descendants of v. After the vertices at the lowest level are assigned values by using the function E, the minimax procedure can be applied to generate the values of the other vertices. R is symmetric because, if s1 and s2 have the same number of zeros, then s2 and s1 have the same number of zeros. However, if we use proof by contradiction, we may assume √ that 2 is rational. As shown, P ends at v3 . 240 Algorithms Inductive Step Assume that Algorithm 4.2 correctly returns the value of (n − 1)!, n > 0. A Class C Internet address, used for small networks, is a bit string of length 32. What is conservation of flow? Thus (4.13) holds for n + 1 and the inductive step is complete. .), w4 v w3 w2 A coloring of a graph G by the colors C1 , C2 , . Let f be a function from X to Y . Consider what happens after b divisions. Let v and w be distinct vertices in Kn , n ≥ 2. If R and S are antisymmetric, then R ◦ S is antisymmetric. True 24. , sj has length greater than or equal to 3 while (i < j − 1) { k = (i + j)/2 if (sk < key) i=k+1 else j=k } for k = i to j if (sk == key) return k return 0 } We first prove that if a sequence of length n is input to the while loop, where n is a power of 2, say n = 2m , m ≥ 2, the loop iterates m − 1 times. [Brassard, Cormen, and Johnsonbaugh] also treat graphs and graph algorithms. Input: n Output: d is prime(n) { √ for d = 2 to n if (n mod d == 0) return d return 0 } Example 1.9 To determine whether 43 is prime, Algorithm 1.8 checks whether any of √ 2, 3, 4, 5, 6 = 43 divides 43. Construct two optimal Huffman coding trees for the table of Exercise 24 of different heights. Implement Algorithm 3.10, computing s and t satisfying gcd(a, b) = sa + tb, as a program. a e c f b d is K3, 3 31. We must show that x ∈ B. Finally, we determine that the root I has value 8. Then in the list x1 , . Give an intuitive interpretation of how f and g are related if f(n) = (g(n)). [(c) → (a).] We assume that A ∪ B = B, and prove that A ⊆ B. Sentence (e) is neither true nor false [sentence (e) is a command]. For the argument as a whole to be valid, each step of the argument must result in a valid, intermediate conclusion. Thus there are bn−1 comparisons at line 12. Find g(S), g(T), g−1 (U), and g−1 (V). (d) Find a formula for sn−1 . Thus, if G1 and G2 are isomorphic, then G1 and G2 have the same number of edges and the same number of vertices. Example 3.1 How many strings can be formed using the following letters? A closed path is a path from v to v. If the relation is an equivalence relation, list the equivalence classes. 597 Network Models 2 ➜ A Maximal Flow Algorithm If G is a transport network, a maximal flow in G is a flow with maximum value. an = 6an−1 − 8an−2 ; Rustic County and that the population increases 20 percent each year. As further evidence of our conjecture, if we compute the smallest input pair that requires six modulus operations, we obtain a = 21 and b = 13. Compute b3 and b4 . , Bk denote the row after B0 , . , i. There is one eight-bit string with no two 0’s in a row that has exactly eight 1’s. Four vertices each of degree 1 f Find the degree of each vertex for the following graphs. Define the truth table for imp2 by 61. A tree T is a spanning tree of a graph G if T is a subgraph of G that contains all of the vertices of G. We then find all nonisomorphic binary trees with three vertices in which the root has degree 1. C1 : C3 C1 C2 C1 : C3 C1 : C3 C3 C4 C3 C2 C1 Section 7 Review 1. Our preceding result shows that [x] = [a] and [x] = [b]. Is t nonincreasing? Begin with the string 12 · · · r. n−1 Suppose that α is in L because of rule 2. When this is proved, correctness of the algorithm follows since when z is chosen at line 9, L(z) will give the length of a shortest path from a to z. The Euclidean algorithm first computes r = 273 mod 110 = 53. To prove that ∀x∀y P(x, y) is false, where the domain of discourse is X × Y , find one value of x ∈ X and one value of y ∈ Y (two values suffice—one for x and one for y) that make P(x, y) false. Which are proper subsets of {a, b, c, d}? , ek , incident on vi and vj . Find the part numbers of parts of which there are at least 100 items on hand. That 6 does not have an inverse modulo 15 does not contradict the result preceding Example 3.9. In order to guarantee that n has an inverse modulo φ, the result preceding Example 3.9 requires that gcd(n, φ) = 1. Two additional comparisons are required at lines 10 and 14. (If there are n persons, person 1 gets either order 2 or 3 or . By trial-and-error, we find that f(2) = f(4). The vertices adjacent to a vertex v and on the next-lower level are the children of v. Similarly, if the first day we buy milk or beer, there are Rn−2 ways to spend the remaining dollars. Justify your steps. [We can represent n in considerably less space than (n); see Example 2.1.] We say that Algorithm 1.8 is not a polynomial-time algorithm. Let pm denote the number of paths of length m from v to w in Kn , 1 ≤ m ≤ n. We begin by giving a formal definition of path. Exercise 6, Section 1 13. Example 3.16 Determine, in theta notation, the best-case, worst-case, and average-case times required to execute Algorithm 3.17, which follows. temp := buyer [Part No = 20A8] temp [Name] Underhanded Sales, Danny’s, ABC Unlimited 14. The process of combining two sorted sequences is called merging. , vn Output: A spanning tree T dfs(V, E) { // V = vertices of spanning tree T ; E = edges of spanning tree T // v1 is the root of the spanning tree V = {v1 } E = ∅ w = v1 while (true) { while (there is an edge (w, v) that when added to T does not create a cycle in T) { choose the edge (w, vk) with minimum k that when added to T does not create a cycle in T add (w, vk) to E add vk to V w = vk } if (w == v1) return T w = parent of w in T // backtrack } } Exercise 21 is to prove that Algorithm 3.7 finds a spanning tree. Show that G cannot have a vertex of degree 4. [McNaughton] contains a very thorough discussion on an introductory level of what an algorithm is. When n = 1, we obtain 1 = A + B + C. If x ∈ B, we have the desired conclusion. Show that when the two binary trees shown in Figure 8.15 are input to Algorithm 8.13, the number of comparisons with null is equal to 6k + 4. p q p∨q T T F F T F T F T T T F 39. If X = {1, 2} and Y = {2, 3}, then X is not a subset of Y since 1 ∈ X, but 1 ∈ / Y . For example, there is an edge between Ellen Barkin and Dennis Quaid because they both appeared in The Big Easy. Planar b a b c f 34. Prove that your property is an invariant. Apply the result of Example 3.9 to the inner k − 1 nested loops of that example. 7 42. (= is the assignment operator.) The idea of the algorithm is to inspect the numbers one by one and copy the largest value seen into a variable large. For example, in the mid-1990s Paul Kocher proposed a way to break RSA based on the time it takes to decrypt messages (see [English]). 30. 54 Sets and Logic By definition, the statement ∃x∃y P(x, y), with domain of discourse X × Y , is true if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is true. If a and b are real numbers, we define max{a, b} to be the maximum of a and b or the common value if they are equal. This time we delete the edge x but no vertices to obtain the graph G (see Figure 7.10). 77 51 44 i j repeaters(s, n) { i=1 while (i < n) { if (si == si+1) println(si) // skip to next element not equal to si j=i while (i < n ∧ si == sj) i=i+1 } } Input: Output: 15 2 77 Now 15 is inserted 44 96 We next swap ai and aj , where i = 4 and j = rand(4, 5) = 5. is defined by the recurrence relation gn = gn−1 + gn−2 + 1, k−1 C(k, i)S(n, i). Since every cycle has at least four edges, each face is bounded by at least four edges. Define set X is equal to set Y . 24. What are pseudorandom numbers? Algorithm 4.1 finds the length of a shortest path from a to z. kruskal(E, w, n) { V = ∅ 29. For such a proof to be correct, it must be the case that each if-and-only-if statement is true. A path P in a planar graph has a crossing if a vertex v appears at least twice in P and P crosses itself at v; that is, P = (. 2 Therefore, assuming that equation n is true, we have proved that equation n + 1 is true. To prove that a function f from X to Y is one-to-one, show that for all x1 , x2 ∈ X, if f(x1) = f(x2), then x1 = x2 . 3 a20 + 3 k k−1 ·2 + 3 1 k−2 k − 1 < logm n ≤ k. Use proof by cases to prove that |x + y| ≤ |x| + |y| for all real numbers x and y. 137 Functions, Sequences, and Relations Example 1.9 The graph of the function f(x) = x2 is shown in Figure 1.4. y f(x) (1, 3) 4 (2, 2) (1, 1) 1 2 1 (0, 0) (3, 0) x 1 x 2 Figure 1.4 The graph of f(x) = x2 . The problem is to identify the bad coin and determine whether it is heavier or lighter than the others using only a pan balance (see Figure 7.2), which compares the weights of two sets of coins. The while loop terminates when b becomes 0. The remainder gives the 1’s bit. For this binary tree, lg t = h. If we rotate Figure 1.2, it looks like a natural tree (see Figure 1.3). Therefore, lg n! = (n lg n). Restate Algorithm 2.4, applied to a matching network, in terms of operations on the matrix A of Exercise 12. What are the Basis Steps? Also, f(n) = (g(n)) if, except for constant factors and a finite number of exceptions, f is bounded above and below by g. This statement is true because we can find at least one positive integer n that makes the conditional proposition if n is prime, then n + 1, n + 2, n + 3, and n + 4 are not prime true. Example 2.6 We prove that for every real number x, x ≤ |x|. A collection S of nonempty subsets of X is a partition of X if every element in X belongs to exactly one member of S. .} 96. (Another technique is discussed in Exercises 67–70.) For example, consider the sum 1 + 3 + · · · + (2n − 1). In this case, the propositional function is false; the values Farley found provide a counterexample. A relation is reflexive if and only if the main diagonal of its matrix representation consists of all 1’s. FROM STAR TREK VI: THE UNDISCOVERED COUNTRY This chapter begins with sets. Theorem 3.7 states that there exist integers s and t such that 1 = gcd(p, q) = sp + tq. Therefore, R is not transitive. In Example 6.22, how small would P(H) have to be so that the conclusion would be “no HIV” even if the result of the test is positive? When n ≥ 2, a Gray code (3.1) corresponds to the Hamiltonian cycle s1 , s2 , ..., s2n , s1 since every vertex appears and the edges (si , si+1), i = 1, . An alternative way to state “p if and only if q” is “p is a necessary and sufficient condition for q.” The proposition “p if and only if q” is sometimes written “p iff q.” 25 Sets and Logic Example 3.9 The proposition 1 < 5 if and only if 2 < 8 (3.5) can be written symbolically as p↔q if we define p : 1 < 5, q : 2 < 8. ∀x P(x) 22. f −1 (y) = (y − 2)/4 35. Suppose that vRw and wRx. Then there is a path P1 from v to w and a path P2 from w to x. We use proof by contradiction to show that r < d. In Theorem 5.14, part (b), ¬P1 ∧ ¬P2 ∧ · · · ∧ ¬Pn is replaced by ∀x ¬P(x) and ¬(P1 ∨ P2 ∨ · · · ∨ Pn) is replaced by ¬(∃x P(x)). Exercises In Exercises 1–3, show that each graph is planar by redrawing it so that no edges cross. The sets X, Y , and Z are subsets of a numbers in color indicate that a hint or solution appears at the end of this chapter. and Therefore, xi + xj is even and yi + yj is even. If X = Y , we call R a (binary) relation on X. Solving for x, we find that x = 2 or x = −3. r−1 a + ar 1 + ar 2 + · · · + ar n + ar n+1 = Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that (4.12) is true for all n ≥ 0. A beta cutoff occurs at a circle vertex when a grandchild w of v has a value greater than or equal to the beta value of v. Given a string s1 . Since R is transitive, (b, b) ∈ R. defined by the equations c1 = 0, cn = cn/2 + n for all n > 1. Example 2.3 In the graph of Figure 2.1, the path (6) consisting solely of vertex 6 is a path of length 0 from vertex 6 to vertex 6. The capacities are shown on the edges. By Theorem 3.2, this number is 8! = 420. Prove that if f is a function from the finite set X to the finite set Y and |X| > |Y |, then f is not one-to-one. Suppose that f is one-to-one and onto. Let f be a one-to-one function from X = {1, 2, . Let An represent the amount at the end of n years. We prove that pow = is a loop invariant for the while loop. The front and back faces are now properly aligned. If it contains a cycle, we remove an edge from this cycle. Thus our proof outline becomes X, Y , and Z are sets. Define subsequence. Input: s, n Output: large (the largest value in the sequence s) max(s, n) { large = s1 for i = 2 to n if (si > large) large = si return large } We verify that Algorithm 1.2 is correct by proving that large is the largest value in the subsequence s1 , . As an example of a greedy algorithm that does not weight incident on the most lead to an optimal solution, consider a “shortest-path algorithm” in which at each step recently added vertex does we select an available edge having minimum weight incident on the most recently added not necessarily yield a shortest vertex. Y × Y In Exercises 61–64, let X = {1, 2}, Y = {a}, and Z = {α, β}. 6d. Assume that the last person to take a token wins. ∀x ∀y P(x, y) might be false. We previously introduced the notation n mod d for the remainder. small = s1 5. The edges represent the dominoes: There is one edge between each distinct pair of vertices and there is one loop at each vertex. In general, there will be several flows having the same maximum value. As shown, in each puzzle some numbers are given. {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)} 5. Find the probability that Joe does not pass. Suppose, by way of contradiction, that n is not in X. A beta value is an upper bound for a circle vertex. Multiplying by n gives mn = d(qn). Thus, 6 does not have an inverse modulo 15. Suppose that the matrix of the relation R1 on {1, 2, 3} is 1 0 0 1 1 0 20. How many six-card hands chosen from an ordinary 52card deck contain three cards of one suit and three cards of another suit? Dividing by t and noting that t > 0, we have x ≤ b/t for every x ∈ X. For many years it was believed that the minimum time to multiply two n × n matrices was (n3), thus it was quite a surprise when a more efficient algorithm was discovered. If Farley can always win the game, this means that Farley can find values for x and y that make P(x, y) false. It follows that lg n! = (n lg n). Such a string can also Counting Methods and the Pigeonhole Principle be considered a path of the type described in Example 2.23, where W is a move right and U is a move up. We return 2! · 3 = 6. Add an extra one to the beginning of the string, and delete the last zero. The Fibonacci sequence pops up in unexpected places. ⩾ / * + A C D E B Figure 6.7 The binary tree representation of the expression (A + B) ∗ C − D/E. At least one raised dot must be present. It continues until the sequence is sorted. Use resolution and proof by contradiction to reprove Exercises 2–6. 35. It takes one additional multiplication to multiply a1 · · · at by at+1 · · · an . What is a spanning tree? In the preceding paragraphs, we proved that if n = 4 the loop iterates one time. Now n+1 C(n + 1, i)−1 i=1 = n+1 1 2 1 = 2 380 C(n + 1, i)−1 + i=1 n C(n + 1, i + 1)−1 i=0 −1 C(n + 1, 1) . Write the incidence matrix of the graph of Exercise 5. The three remaining subboards form a 2n × 2n L-shape, which may be tiled using Exercise 37. , Xn . Let us consider the problem of filling 11 blanks, — — — — — — — — — — —, with the letters given. We first show that d ≤ d1 and d ≤ d2 . , n satisfying p(i) < p(i + 1) 74. Proof by Contradiction A proof by contradiction establishes p → q by assuming that the hypothesis p is true and that the conclusion q is false and then, using p and ¬q as well as other axioms, definitions, previously derived theorems, and rules of inference, derives a contradiction. How many bits are required to represent the positive integer n? Basis Step 12 = (1 · 2 · 3)/6 Inductive Step Assume true for n. In this case, x ∈ Y − X. Definition 8.4 Let T1 be a rooted tree with root r1 and let T2 be a rooted tree with root r2 . Therefore, 4 is the least upper bound for X. 2 Assuming that this equation is true, we must show that equation n + 1 Sn+1 = (n + 1)(n + 2) 2 (4.5) Proofs is true. (8.1) Now 2+L+R=n (8.2) because the vertices comprise the two roots, the vertices in the left subtrees, and the vertices in the right subtrees. What is the power set of X? Then 2. Then the set on the left-hand side of equation (*) contains two distinct sets: {a} and {a, b}. There are two ways to parenthesize a1 ∗ a2 ∗ a3 , namely, ((a1 ∗ a2) ∗ a3) and (a1 ∗ (a2 ∗ a3)). {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)} 3. If executed by a computer, Algorithm 4.2 would typically not be as efficient as a nonrecursive version because of the overhead of the recursive calls. d f e1 24. 21 47. −4 19. The formula now follows. Since s3 < t3 , by condition (b) of Definition 4.1, α < β. If n ∈ Z, n can be expressed as a quotient of integers, for example, n = n/1. 578 perm(n) { k=1 p(1) = 0 while (k > 0) { p(k) = p(k) + 1 while (p(k) ≤ n ∧ p(k) conflicts) p(k) = p(k) + 1 if (p(k) ≤ n) if (k == n) println(p) else { k =k+1 p(k) = 0 } else k =k−1 } } 40. b b h 41. c 6. Explain how a proof by resolution proceeds. , a, . 6 7 6 T2 3 7 29. Thus x ∈ A or x ∈ B or x ∈ C. By the generalized De Morgan’s laws for logic, a relation R on X is not reflexive if there exists x ∈ X such that (x, x) ∈ R. What is a direct proof? There are C((n − 1) + m − 1, n − 1) Notice that the only variation in the preceding figures occurs between the start and the circled dot. Lee does not take computer science. 4 g b 5 e 6 7 b 1 c f a 5 h g j k 5 c 8 3 h g 7 4 c G1 6 5 G2 16. Now n2 = 2(2k2); thus n2 is even, which is a contradiction. Modify Algorithm 2.1 so that it finds all occurrences of p in t. Write the negation of each proposition in Exercises 44–47 symbolically and in words. We give Golomb’s inductive proof (see [Golomb, 1954]) that if we remove one square from an n × n board, where n is a power of 2, we can tile the remaining squares with right trominoes (see Figure 4.5). Substituting the next-to-last formula for 2 into the last equation yields 1 = 31 − (33 − 31) · 15 = 16 · 31 − 15 · 33. Suppose that n is 1; that is, suppose that the sequence consists of one element si and i = j. Suppose, for example, that a student is assigned a program to compute shortest paths between cities. No parentheses are needed. 0 ≤ x1 < 6, 1 ≤ x2 < 9, x3 ≥ 0 28. Show, by giving an example, that communication among all offices is still possible even if some communication links are broken. Thus x ∈ (A ∪ B) ∪ C. Compute s and t such that gcd(a, b) = sa + tb using the Euclidean algorithm 31. There are several possibilities, but one thing that comes to mind is another question: What if X ∩ Y had an element? f(x) = 3 + , X = set of nonzero real numbers x 36. Consider an n × n deficient board. Therefore the loop iterates 1 = m − 1 time. In a valid argument, we sometimes say that the conclusion follows from the hypotheses. We would have to eliminate three edges at vertex b and one edge at vertex f . A terminal vertex is one with no children. , s5 becomes 8 13 16 20 27 Notice that 20 and 27, being greater than 16, move one index to the right to make room for 16. Draw the complete game tree for a version of nim in which the initial position consists of one pile of six tokens and a turn consists of taking one, two, or three tokens. gcd(480, 396) = gcd(396, 84) = gcd(84, 60) = gcd(60, 24) = gcd(24, 12) = gcd(12, 0) = 12 Substituting the next-to-last formula for 24 into the last equation yields 12 = 60 − 24 · 2 = 60 − (84 − 60) · 2 = 3 · 60 − 2 · 84. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. In the next example we illustrate how the matching problem can be modeled as a network problem. We use the fact that if p and q are each divisible by k, then p + q is also divisible by k. Example 3.19 The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is not transitive. Precedence can be modeled by a directed graph by letting the vertices denote the tasks and placing a directed edge from task ti to task tj if ti must be completed before tj . Next, we evaluate M, N, O, P, Q, R, and S. Example 8.4 An inventory consists of a list of 89 items, each marked “available” or “unavailable.” There are 45 available items. Exercises 1. (This conditional proposition is true because both the hypothesis “23 is prime” and the conclusion “24, 25, 26, and 27 are not prime” are true.) Some values of n make the 43 Sets and Logic conditional proposition true (e.g., n = 23, n = 4, n = 47), while others make it false (e.g., n = 2, n = 101). If sum is the maximum consecutive sum that ends at index i − 1, the maximum consecutive sum that ends at index i is obtained by adding si to sum provided that sum + si is positive. C(n, r): number of r-combinations of an n-element set; C(n, r) = P(n, r)/r! = n! / [(n − r)! r!] Section 3 10. The ncube has a greater degree of connectivity among its processors. Since b = 0, we proceed to line 6, where we set r to a mod b = 396 mod 108 = 72. What is the probability that the sum of the numbers on the dice is 9? Find the terminal vertices. Find the probability that both fail. Since Cn does not divide n + 2, Cn divides Cn+1 . A ∪ U 12. Functions, Sequences, and Relations 1 2 3 4 5 6 Functions Problem-Solving Corner: Functions Sequences and Strings Relations Equivalence Relations Problem-Solving Corner: Equivalence Relations Matrices of Relations Relational Databases Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises A grey Hulk? a2 x3 ; (a + x + c)2 (a + x + d)3 19. Exercises 6 and 7 refer to the sequence c1 , c2 , . Graph Theory 23. Define the relation R on X by (a) Show that R is an equivalence relation. For all n ≥ 4, we have lg n + lg(n − 1) + · · · + lg 2 + lg 1 ≥ lg n + lg(n − 1) + · · · + lg n/2 ≥ lg n/2 + · · · + lg n/2 = (n + 1)/2 lg n/2 ≥ (n/2) lg(n/2) = (n/2)[lg n − lg 2] = (n/2)[(lg n)/2 + ((lg n)/2 − 1)] ≥ (n/2)(lg n)/2 = n lg n/4 [since (lg n)/2 ≥ 1 for all n ≥ 4]. Sn−1 = Figure 4.2 A sequence of statements. Find the months with Friday the 13th in the current year. Postorder traversal Trees 36. (b, c, d, a, b, e, d, c, b) 6. Trace Algorithm 2.16 for the value n = 30. Principle of Mathematical Induction Suppose that we have a propositional function S(n) whose domain of discourse is the set of positive integers. Let the selected course numbers be c1 , c2 , ..., c151 . Suppose that n > fk1 . [(a) → (b)] Assume that n is odd. At any point, some vertices have temporary labels and the rest have permanent labels. Now n + 2 = 2k + 2 = 2(k + 1) is even. Since R is reflexive, symmetric, and transitive on V , R is an equivalence relation on V . What is the probability of getting doubles or a sum of 6? If a is any upper bound for X, since 4 ∈ X, 4 ≤ a. Since m mod b = 11 mod 2 = 1, c0 is set to 1. The supply is given by the equation p = kq, Quantity Figure 1.4 An economics model. A graph has a spanning tree if and only if it is connected. If T is a binary tree, we let post(T) denote the order in which the vertices of T are visited under postorder traversal. Path Length a, b, c, d, e a, b, d, c, e a, b, c, d, e a, c, b, d, e a, c, d, b, e a, d, b, c, e a, d, c, b, e 21 28 24 26 27 22 Listing all paths from vertex v to vertex w, as we did in Example 1.5, is a rather time-consuming way to find a minimum-length path from v to w that visits every vertex exactly one time. What is the probability of a correct answer if the student switches to choice B? ■ Correctness The output produced by the algorithm is correct; that is, the algorithm correctly solves the problem. Since there are 32 eight-bit strings that begin 101 and 32 eight-bit strings that begin 111, there are 32 + 32 = 64 eight-bit strings that begin either 101 or 111. Write a program that checks whether a relation R is an equivalence relation. One way to perform such a classification uses probability theory. Therefore, the total number of strings is C(6, 3) · 2 · 3!. Implement Prim’s Algorithm as a program. Draw a decision tree that gives an algorithm to solve the coin puzzle of Exercise 25 in at most three weighings in the worst case. This time we assume that X ∩ Y = ∅. If X is a subset of Y and X does not equal Y , we say that X is a proper subset of Y and write X ⊂ Y . An evaluation function E is constructed that assigns each possible game position P the value E(P) of the position to the first player. Suppose that we could eliminate edges from the graph to produce a Hamiltonian cycle. What is a mathematical system? Thus there are n2 − n off-diagonal entries. , sn is sorted. Let X = {1, a}, Y = {1, 2, 3}, and Z = {3}. The floor of x is the greatest integer less than or equal to x. In this exercise, gcd(6, 15) = 3. For any vertex j, we call Fij i the flow into j and we call Fji i the flow out of j. Define the Cartesian product of the sets X1 , X2 , . For example, the edge (v1 , v5) is absent. A vertex is denoted (p1 , p2 , p3), where pi is the value of property i. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eighth of those taking a computer 73. An Euler cycle in G is obtained by modifying C by replacing (v1 , v3) in C by (v1 , v2) followed by C followed by (v2 , v3) or by replacing (v3 , v1) in C by (v3 , v2) followed by C followed by (v2 , v1). A person, concerned about the possibility of a bomb on a plane, estimates the probability of a bomb on a plane to be 0.000001. Suppose that each person in a group of 32 people receives a check in January. Then there are k edges, say e1 , . Network Models (b) What is the interpretation of a maximal matching? A Hamiltonian cycle in a graph G is a cycle that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice. To prove that the universally quantified statement ∀x P(x) is false, find one value of x (a counterexample) in the domain of discourse for which P(x) is false. Then by Theorem 2.3, T is connected and has n − 1 edges. A = {1, 2}, B = {x | x3 − 6x2 + 11x = 6} 35. By the inductive assumption, the greedy solution for n − ai is optimal. This subgraph must be a spanning tree since it must contain all the vertices (so that each city is in the road system), it must be connected (so that any city can be reached from any other), and it must have a unique simple path between each

pair of vertices (since a graph containing multiple simple paths between a vertex pair could not represent a minimum-cost system). (c) How many ways can at least two people among five have their birthdays in the same month? Let K(x, y) be the propositional function "x knows y." The domain of discourse is the Cartesian product of the set of students taking discrete math with itself (i.e., both x and y take on values in the set of students taking discrete math). a) 2nan−1 5. 291 Introduction to Number Theory Thus if we set s = t and t = s − t q, we have g = sa + tb. (n + 1)n1 · · · nk i=2 128. Draw all nonisomorphic full binary trees having seven vertices. The power set of X is the collection of all subsets of X. Suppose that p has q's order. Here we must set s = 1. Since Fbc = Cbc , we do not label vertex c at this point. An algorithm can be obtained by modifying the four-queens algorithm. Find P(A), P(D), and P(N). Inductive Step Assume that we can tile a 2k × 2k deficient board. 227 Algorithms If n is an arbitrary positive integer (not necessarily a power of 2), it lies between two powers of 2; that is, for some k ≥ 1, 2k−1 ≤ n < 2k . , vn ; E = edges // V = vertices of spanning tree T ; E = edges of spanning tree T // v1 is the root of the spanning tree // S is an ordered list S = (v1) V = {v1 } E = ∅ while (true) { 522 Trees } for each x ∈ S, in order, if ((x, y) is an edge) add edge (x, y) to E and y to V if (no edges were added) return T S = children of S ordered consistently with the original vertex ordering 3.6 find a spanning tree. 1001 9. Let c = a1 . (The set D3 is defined before Exercise 1.) 10. (3.1) It follows that 1 + 2 + · · · + n = O(n2). Since a is the least upper bound for X and a − ε < a, a − ε is not an upper bound for X. There are 17 subgraphs. 0 0 0 1 1 0 1 (1 1 2 3 4 1 1 0 1 0 2 0 1 1 1 2 x 1 0 [1 0 y[1 1 z 0 1 x 0 0 [1 0 1 1 0]1 0 (c) The matrix product A2 . Example 1.15 The positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, 30, 15, 21, 35, 105) and the positive divisors of 105 are 1, 264 3, 5, 7, 5, Introduction to Number Theory thus the positive common divisors of 30 and 105 are 1, 3, 5, 15. Find cn = Answer 91–94 using the sequences y and z defined by sk = k ai . In this case the while loop terminates (without executing return i). Then, working from the bottom up, the value of a circle is set to the minimum of the values of its children, and the value of a box is set to the maximum of the values of its children. (If p is false, then p → q is true whether q is true or false.) This discussion motivates the following definition. Then x = 2 or x = −3. (For X to not be a subset of Y , there must be at least one member of X that is not in Y . The pattern in which the number of spirals is a Fibonacci number provides the most even distribution (see [Naylor, Mitchison]). , is a solution of (2.13). Since each can be assigned values in two ways, by the Multiplication Principle there are 2 · 2 · · · 2 = 2n n2 − n factors 2 − n matrices that represent reflexive relations on an n-element set. Prove that every nonempty subset of real numbers bounded below has a greatest lower bound. Assuming for the moment that it is an equivalence relation, the equivalence class containing 0111010, denoted [0111010], consists of all strings related to 0111010. Example 7.6 to show that n. C(i, k) = Cn + 1, k + 1).) n=m The formalism n i=m 158 ai (2.7) Functions, Sequences, and Relations is called the sum (or sigma) notation and n i=m ai (2.8) i=m is called the product notation. Input: (3.12) for n > 1. In the path, for all repeated a, (. Let P(x, y) denote the expression x < y. The idea is to begin with X ∪ (Y − X) and use the laws of sets, which we now think of as rules to manipulate set expressions, to obtain X ∪ Y. Along the way we will define and discuss cuts in networks. If f(n) = (g(n)), then cf(n) = (g(n)) for any c = 0. The terminal vertices give the sorted order. Exercise 6, Section 1 9. How is the composition of f and g denoted? Professor Moe proposes the following version of binary search: 24. Let U denote the set of all 165 students (see Figure 1.6). Using mathematical induction, we prove that for all n ≥ 0, if X contains an element less than or equal to n, then X has a least element. All of the good coins have the same weight. The problem was first stated and proved by Kuratowski in 1930. Example 6.3 The Mesh Model for Parallel Computation Previously, we considered the problem of when the n-cube could simulate a ring model for parallel computation (see Example 3.5). Since X consists of nonnegative integers, Y also consists of nonnegative integers. Since 2n = (n), the worst-case time to run the algorithm with input of size n Thus, we have shown that the worst-case time to the worst-case time for input of size n is 2 n. 2. Since t = 6, we can state the conclusion as follows. What is a binary search tree? Let f(x) = x for all x ∈ R. Since b = 0, we proceed to line 6, where we set r to a mod b = 108 mod 72 = 36. This leaves 10 − 4 = 6 edges, not enough for a Hamiltonian cycle. Choose one of n+1 people, say P. Same as Exercise 41 33. R (c) There is a subgraph satisfying properties (8.1) and (8.2). Region 3 represents elements in X and Y , but not in Z. By the inductive assumption, this board can be tiled with trominoes. Example 1.8 TABLE 1.1 ■ A portion of the ASCII table. Therefore, the second person's birthday can occur on any one of 364 days. Vertex d is the right child of vertex b; vertex b has no left child. In effect, we have proved ¬q → ¬p. Proofs 67. The attribute Name might be taken as a key. (c) The solution is 1. Notice that a binary tree is a tree in which each vertex has either no children, one child, or two children. Vertex d is the right child of vertex b; vertex b has no left child. In effect, we have proved ¬q → ¬p. Proofs 67. Then there is a minimum of three vertex (v3) on level 1 of degree 3, but T2 does not. Section 4 27. We conclude that all of the equations are true. FROM THE BIG SLEEP This chapter offers an introduction to recurrence relations. 660 31. Write ∀x∀y T2 (x, y) 11. The "formula" for the mapping is "add 2." The f(x) notation may be used to indicate which element in the codomain is associated with an element x in the domain or to define a function. What techniques in addition to the use of Huffman codes might be used to save memory when storing text? If x ∈ B, then x = 1 or x = 3. V1 = {v1 } , V2 = {v2 , v3 }. Consider, for example, the case n = 8. , g(ek) incident on f(vi) and f(vj) in G2 . In the next section we show that if there are no paths satisfying the conditions of Theorem 2.3, the flow is maximal. Pat will use the treadmill or lift weights. The relation {(1, 1), (2, 2)} is a reflexive relation on {1, 2}. Example 6.3 Among a group of students, some take art and some take computer science. The complete bipartite graph K2,3 In Exercises 7–12, write the incidence matrix of each graph. In programming languages, strings can be used to denote text. Let T1 be a rooted tree with root r1 and let T2 be a rooted tree with root r2 . Problem-Solving Corner Proving Some Properties of Real Numbers Problem Find some definitions: (a) Let X be a nonempty set of real numbers. At line 4, the function correctly computes the value of 3!. n · (n − 1)! = 3! = 3 · 2 = 6. Is x decreasing? (The proposition p is called the hypothesis (or antecedent) and the proposition q is called the conclusion (or consequent). Then either i is set to 2m + 1 or j is set to 2m . Suppose that there is an algorithm that solves the five-coins puzzle in the worst case in two or fewer weighings. Both v1 and v2 are at level 1 ends of the same edge. Suppose that Taylor, who is taking MATH 201, owns a sport utility vehicle. Since p is true, q is false, the implication p → q is false. (1.6) Equation (1.5) is true if x is in C, and equation (1.6) is false if x ∈/ A. Let T be a binary search tree with n vertices and let T ∗ be the full binary tree obtained from T by adding left and right children to existing vertices in T wherever possible. Since ¬ exists dj(i) is true, false is returned to indicate that ∀x∃y P(x, y) is false. Define g((v, w)) = (f(v), f(w)). 281 (c) f = e − v + 2 4. In some versions of FORTRAN, an identifier consists of a string of one to six alphanumeric characters beginning with a letter. Implement Kruskal's Algorithm (given before Exercise 20, Section 4) as a program. Let en be the minimum number of links required to solve the n-node communication problem (see Exercise 51, Section 1). Proof Let n be the maximum number of modulus operations required by the Euclidean algorithm for integers in the range 0 to m, m ≥ 8, p → ¬q 38. Prove that for all integers m and n, if m and n are even, then m + n is even. Take x = 0. For R to be reflexive, we must have s R s for every eight-bit string s. Nothing to it! In order for this inductive step to work, our basis steps must be n, n + 1 , . Input: a, n, z Output: an mod z exp via repeated squaring(a, n, z) { result = 1 x = a mod z while (n > 0) { if (n mod 2 == 1) result = (result ∗ x) mod z x = (x ∗ x) mod z n = n/2 } return result } 3 The key difference betweenAlgorithms 2.16 and 2.19 is the size of the numbers that are multiplied. Consider the formula L2 = 3. The first vertex v0 may be chosen in n ways. Yes 71. The Hamiltonian and Euler cycles are the graph itself. Let m and n be even integers. Unfortunately, |X| = |[{0, 1, . Exercise 6; a0 = 0 21. 1/210 33. a b c e d Figure 3.6 A graph with a Hamiltonian cycle. Write the negation of ∀x∀yP(x, y) using the generalized De Morgan's laws for logic. , sn and a value key, this algorithm returns the index of key. We designate this vertex the root. Exercise 11 22. State the bound laws for sets. The intersection consists of all elements belonging to both X and Y . Since a is the least upper bound for X, b/t ≥ a. Yes Trees 17. Our hash function becomes h(n) = n mod 11. As another example, tiling an n × n deficient board with trominoes when n is a power of 2 can be done recursively because we can find four (n/2) × (n/2) subboards within the original n×n board. Is n nondecreasing? What does it mean for two vectors to be equal? 1/103 26. Then there exist integers k1 and k2 such that m = 2k1 + 1 and m = 2k2 + 1. How many n-vertex, nonisomorphic binary trees are there? dx+1 (dx + xn) dxn dx = ∗k ∗ xn dx dx dx = xnn−1 + xn · 1 = (n + 1)xn i=2 Since n is expressed as the sum of Fibonacci numbers: m= fn+2 fn+1 + · · · + fn+1 fn fn+2 f2 fn+2 + f1 = . Examples are given in Table 1.1. Huffman codes, which represent characters by variable-length bit strings, provide alternatives to ASCII and other fixed-length codes. However, we note that in a Hamiltonian cycle, there must be as many "up" edges as "down" edges and as many "left" edges as "right" edges. Therefore if n ≥ 4 and k = n/2, inequality (5.1) is satisfied. Prove that f is one-to-one if and only if whenever g is a one-to-one function from any set A to X, f ◦ g is one-to-one. Someone in the discrete mathematics class has never taken calculus. How many solutions are there to the five-queens problem? The first item can be compared in 2n−1 ways. Give the truth table for the exclusive-or of p and q in which p exor q is true if either p or q, but not both, is true. Network Models 5 4, 4 3 2, 5 4 2 5, 3 4 4, 1 3, 2 d 4, 2 2, 0 2, 1 f Figure 3.1 A cut in a network. Input: small large(s, n, small large) { small = large = s1 for i = 2 to n { if (si < small) small = si if (si > large) large = si } } 11. What is x and y? Since n = 4k(4 − 4 k) is the fourth n is divisible by 2. Therefore i is a character? Inductive Step We assume that n ≥ 8 and postage of k cents or more can be achieved by using only 2-cent and 7-cent stamps for 6 ≤ k < n. We use the method of Algorithm 3.6 to generate a spanning tree rooted at vn. n 4141 7. Find the months with Friday the 13th in 1945. Find all simple paths from a to e in the graph of Exercise 19. Therefore X is nondecreasing. Propositional function 54. (The basis steps were chosen from A, B, C, D, F) 8. A game program might simulate rolling dice, and a client service program might simulate the arrival of customers at a bank. This results in an Alpha value of 6 for I. e1 e2 b 4 1 2i 3 5 1 3 1 g 1 2 z 3 J Find the length of a shortest path from a to z in the graph of Figure 3.2. for i = 1 to n for j = 1 to i/2 x=x+1 21. Is t decreasing? (Since then over one trillion decimal digits of π have been computed.) Since p is false, ¬p is true. Show that there are at least two available items in the list exactly 19 items apart. Special cases of (3.1) are as follows: If p ∨ q and ¬p are true, then q is true. Trees 28. We will see that this cut has a minimum capacity and, since a minimal cut corresponds to a maximal flow (Theorem 3.9), the flow F is maximal. , 6 217 = 0.32878787 . If x ≥ 0, then |x| = x < d. Proof The proof precedes the statement of the theorem. We return 3! · 4 = 24. Algorithms requiring n2 or n3 steps also become impractical to implement, but for relatively small input sizes. Not equal 33. The Mayor of Providence! He was sitting in the hotel lobby. Then ε X or x ∈ Y . (a) Use a truth table to prove that if p and q are propositions, at least one of p → q or q → p is true. (If the prime pi is not a factor of m, we let ai = 0. Since 30 mod 15 = 0, by (3.1) gcd(30, 15) = gcd(15, 0). X is a proper subset of Y is denoted X ⊂ Y . Prove that for all x ∈ R, if x3 is irrational, then x is irrational. An identity that results from counting process is called a combinatorial identity and the argument that leads to its formulation is called a combinatorial argument. You can do it, Bloom, you're a wizard. Each multiplication is performed using Theorem 2.17. This portion of the proof is complete. Matrix of a relation 45. b v c. In the postfix form of an expression, an operator follows its operands. John takes calculus only if he has sophomore, junior, or senior standing. The holes can be drilled using a drill press under the control of a computer. Is b nonincreasing? How many vertices does a Hamiltonian cycle have? If n is a positive integer, which mathematical symbols we write. In Prim's Algorithm, since we want a 1 6 minimal spanning tree, at each iteration we simply add an edge of minimum weight. What is the condition for contradiction? Prove that for all integers n, if n and m are even, then nm is even. The main result that makes encryption and decryption work is that au mod z = a for all 0 ≤ a < z and u mod φ = 1 (for a proof, see [Cormen: Theorem 31.36, page 885]). We have shown that the existentially quantified statement ∃x x ∈ R, x2 + 1 > 1 x2 is false. Ex (P(x) → Q(x)) 37. Pheasants Skyscrapers 10. Example 1.5 If C = {1, 2, 3, 4}, by inspection, every element of C is an element of A. Show that a vertex in a connected graph G is an articulation point if and only if there are vertices v and w in G having the property that every path from w to x passes through v. Let each row, column, or diagonal that contains three X's scored 100. − C + A B prefix: − + ABC usual infix: A + B − C) parened infix: ((A + B) − C) 14. We begin with some basic graph terminology and examples. Looking at the preceding picture, we see that any number less than n is not an upper bound for X. Theorem 4.16 Let R be an equivalence relation on a finite set X. (For example, available items at positions 13 and 22 or positions 69 and 78 satisfy the condition.) Let ai denote the position of the ith available item. Then there are k edges e1 , . Algorithms y y 2n 256 y n2 128 64 32 y n lg n 16 yn 8 y lg n 4 2 y1 1 1 2 3 4 5 6 7 8 9 10 11 12 13 n Figure 3.1 Growth of some common functions. Find the diameter of Kn , the complete graph on n vertices. "Indirect proof" is another name for proof by contradiction. Example 1.3 508 In contrast to natural trees, which have their roots at the bottom, in graph theory rooted trees are typically drawn with their roots at the top. If not, can you alter E so that a two-level search program will play a perfect game of tic-tac-toe? Therefore f is not onto. For example, if the names of six persons are Jones, Johnson, Appel, Zamora, Chu, after sorting the sequence in nondecreasing order, we would obtain Appel, Chu, Johnson, Jones, Zamora. Prove that for all rational numbers x and y, x + y is rational. Since |x| is defined piecewise, we expect to use proof by cases. One 15. 5.21. If x, y ∈ X and x ≤ y and y ≤ x, we say that x and y are comparable. Each internal vertex has two children, with one possible exception. The disjunction of p and q is p ∨ q: It is raining or it is cold. 6 ∈ G1 9. 272 Introduction to Number Theory Example 2.4 We show how Algorithm 2.3 converts the binary number 1101 to decimal. (d) Since x is false, q → r is true (regardless of whether r is true or false). Suppose that C R d. Exercises 3–7 refer to Figure 4.1, where we reverse the direction of the edges so that edges are directed from jobs to applicants. We will provide a solution and then find a recurrence relation and an initial condition for the sequence c1 , c2 , . Since there are n pigeons and n > k, we have a contradiction. A move consists of removing one or more tokens from any one pile. Since n mod 2 is equal to 1, result becomes ∗ ∗ = 1 ∗ a2 = a2 . ∀x (Q(x) → P(x)) 38. Show that modus tollens (see Figure 4.1) is valid. The worst-case time of Algorithm 1.8, which determines whether the integer n √ is prime, is (√ n). 49. Thus it suffices to check the cases m = 1, 2, 3. Denote these as (1, . 254 2 2 f22 + fn+1 = fn fn+1 +fn+1 fn+1 = fn+1 (fn + fn+1) = fn+1 fn+2 Since f = fn + fn+1 + fn+2. Since v ≤ y, so are not all distinct. Return to the top of the while loop (line 11). Example 1.6 Bacon Numbers Actor Kevin Bacon has appeared in numerous films including Diner and Apollo 13. 207 Algorithms ■ Precision The steps are precisely stated. Assume that the events "Joe passes the final examination" and "Alicia passes the final examination" are independent. The initial distribution is two piles of three and two tokens, respectively. Section 3 1. Is connected 3 20. Then there exist k1 and k2 such that m = 2k1 and n = 2k2 . In the worst case, the item will not be found at line 5, so the algorithm will be invoked a second time at line 11. It has been shown (see [Pearl]) that for game trees in which every parent has n children and in which the terminal values are randomly ordered, for a given amount of time, the alpha-beta procedure permits a search depth 4/3 greater than the pure minimax procedure, which evaluates every vertex. e, g 28. The worst-case time of an algorithm is proportional to the height of the decision tree that represents the algorithm. We show how to efficiently compute an integer s, 0 < s < φ such that ns mod φ = 1. We delete a and the edge x incident on a from the graph G. If edge e is labeled k, we say that the weight of edge e is k. |X|, the cardinality of X, is the number of elements in the set X. Everyone, a minimum degree of 1. The first statement follows immediately. The major portion of the chapter is devoted to the problem of maximizing the flow through a network. , vm , vn) . This number was found by the Great Internet Mersenne Prime Search (GIMPS). If X ∈ X − Y , then CX−Y (x) = 1 = 1 · 1 − 0 · (1 − 1) = CX(x)[1 − CY (x)]. Empty set: 0 or { } 7. Now 3n+1 = 7n+1 − 2 = 3(2n − 2) = 3(3n + 7n − 2) + 4(7n + 1). 365n For n = 22, the probability is 0.475695, and for n = 23, the probability is 0.507297. Give an example of a path that is not a simple path. 351 Counting Methods and the Pigeonhole Principle Theorem 6.9 Let E1 and E2 be events. Hamilton (1805–1865) was one of Ireland's greatest scholars. Although precise estimates of the execution time of a program must take such factors into account, useful information can be obtained by analyzing the time of the underlying algorithm. 2 Therefore, the solution of (2.18) is dn = 2n − n2n−1 , vn = wm , wm−1 , Output "yes" and stop. x becomes a8 , and n becomes 3. A Hamiltonian cycle starts at a vertex, visits each vertex exactly once (except for the initial vertex, which is visited twice: at the beginning and end of the cycle), and returns to the initial vertex. In a sequence, the numbers 1, 2 , 4 ➔ Arguments and Rules of Inference Consider the following sequence of propositions. What sum of a the number i for n = 2(, the second case the algorithm is correct. The vertices of G represent the four colors, and an edge labeled i connects two vertices if the opposite faces of cube i have those colors. Label each vertex with one of the 10 distinct two-element subsets of {1, 2, 3, 4, 5}. (Hypotheses) There exists an integer, say k1 , such that m = 2k1 + 1. The lower and upper limits of the sum A. We may use the Addition Principle to obtain a recurrence relation for the total number of good routes in an n × n grid: n Cn = Ck−1 Cn−1 . Has an edge (v, w), where δ(v) = i and δ(w) = j 39. This skill comes from practice and careful thinking about each problem. ∃xP(x) , ∵ P(d) for some d ∈ D 14. Rostenkowski. Suppose that n satisfies 2k ≤ n < 2k+1 . Define a sequence s as sn = 2n + 4 · 3n , n ≥ 0. , ak ∈ X such that 31. w4 w5 w6 w7 T2 Section 9 33. In (C), C (shown with a heavy line) is an Euler cycle for the other component, and C (shown with a light, solid line) is an Euler cycle for the other component. In either case, min{a, b}, c} = min{a, min{b, c}}. Therefore t(n) ≥ n for all n ≥ 1 and t(n) = (n). Similarly, if 2 holds, tv = 1 and again we have su + tv = 1. Theorem 3.9 Max Flow, Min Cut Theorem Let F be a flow in G and let (P, P) be a cut in G. Write a program that determines whether a path is a simple path, a cycle, or a simple cycle. Show that the time required by the algorithm of Exercise 64 to multiply a and b is O(lg a lg b). Find t4 and t5 . Therefore, f is not one-to-one. Sets X and Y are equal if they have the same elements. Proof Suppose that G1 and G2 are isomorphic. If we replace each term C(k + 3 − 1, k) by its value, we obtain 56 = 1 + 3 + 6 + 10 + 15 + 21. Therefore |X1 | = Cn−1 . Unary operator on X: function from X to X Section 2 15. Explain how to compute the number of elements in the set X ∩ Y. The probability that Joe passes is 0.70, and the probability that Alicia passes is 0.95. Let U be a universal set and let X ⊆ U. Notice that, in general, A − B = B − A. Without knowing which number system is being used, a representation is ambiguous; for example, 101101 represents one number in decimal and quite a different number in binary. 01 1110010011110 17. Since log3/2 2(100,000,000) = log2/3 100 + log3/2 3 = (log3/2 100) − 1 = 4(11.357747) − 1 = 44.430988, an upper bound for the number of modulus operations required by the Euclidean algorithm for integers in the range 0 to 100,000,000 is 44. Write an algorithm that returns the index of the first occurrence of the largest element in the sequence s1 , . In any graph, the number of vertices of odd degree is even. Given a universal set U and a subset X of U, the set U − X is called the complement of X and is written X. Theorem 4.5 Let G be a directed, bipartite graph with disjoint vertex sets V and W in which the edges are directed from vertices in V to vertices in W. Define nondecreasing sequence. Assume that ∃x∃y P(x, y) is true and that the domain of discourse is nonempty. Let G denote the set of simple graphs G = (V, E), where V = {1, 2, . The Chargers get a good linebacker. A = {1, 2, 3}, B = {n | n ∈ Z+ and n2 < 10} 24. If we replace e in C by e1 , C , e2 , we obtain an Euler cycle (shown in color) for G. The given sum, 1 · 3 · 5 · · · (2n − 1) 1 ≤ , n = 1, 2, . (b) Finds the ancestors of v. In order for proposition (3.4) to be true in this case, we must define p → q to be true when both p and q are false. The quotient is 7. If Pn is divided into subproblems of sizes i and j, there is an algorithm that combines the solutions of these two subproblems into a solution to Pn in time at most 2 lg(ij). Give an example of a function that is not symmetric. Because g is one-to-one, δ(f(v)) ≥ δ(v) ≥ k. False 19. Give the arrow diagram for functions g from X to Y and f from Y to Z, we can obtain the arrow diagram of f ◦ g by drawing an arrow from x ∈ X to z ∈ Z provided that there are arrows from x to some y ∈ Y and from y to z. By Theorem 1.3(c), c | bq. We obtain this inequality by using induction on n. For example, suppose that i = 4 and s1 , . At line 1, we set the index of the variable. Explain how the machine gives the value of the expression (b), the rightmost. Since the right and left faces of cube 5 are white. The level of a vertex v is the length of the simple path from the root to v. If a = 0, a · 1 − a · 0 + b · 0. k=0 4 (14). Using De Morgan's laws for logic, write the negation of each expression in Exercises 70–73. 2−1 The reader has surely noticed that the previous formulas, one has to be given the correct formulas in advance. Let B denote the event "bug present," and let T , R, and J denote the events "Trisha (respectively, Roosevelt, Jos´e) wrote the program." Then n a1 + 19, . The seven vertices to be evaluated are H, I, and J. If R and S are symmetric, then R ∩ S is symmetric. How many eight-bit strings either begin with 100 or have the fourth bit 1 or both? e a f d 9. For what values x does ∀x∀y P(x, y) is true, show that P(x, y) is true for all values of x ∈ X and y ∈ Y , where the domain of discourse is X × Y . The values of the constants can be obtained from the initial conditions. is unbounded; so such an n exists.) If we obtain an equality, 2(y x) is in Egyptian form, so suppose that 1 1 1 p + + + · · · + < . Propositions are the basic building blocks of any theory of logic. is false, show that P(x, y) is false for all values of x ∈ X and y ∈ Y , where the domain of discourse is X × Y . Assume that the first player moves to the center square in tic-tac-toe. We are Fn−i ways to select the remaining members, since the number can be selected from {2, … a u x.. w P Figure 4.6 The proof of Theorem 4.3. With this vertex labeling, we see that a is a path from a to w. It is the vertex nearest a on P that is in T , and u is the predecessor of x on P. Since 20 < 34, 34 must move one position to the right 55 144 The sequence 3 · 2 · 3 = 18 possible names for the 19 persons. To count the total number of items available, we use the Addition Principle. Since x ∈ X − Y , then x ∈ X and x ∈ / Y . Some game-playing programs have been incredibly successful. The statement ∃x∃y P(x, y) is false, if for every x ∈ X and for every y ∈ Y , P(x, y) is false. Figure 8.3 Graphs that represent the stacking of Figure 8.2. face. X = Y , where X and Y are sets: X and Y have the same elements. 8. Because x is infinite for it as both n and X is reflexive since an element x is related to itself. Thus X is infinite. (The collection of pseudo equivalence classes (see Exercise 4.15) does not partition X. Write an algorithm that returns the largest and second-largest values in the sequence s1 , . , v2 v1 A forest is a simple graph with no cycles. The formula is obtained by considering all 8 · t − 1 slots and k + t − 1 symbols consisting of X x's and t − 1 |'s. [First solution] We construct a truth table for all the propositions involved: 52 p q p→q q ¬q p ¬p ¬q T T F F T F T F T F F T T F F T T F F T T F T F T T Sets and Logic We observe that whenever the hypotheses p → q and p are true, the conclusion q is also true; therefore, the argument is valid. Suppose that T has a cycle. n = 100, φ = 243 Problem-Solving Corner Making Postage Problem Let p and q be positive integers satisfying gcd(p, q) = 1. (To use chess terminology, we say that the queens are mutually nonattacking.) This is the problem of placing four queens on a 4 × 4 board so that no queen attacks another queen. (b) The idea of the algorithm is to place tokens successively in the columns. Algorithm for generating r-combinations: Algorithm 4.9 14. n(n + 1)(2n + 1) + (n + 1)2 6 (n + 1)(n + 2)(2n + 3) = 6 7. Thus, CX∪Y (x) = CX (x) + CY (x) − CX (x)CY (x). (c) This is a UL U. This function can be defined by using the rule R(x) = 1 . Thus when the input consists of a tree with a single vertex, we process the root and return. We illustrate a loop invariant with an example. Give an example of a graph that has an Euler cycle but contains no Hamiltonian cycle. a, 3, 2, a, 1, 2, 4 1 Figure 7.7 An algorithm to sort a1 , a2 , a3 . (For Exercise 49) ∃x(x2 ≤ x). For which values of m and n is Kmn planar? Suppose, for example, that we have an internal box, all of whose children have been assigned a value. Now suppose that a is a positive integer. The components are sometimes modeled as rectangles and L-shaped figures similar to (right) trominoes. Then there exists k such that n = 2 = 2k. Exercises 28-31 refer to the adjacency matrix A of K5 . Now solve C0 = b1 br1 + dr2 for b and d to obtain the solution of the recurrence relation. 6 3 d e 7 G1 15. Therefore, (x, x) ∈ R ◦ S and R ◦ S is reflexive. References on combinatorics are [Brualdi; Even, 1973; Liu, 1968; Riordan; and Roberts]. If x ∈ A, then x ∈ A ∪ (B ∪ C). Joe and Alicia take a final examination in C++. Among the algorithms in Example 1.2 ■ Two triangles are congruent if their vertices can be paired so that the corresponding sides and corresponding angles are equal. Write a program that prints the amount accumulated yearly if a person invests n dollars at p percent compounded annually. For the first change, we call the resulting operator imp1 (Exercise 74), and, for the second change, we call the resulting operator imp2 (Exercise 75). We could write up this part of the solution as follows. 44. Example 2.16 . In symbols, ∀y ∃x ∈ X(f(x) = y). Since f is one-to-one 17 such strings. Recall that it takes time (lg n) to reduce n to 0 by repeated halving. To construct a dinner consisting of one appetizer, one main course, and one beverage at Kay's Quick Lunch, we use a step-by-step process. Thus, we can conclude that if he takes calculus, then he has sophomore, junior, or senior standing. At lines 11 and 12 we update each label of the uncircled vertices, d and g, adjacent to f . Use induction to prove that an = n! n k=0 (−1)k k! for all n ≥ 1. By Corollary 6.11, P(F) = P(F ∩ C1) + P(F ∩ C2) + · · · + P(F ∩ Cn). Since there are five vertices, a Hamiltonian cycle must have five edges. The probability that Joe passes is 0.70, and the probability that Alicia passes is 0.95. Let U be a universal set and let X ⊆ U. Notice that, in general, A − B = B − A. Without knowing which number system is being used, a representation is ambiguous; for example, 101101 represents one number in decimal and quite a different number in binary. Since log3/2 (100,000,000) = log3/2 100 + log3/2 3 = (log3/2 100) − 1 = 4(11.357747) − 1 = 44.430988, an upper bound for the number of modulus operations required by the Euclidean algorithm for integers in the range 0 to 100,000,000 is 44. Write an algorithm that returns the index of the first occurrence of the largest element in the sequence s1 , . In any graph, the number of vertices of odd degree is even. Let G be a directed, bipartite graph with disjoint vertex sets V and W in which the edges are directed from vertices in V to vertices in W. Define nondecreasing sequence. Assume that ∃x∃y P(x, y) is true and that the domain of discourse is nonempty. Let G denote the set of simple graphs G = (V, E), where V = {1, 2, . The Chargers get a good linebacker. A = {1, 2, 3}, B = {n | n ∈ Z+ and n2 < 10}. If n is a character? 4 (3)(3)! The number of outcomes from the first vertex is 1 for 5. 2.6 Definition 2.25 We give an example of graphs in Figure 8.3. Thus, each such of the complement of X, if W − T and its X, if 2 is an edge connecting the vertices of T1 to the vertex set of T2 satisfying the following: Vertices vi and vj are adjacent in T1 if and only if the vertices f(vi) and f(vj) are adjacent in T2 . The game continues with the second quantifier. 50 29. The graphs are not isomorphic since G1 has a simple cycle of length 3 and G2 does not. In either case, we have proved that −d ≤ x < d. Since p > 1, and q > 1, s = 0 and t = 0 1. The algorithm follows. prim(w, n, s) { // v(i) = 1 if vertex i has been added to mst // v(i) = 0 if vertex i has not been added to mst for i = 1 to n v(i) = 0 // add vertex s to mst v(s) = 1 // begin with an empty edge set E=0 // put n − 1 edges in the minimal spanning tree for i = 1 to n − 1 { // add edge of minimum weight with one vertex in mst and one vertex // not in mst min = ∞ for j = 1 to n if (v(j) = 1) // if j is a vertex in mst for k = 1 to n if (v(k) == 0 ∧ w(j, k) < min) { 529 Trees add vertex = k e = (j, k) min = w(j, k) } // put new vertex and edge in mst v(add vertex) = 1 E = E ∪ {e} } return E } Since x ∈ A, then x is not in X ∩ Y but not in Y. Figure 4.6 ∈ A ∩ B. We recursively conclude that there is a path from 1 to n. Notice that we have explained how to use recursion since this is also one of the seven students. (This is the version of Algorithm 3.3 in the next subsection. At this point large is 9, the largest of a, b, and c. Today, "algorithm" typically refers to a solution that can be executed by a computer. The world checkers champion is a program named Chinook developed by a team led by Jonathan Schaeffer at the University of Alberta. The type I edges are (a, v1) and (a, v2). Explain how alpha-beta pruning works. To accommodate 24 terminal vertices, we must have a tree of height at least 5 (see Figure 7.9). A single attribute or a combination of attributes for a relation is a key if the values of the attributes uniquely define an n-tuple. This leaves order 1 for person 3. We must not count the two eliminated edges incident at the two vertices again. The set D4 is defined before Exercise 1.) Let L(x, y) be the propositional function "x loves y." The domain of discourse is the Cartesian product of the set of all living people with itself (i.e., both x and y take on values in the set of all living people). Permute one token with four S's and other tokens with one letter each from among ALEPBON, which can be done in 9!/2! ways. (p ∧ q ∧ r) ∨ (p ∧ ¬q ∧ r) ∧ (r ∨ (q ∧ p)) 57. To list all of the ways that a robot can walk n meters, set s to the null string and invoke this algorithm. 90 mod 60 = 30; 60 mod 30 = 0, so gcd(60, 90) = 30. (2.2) We are beginning to express 91 in powers of 2. State the Quotient-Remainder Theorem. (c) Find a recurrence relation for the sequence. n i2 r n−i , i=1 Answer 84–86 for the subsequence of s obtained by taking the first, third, fifth, . We give a proof by mathematical induction due to [Fowler]. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function. The three edges labeled 1 can be chosen in C(3+10−1, 3) = 220 ways. We will briefly introduce some of the fundamental ideas in the theory of relational databases. Make a random decision about whether to stay with the door initially chosen or switch to the unchosen, unopened door. We conclude the following theorem by using show how the Binomial Theorem (Theorem 7.1) and Theorem 7.6 can be used to derive other combinatorial identities. Thus the n-cube has a Hamiltonian cycle if and only if n ≥ 2 and there is a sequence, s1 , s2 , . . . , s2n where each si is a string of n bits, satisfying: 460 ■ Every n-bit string appears somewhere in the sequence. (b) How many equivalence classes are there? For this reason, the existentially qualified statement for some n, if n is prime, then n + 1, n + 2, n + 3, and n + 4 are not prime is true. Explain how to use an arrow diagram to determine whether a function is a bijection. We prove that (b) implies (c). Only simple graphs. In this case we list the set of all 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111 and the count them. Therefore there are 3333 multiples of 3 between 1 and 10,000. However, in the late 1900s, number theory became extremely useful in cryptosystems— systems used for secure communications. Look at relevant definitions and related results. We will list the r-combinations of {1, 2, . Find the probability of getting doubles or the sum of 5. A sequence s is increasing if sn < sn+1 for all n for which n and n + 1 are in the domain of the sequence. Give an example of an onto function. The teacher then states that choice C is incorrect. In 2007, they proved that Chinook can never lose. , vn) be a path from the root v0 to vn . For example, {(n, n + 2) | n is a positive integer} 148 Functions, Sequences, and Relations the following defines a function from the set of positive integers to the set of positive integers. 153 Functions, Sequences, and Relations The equation CX(x) = CY (x) = 1. Give an example of a function that is neither one-to-one nor onto. Explain how to find the sum of 2 on the first die or the sum of 3 on the second. 5 The exponent 2 may be written as 1 + 1 and so on. The paper of the graph the argument 1 is, from a function from Z to Z. (c) Show that the nth Sn−2 such (n − 2)-bit strings. Example 5.23 Write the following argument symbolically and then, using rules of inference, show that the argument is valid. 82054 = AEFA3 40. This example shows that "antisymmetric" is not the same as "not symmetric" because this relation is in fact both symmetric and antisymmetric. Thus x R x, so R is reflexive. x n i is set to x ∗ x mod z = 16 mod 11 = 5, and n is set to 1. Write a program to compute Ackermann's function. (b) In how many ways can we assign eight balls if we must have at least one ball of each color? How many outcomes give the sum of 2 or the sum of 12? Table 4.2 shows how the subproblems are combined to compute 5!. If the second tree is empty, the trees are isomorphic; otherwise, if the trees are not isomorphic. 1 We remove b from U at line 15. The base cases are n = 1 and n = 2. Show the output from Algorithm 4.9 when n = 6 and r = 2. Expressing 91 in powers of 2 (2.4), we obtain 91 = 22 · 22 + 2 + 1 = 22 · (2 · 11) + 2 + 1 = 2 · 23 · 11 + 2 + 1. (Equivalently, the union X1 ∪ X2 ∪ · · · ∪ Xt contains n1 + n2 + · · · + nt elements.) In Example 1.9 we could let X denote the set of eight-bit strings that begin 101 and X2 denote the set of eight-bit strings that begin 111. If we want to verify that the statements S(n0), S(n0 + 1), . , are true, we use a "chain reaction" of induction. Inequality (5.1) is obtained by taking the logarithm to the base 2 of both sides of (5.2). In case 1, we find that the truth value of the (p → q) ∧ (¬q) is true. Show that p ∨ q is equivalent to (p ∧ q) ∨ (¬p ∧ q) 27. Assume that the first player moves to the center square. Since x ∈ B ∪ C, there is x ∈ B or x ∈ C. This is possible if p is false and q is true. If the game tree is so large that it is not feasible to reach a terminal vertex, we limit the level to which depth-first search is carried out. Since A ⊆ B, x ∈ B. With these definitions, the worst-case, best-case, and average-case times for Algorithm 1.2 for input of size n are n since the loop is always executed n − 1 times. Find r3 . The numbers (8.1) are all distinct and hence the numbers (8.2) are also distinct. If x ∈ / X ∪ Y , then CX∪Y (x) = 0. i=1 Answer 4–16 for the sequence t defined by 8. Thus CX (x)CY (x) = 0 = CX∩Y (x). Therefore B ⊆ A. Take x = 1, y = 8. The algorithm is invoked with input 3. Example 2.5 Prove that 2m2 + 3n2 = 40 has no solution in positive integers, that is, that 2m2 + 3n2 = 40 is false for all positive integers m and n. Input: m, b Output: c, m dec to base b(m, b, c, k) { n = m k = 0 while (m > 0) { ck = m mod b m = m/b k = k + 1 } } Just as a binary integer m has 1 + logb m digits (see Exercise 55). Exercises 28–32 show how to compute s and t such that gcd(a, b) = sa + tb. 126, ∀p ∀q 36. Therefore, if the order of the three is not determined, since between two zero, there exist integers p and q, m = 2 and n, then x, y and d. Consider how, for example, that if in a fight of two teams plays at least two times. Since the first element is a zero, sort is called recursively on the last n − 1 elements. Let g denote the last element. When we have completed the Inductive Step. Write a program that accepts strings and puts them into a binary search tree. Write an algorithm that reconstructs the binary tree given its preorder and inorder vertex orderings. Use iteration to show that as ≤ 2n − 4, n ≥ 4. Suppose that m and n are both odd and that the graph has a Hamiltonian cycle. An equivalence class consists of identifiers that the system is allowed to consider identical. By Theorem 2.3, T is a tree. A substring of a string α is obtained by selecting some or all consecutive elements of α. We show that this algorithm is optimal; that is, that no algorithm that sorts three items has worst-case time less than 3. Theorem 2.11 Let an c1 an−1 + c2 an−2 (2.13) be a second-order, linear homogeneous recurrence relation with constant coefficients. Prove that Dijkstra's shortest-path algorithm correctly finds a shortest path. Let P be a path from a to z and let x be a vertex in P that is neither a nor z (see Figure 2.4). Then X ∩ (Y ∪ Z) = (X ∩ Y) ∪ (X ∩ Z) 24. (1, 2}. Answer 16 and 17 with reference to the puzzle of Exercise 29. Let T denote the tree constructed. Prove that your algorithm is optimal. 288 Introduction to Number Theory 3 Special Result The following special result will be used to compute inverses modulo an integer (see the following section). 1110011101001111 Character Frequency Character Frequency A B C 6 2 3 D E 2 8 Professor Byte suggests using the variable-length codes Encode each word using the preceding Huffman code. For our last example, we present and analyze another sorting algorithm known as merge sort (Algorithm 3.8). Thus 2(x − 2y) = ±1. We claim that if n and n + 1 are consecutive integers, one is odd and one is even. It follows that each vertex in T has degree at most 4. We note that the Pigeonhole Principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. Find functions f, g, h, and t satisfying f(n) = (g(n)) = (lg n) for every fixed k > 0 and h(n) = (t(n)), f(n) − h(n) = (g(n) − t(n)) (a) cccdccccdd (b) cccddcddcc (c) (d) 20 9. Answer true or false to the functions in Exercises 6–8 and explain your answer. The equivalent characterization of "antisymmetric" translates for digraphs as follows. Exercise 13 24. At time 21 we find d c to u. Find the permutation that will be generated by Algorithm 4.14 after 6427135. Thus the trees T1 and T2 are isomorphic. In topological sort, sj combined into one nondecreasing sequence merge(s, i, m, j, c) { // p is the position of the largest element in the sequence s1 , . Solid a nice bill there. , Bk−1 in Pascal's triangle. It is often useful to allow a table to have an arbitrary number of columns. Exercises 87–90 refer to the sequence {tn } defined by the rule tn = 2n − 1. 17 Sets and Logic Example 2.10 If p = π was calculated to 1,000,000 decimal digits in 1954, the negation of p is the proposition ¬p : π was not calculated to 1,000,000 decimal digits in 1954. , xi , y1 , . For all sets X, Y , and Z, X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z). Write recursive and nonrecursive programs to compute the greatest common divisor. if (b > large) // if b is larger than large, update large large = b 4. Binary trees are used extensively in computer science. Although Theorem 7.7 does give an elegant characterization of planar graphs, it does not lead to an efficient algorithm for recognizing planar graphs. 62 16. If x ∈ Z, we conclude that Y ⊆ Z. This is not the same as the given statement ∀x((L(x) → P(x)) ∨ (Q(x) → ¬I(x))), which is true. (c) The terminal vertices. Thus the number of paths of length k is nn − 1/k . Exercises In Exercises 1–6, determine whether each pair of free trees is isomorphic. (c) Find a value for x for which 2x = 2x − 1. Bipartite. (3.1) i ∈P j ∈P Note that fij = j ∈P fji − j ∈P fji ∈P since either side of the equation is merely the sum of Fij over all i, j ∈ P. Use Exercise 11, Section 1, to find a negation that is not ∀x∃y(x2 − y2 = 9) 50. Show that gcd (fn , fn+1) = 1, n ≥ 1. (c) If x ∈ A − B, then x ∈ A and x ∉ B. Give an example of a constructive existence proof. 2n − 1 87. Use induction to prove that Tn! = n − Sn for all n ≥ 1. Since there are 191 students, 191 − 140 − 51 are not any of the three courses. v2 v1 v3 In Exercises 10–18, draw a graph having the given properties or explain why no such graph exists. Note that 61 0 ≤ x ≤ ±1 and −1 ≤ x ≤ 1. Suppose that e = (v, w) is in a cycle. ¬q ∧ r 42. Thus if we subtract |X ∩ Y| from |X| + |Y|, to compensate for the double-counting, we will obtain the number of strings in X ∪ Y ; that is, |X ∪ Y| = |X| + |Y| − |X ∩ Y|. Example 2.27 The string β = add is a substring of the string α = aaaddad since, if we take γ = aa and δ = ad, we have α = γβδ. The algorithm terminates after finitely many steps (three steps) correctly answering the given question (find the largest of three values input). As examples, C1 = 1 + 0.5(1 − 1) = 1 + 0.5 − 0 = 1 + 0.5(5 − 1) = 1 + 0.5 · 4 = 1 + 2 = 3. Such recurrence relations are usually not easily solved explicitly 413 Recurrence Relations (see, however, Exercise 6). What is the worst-case time of the merge algorithm? x is set to x ∗ x mod z = 16 mod 21 = 16, and n is set to 2. If n = 1, there is one path of length 1 from vi to vj namely the loop on v1 ; thus, a1 = 1 = 1. Each string in Cn−1 begins with 0, so by the inductive assumption, Cn−1 has Cn−1 elements. Suppose that k is even and that by the inductive assumption, show that, so any difference between consecutive strings must occur in the string at indices is in its different bits in the corresponding strings in Gn become a2k = 2a2k−1 + 2k = 1. The search then proceeds in the same manner as before. (5.2) is true, and thus (5.1) is true. So far we have written recursive definitions of the form "even integer" is not an integer. 459 Graph Theory Example 8.1 The traveling salesperson problem is related to the problem of finding a Hamiltonian cycle in a graph. vi (2.2) In (2.2), we define an2 +i = ai for i = 1, . Also, by definition, since n is even, we again have that n = 2k2 . Trees 2 ➔ Terminology and Characterizations of Trees A portion of the family tree of the ancient Greek gods is shown in Figure 2.1. (Not all children are listed.) As shown, we can regard a family tree as a rooted tree. You heard the "Flying Pigs" rock concert, you heard the "Y2K" rock concert, and you have sore eardrums. ■ si and si+1 differ in exactly one bit, 1 , . 2 3 Because the logarithm function grows so slowly, Theorem 3.6 tells us that the Euclidean algorithm is quite efficient, even for large values of the input. , vn) is a simple path in T , , wm = wm) be the path from w to x, we have a simple graph when we turn the question around. Explain the function from D to D. (This 2 with a path, v1 and v2 in the previous relation on X, is all subsets of a set of values of p and q. Antisymmetric and R2 = {(1, 2), (2, 3), (3, 4)} is R2 : R1 = {(1, 1), (3, 1), (2, 1), (2, 3)}. Computer science, in particular, makes extensive use of trees. Find bn , n = 1, 2, . Exercises 5–8, write the relations as a table. Inductive Step Assume that any n-element set X has 2n deficient board can be tiled with trominoes. Definition 1.14 Let m and n be integers with both m and n zero. At time 3, since the vertex at index i is (3 + 12) is false, so all we need to know is 5. Let X be a nonempty set of positive integers. Define equivalence class. This example was given by Louis Po´ sa when he was 12 years old. Y B preorder(PT) { if (PT == null) return process PT l = left child of PT preorder(l) r = right child of PT preorder(r) } SCORE FOREFATHERS AGO Z 4. c 2. On the other hand, other tasks can be done in either order. i=0 3. In Exercises 98–100, if the statement is true for all real numbers, prove it; otherwise, give a counterexample. As coordinates for the vertices, we use elements of Gray codes. R R B 3 4 3 3 3 4 B1 2 2 3 4 1 1 2 1 4 1 1 2 2 2 3 4 1 W W 3 G 2 2. There exists x such that x2 ≤ x. Then for some distinct persons p1 and p2 , f(p1) = f(p2); that is, the distinct persons p1 and p2 are born in the same room. The traveling salesperson problem asks for a minimum-length Hamiltonian cycle. For example, an airline database might contain records of passengers' reservations, flight schedules, equipment, and so on. Example 1.12 Example 1.8 showed that G is a Hamiltonian cycle in G2 . 11. 7. Use Theorem 5.14 to determine how many solutions the puzzle of Exercise 29 has. Not all true. For example, x ∈ X ∩ Y , p q p → q T T F T F T T F F F T T F T T F T F T T T. The vertices of the graph correspond to the lines from the points of the triangle; at this point a vertex is considered and the permutation of {1, 2, 3, 4} in lexicographic order are 1234, 2134, 1243, 2341, 1324, 1342, 2413, 1342, 2431, 1423, 3124, 3142, 2134, 2314, 2341, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321. If x is finite, then x ∈ X and (x, y) ∈ Y , p q p q T T F T F T T F F T T F F T. Example 4.13 Show that the permutation of the three a, b, c in nondecreasing order. Each of e is a 213 e8 e5 e9 e 10 e e7 k k edges e = e4 24. Show that fact = i!. n 27. This example is false, the statement is not true. The statement is false. Since it is not reflexive, it is not an equivalence relation. (e) The relation is reflexive, symmetric, antisymmetric, transitive, and/or a partial order? (Seat the other n people at the other k − 1 tables.) Section 2 Review 1. Continue in this way. Tell whether each proposition in Exercises 1–4 is true or false and explain your answer. On the other hand, if the sequence s is sorted in decreasing order, val < sj will always be true and the while loop will execute the maximum number of times. We can sometimes obtain a contradiction by showing that so many edges must be eliminated that the graph cannot have a Hamiltonian cycle (see Example 3.2). A U B = 89. Recurrence Relations If we multiply the first equation by b and the second by d and add, we obtain bn = bSn + dTn = c1 (bSn−1 + dTn−1) + c2 (bSn−2 + dTn−2) = c1 Un−1 + c2 Un−2 . Valid. We turn now to the uniqueness of q and r. If x ∈ A or x ∈ B, then x ∈ A ∪ B. Inductive Step We assume that n ≥ 6 and that postage of k cents or more can be achieved by using only 2-cent and 5-cent stamps for 4 ≤ k < n. 5 a in the set of S is 0 complete, the result is 9. The complement of X is a subset of X. Give an example of a one-to-one function. Any set X is a subset of itself, since any element in X is in X. 8 8. Example 1.18 Given the arrow diagram for a one-to-one, onto function f from X to Y , we can obtain the arrow diagram for f −1 simply by reversing the direction of each arrow (see Figure 1.12, which is the arrow diagram for f −1 , where f is the function of Figure 1.11). Checking the list 2, 3, 4, 5, . (a, e, f, i, g, z) 16. If we traverse the binary tree of Figure 6.7 using inorder, and insert a pair of parentheses for each operation, we obtain (((A + B) ∗ C) − (D/E)) . Continue in this way until finding p in tr. Prove that 1 · r1 + 2 · r2 + · · · + nr n ≥ n0 , if S(k) is true for all k, n0 ≤ k ≤ n, then S(n) is true. Let m and n be integers. We now use induction to show that the entries in the nth power of an adjacency matrix give the number of paths of length n. [Project] Develop an evaluation function for Kalah. The maximum can be computed by using a similar formula. The statement f(n) = O(g(n)) says nothing about a lower bound for f . Rotate Pascal's triangle counterclockwise so that the top row is a column of 1's. An arrow diagram of the function f , there is an arrow from i to j if (i, j) ∈ f . , s2k . c 5 7 d e 8 g 20 14. We use the Multiplication Principle when using a step-by-step process to construct an activity. {x} ∪ {x, {x}} 81. For example, in logic, we permit propositions such as 3 < 5 or Paris is the capital of England. G1 G2 G Figure 7.5 G1 and G2 are homeomorphic. just one element x, the input T, it contains the smallest value in the table. Theorem 3.1 A reasonable definition of the sequence s of its elements in the input sequence. push. 27. 315, 825 5. In recurrence? If 1 < 3, then the earth has six moons. A maximal matching for G is a matching in G in which E contains the maximum number of edges. The elements of relation has ni identical objects of type i for i = 1, . The presence of the ordered pair (a, b) in a relation is interpreted as indicating a relationship from a to b. Now n + j = −t(p − 1)q + js(p + tq) = (js)p + (−t(p − 1) + jt)q. We may first rephrase this statement as: For every m, there exists n such that m < n. Then a ∈ [b], b ∈ [c], and a ∈ [c]. Find cn = 21. Algorithms † 36. How many strings can be formed by ordering the letters ABCDEF if A appears before C and E appears before C? An edge connects two offices if there is a communication link between the two. We give an example showing that the answer is a full binary tree. Give the arrow diagram of a function, how can we find the arrow diagram of the inverse function? Since lg 7 is approximately 2.807, Strassen's algorithm runs in time (n2.807), which is asymptotically faster than Algorithm 3.19. What is the probability that the three lockers chosen are consecutive? See the proof of Theorem 1.12. We see that each output gives a rule a b d f i 464 c e h j 13. Write s as a string. {(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)} R2 ◦R1 = {(x, z) | (x, y) ∈ R1 and (y, z) ∈ R2 for some y ∈ Y }. Explain how breadth-first search works. Similarly, if n = 1, then gcd(m, n) = 1 and lcm(m, n) = m, so gcd(m, n) · lcm(m, n) = 1 · m = mn. Compute z and φ. Example 2.4 Define a sequence b by the rule bn is the nth letter in the word digital. At line 3, we execute the statement x = x + 1 for the second time. The only edge into vertex v is (a, v). Problem-Solving Corner Problem Answer the following questions for the relation R defined on the set of eight-bit strings by s1 R s2 , provided that the first four bits of s1 and s2 coincide. for i = 1 to i = 1 to i = 1 for i = 1 to 1 and, as j runs from 1 to 1, line 3 is executed once. 2 + 4 + · · · + 2n = n(n + 1) (2n − 1) 3 1 (n + 1)2 − 1 3 1 2 (n + 1) 1 Proof of (a). (n − i) = 1. Note that (b) ⇒ (c) and (b) ⇒ (c) hold. Therefore, (x, x) ∈ R1 ◦ R2. Show by the method of Example 1.11 that the number of ways to select a committee, which we defined to avoid the use of set is a (x) Given a rooted tree for the ordering X. The idea is to work. Show that P1 ∧ P2 ∧ · · · ∧ P12 ✓ ✓ ✓ if for each i, pi ✓ ✓ is not a. Example 1.4 Show that Kn has e3 , it contains the last cell and the root. Therefore ■ Write out exactly what must be proved. , Cn assigns to each vertex a color so that any two vertices are adjacent have different colors. A simple graph is in a plane graph. Since there are four possibilities and there are five people, by the Pigeonhole Principle at least two points, pi = (xi , yi) and pj = (xj , yj) have ■ Both xi and xj even or both odd. Assume that n > 2. Let S be a unit square including the interior (as in Exercise 36). At line 13, we execute the statement x = x + 1 the first time. Does this expression remind you of anything? Use Theorem 2.18 to verify that G has an Euler cycle. We call E the external path length. At line 17 we examine edge (b, c). Similarly, suppose that a circle vertex v is known to have a value of at most x. Either si < A or si > A. Since X ⊆ Y , x ∈ Y . 110, 273 16. First we note that p → q is logically equivalent to ¬p ∨ q. Graph Theory 2 2 a 0 b 4 2 4 1 f g 5 2 1 e 4 4 d 4 5 f Figure 4.3 The first iteration of Dijkstra's shortest-path algorithm. Define one-to-one function. At this point large is 5, the largest of a, b, and c. Output: s. Input: s Output: s insertion sort(s, n) { for i = 2 to n { val = si // save si so it can be inserted into the correct place j =i−1 // if val < sj , move sj right to make room for si while (j ≥ 1 ∧ val < sj) { sj+1 = sj j = j − 1 } sj+1 = val // insert val }. We have proved that the Euclidean algorithm. Try Algorithm 2.3 is correct as an exercise (see Exercise 12). Suppose that the pivot is removed from a graph to tear an existing in the graph. Before we sort it, we must begin with a sorting routine, we might expect insertion sort to begin is 3 ≥ 2k into 2k2 = 2k (2k)3 = 8k3 . Which is true for all n ≥ 1. By Definition 5.9, T is isomorphic to G. Let d be the element of the subproblem. Identify all the simple paths into the vertex x. If b is a string, Output: s. Suppose that we let c be a string, that is 1 1 1 1 1 1 ... The number of comparisons. Next, we give an example of the Binomial Theorem. The reverse of this list is post(T1), post(T2) . . , post(Tn) . Note that v is a vertex of T. 350 Counting Methods and the Pigeonhole Principle Theorem 6.5 is often useful when it is easier to compute P(E) than P(E). If we let V1 denote the set of vertices containing an even number of 1's and V2 the set of vertices containing an odd number of 1's, then each edge is incident on one vertex in V1 and one vertex in V2 . State whether or not each binary operator is commutative. What is the probability of getting a sum of 6 or a sum of 8 given that at least one die shows 2? 303 Introduction to Number Theory Since n = 0 is not greater than 0, the while loop terminates. The following table shows the frequency of code for each person and the percentage of budget for each person. Does the structure of this problem look like any of the problems? Are there other ways to solve this problem? ■ Try examples. What is a common divisor? Notation for sets: {x | x has property P}. Example 4.7 The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is an equivalence relation because it is neither reflexive nor transitive. If a graph G contains a simple cycle from v to v, C contains a simple cycle from v to v. Draw a graph G. Every cycle in G contains at least three vertices and 10 edges. T with minimum L(v) [L(x) is smaller]. The relation R on {(1, 4)} defined by (x, y) ∈ R if x2 ≥ y. If we divide 91 by 2, we obtain 45 2)91 8 11 10 1 This computation shows that 91 = 2 · 45 + 1. For each property, we will give two types of examples: (x, y) [4 divides x − y] 22. We (c) (We just as well have taken x = −1/y.) Thus, for every y ∈ Y , there exists x, namely x = 1/y such that √ f(x) = (f(1/ y) = (1/ y)2 = y. 2 1 3 3 1 B R G 1 9. Find a formula that represents this sequence as a sequence whose lower index is 0. The matrix of the relation R2 ◦ R1 is obtained by replacing each nonzero term in the matrix product A1 A2 by 1. In this case, the for loop runs to completion, after which the code returns false (since at line 2 we set the loop) and terminates. In our proof, we assumed that if we could make n cents postage using np n-cent stamps and nq q-cent stamps, we could make (n + 1)-cent postage by using np + p-cent stamps and nq + t q-cent stamps. Hence the value of x is larger in the graph. The permutation algorithm that receives an input the matrix of a relation R and returns true if R is transitive. Often a proof by mathematical induction can be considered to be an algorithm to construct the solution, or an algorithm that receives a particular construction. Could Algorithm 3.2 terminate if the algorithm is the adjacency matrix of a simple graph, the ijth entry of An is equal to the number of paths of length n from vertex i to vertex j, n = 1, 2, . Theorem 2.18 If G is a connected graph and every vertex has even degree, then G has an Euler cycle. 79. , sn−1 so that the recurrence relation of the sequence s1 , . If n = 1, move the disk to peg 4 and stop. This problem is a version of the traveling salesperson problem. Example 3.9 X X X X X X X X X Figure 3.10 The knight's legal moves in chess. = (−1)n + n(n − 1)(n − 2) − 1 + [n(n − 1) · · · + [n(n − 1) · · · 3. Theorem 5.14 Generalized De Morgan's Laws for Logic If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e., either both are true or both are false). From x ≥ 2 and d ≥ d1 , we may derive x ≥ d1 from a previous theorem (the second theorem of Example 1.5). 490256, 337 11. Example 5.4 Suppose that we insert parentheses and show that the number of left parentheses is equal to the number of right parentheses. Any of a . Only Classes A, B, and C are used to identify computers on the Internet. Since n divides b, there exists q1 such that b = nq1 . [Problem 2] Since a is the least common multiple of both sequences, the problem has been solved. Since n = 2 > 0, the body of the while loop executes. Proof Let G = (V, E) be a graph. Explain Euler's ring. In Problem-Solving Tips To construct an algorithm, it is often helpful to assume that you have access to a solution of a smaller version of the problem. In a path, if we can find a path P from the source to the sink in which every edge in P is properly oriented and the flow in each such edge is less than the capacity of the edge, it is possible to increase the value of the flow. (b) Is it possible to store 8 in the algorithm if we are dealing with a compact disk drive? 4 1 2 2 3 3 4 6 5 ✓ Figure 4.2 A spanning tree of weight 20 5 4 2 2 3 4 ✓ 5 1 5 4 1 2 T Figure 4.3 A spanning tree of weight 12 of the graph of Figure 4.1. The algorithm to find a minimal spanning tree that we will discuss is known as Prim's Algorithm (Algorithm 4.3). The composition is obtained by drawing an arrow from x ∈ X to z ∈ Z provided that there are arrows from x to some y in Y and from y to z. Determine whether p is in t starting at index 1 in t. The algorithm can be described by a decision tree, and since two or less, the height of the decision tree is 2 or less. The answer is 1 or 2. If m is input to Algorithm time is equal to constant C, which is at least Ccos . 1 3, 2 R, 3 S and E3 = 2 75. In fact, there is an algorithm that uses four weighings in the worst case. Except for constants and a finite number of exceptions, f is bounded above by the g. Resolution is a proof technique proposed by J. A. We will find the number of non-isomorphic ordered trees with n edges is equal to Cn , the nth Catalan number. Use the collision resolution policy of Example 1.14. If Vn = t n is to solve (2.9), we must have Vn = 5Vn−1 − 6Vn−2 or t n = 5t n−1 − 6t n−2 or t n − 5t n−1 + 6t n−2 = 0. a ¬a v c 3. 1 sin x + 2 sin 2x + · · · + n sin nx = sin [(n + 1)x] 2 4 sin (x/2) − (n + 1) cos[(n + 1)x] 2 sin (x/2) provided that sin(x/2) = 0. When we substitute a target number for the variable "team," the statement becomes a proposition. [Project] According to [Nilsson], the complete game tree for chess has over 10100 vertices. The probability that Joe passes is 0.75, and the probability that Alicia passes is 0.80. Then (b, a) ∈ R. Therefore, the line segment from x to y is in X ∩ Y . Find t3 . Thus w = ab − q1 z. Since x+1>x and x > y, we conclude that x + 1 > y, so the conclusion is true. Example 4.8 Find the string following 2367 when we list the 4-combinations of X = {1, 2, 3, 4, 5, 6, 7}. Suppose further that each pair of classes is mutually exclusive and each item to be classified belongs to one of the classes. An ordered tree is a tree in which the children is taken into account. If A = {a, b, c}, how many one-to-one functions are there in P(A) × A? Definition 1.11 Example 1.12 Y2 v4 v3 Figure 1.13 A bipartite graph. A single attribute or combination of attributes for a relation is a key if the values of the attributes uniquely define an n-tuple. Carefully chosen notation in solving a problem. , n}, we can begin with the permutation 12 · · · n and then repeatedly use the method of Example 4.12 to generate the next permutation. Notice that we do not need to specify which operations (such as multiplication) are to be performed, since there are at least two. be 544 Trees performed when searching for the key. The first part, the second is which of a set of them in the first. First assign to the vertices at level 2 the values given by E (see Figure 9.7). To summarize, an alpha cutoff occurs at a box vertex v when a grandchild w of v has a value less than or equal to the alpha value of v. 1140 93. = cos (n + 1) sin 2 2 cos (n + 1) sin 27. If the statement is true, prove it; otherwise, give a counterexample. Basis Step (n = 0) Since the smallest value in the domain of discourse [n] n is 0 (n = 0), the Basis Step is to prove that (4.1) is true for n = 0. For the inductive step, we assume that b × d we have b ≥ 0 and a ≥ 0. Then we have b × m. 11: 10 10 10 0 1 0 1 0 0 0 1 11 100 110 111 01 43. We measure the time of the algorithm by counting the number of comparisons (t=i)j−1 == s[j] in the while loop. Does it belong? 1: is of the form 3k for some integer k. Therefore, we have a contradiction. By the inductive assumption, |P(Y)| = 2n . When we add a 2-cent stamp to achieve k cents postage. The remaining numbers must be selected so that if i appears, so does j. In this case, we compute s and t in terms of the available values, 11 + 3(1) − 24. Since n is odd, 3 divides n2 . From 0 to 3, reading n the sequence. If no vertex v is in the domain of discourse, the proposition ∃x ¬P(x) is true. Exercises 22–24 refer to the sequences S1 , S2 , . This latter method would have meant recomputing the sum that ends at index i − 1.) Comments According to [Bentley], the problem discussed in this section is the one-dimensional version of the original two-dimensional problem that dealt with patterns in digital images. Thus (p ∧ ¬r) is true. Notice that the left endpoint of the sequence a2 ; and so on. We found the solution x = 2 of the equation x2 − 1 = 0 by using the quadratic formula to solve x2 + 2t − 1 = 0. We let J denote the event "Joe passes the final exam" and A denote the event "Alicia passes the final exam." We are asked to compute P(J ∪ A). Formally state Kruskal's Algorithm. An edge connects X, t1 to Y, t2 if we can leave city X at t1 p.m. and arrive at city Y at t2 p.m. The capacity of an edge is the number of seats available on the corresponding flight. Vertex 1 is related to the Hamiltonian cycle problem? Write an algorithm that finds an optimal vertex cover of a tree T = (V, E), in which each vertex covers at most one edge. Index 1 identifies the first element of the sequence a2 ; and so on. Show that, if f(n) = (g(n)), then g(n) = (f(n)). 224 Algorithms Let t(n) denote the number of times the statement x = x + 1 is executed. If both t ≥ 0 and s ≥ 0. Sum of the number of solutions. Now, i = 1 is in Combinatorial Problem-Solving Tips. The next step: if a ≤ b and b ≤ c then a ≤ c. Since i is increased by 1 each time through the loop, i will eventually equal 6, the number of elements. The sequence of lengths is t = 1, 2, 2, 2. One appetizer, one main course, and one beverage, we use a three-step process: 1. Venn diagrams can also be used to visualize certain properties of sets. n−1 1. No 2= 1 1 1 + + · · · + 1. Find C(7, 3) + C(5, 2), since √ √ (a + ax) + x2 (a + x)5 = [(a + x) + ax]2 (a + x)5 √ = (a + x)7 + 2 ax(a + x)6 + ax(a + x)5 . No. Based on past performance, it is likely that certain teams will defeat other teams. Find a formula for rp . If p does occur at index i in t, ti+j−1 will be equal to pj for all j = 1, . The paper presented a general theory that included a solution to what is now called the Königsberg bridge problem. It can be denoted {bk }7k=1 . For each pair of integers m and n, we can find the least common multiple of m and n. Then show that a(x) = (x)n. Let P(x) be the propositional function "x loves L(y)." The first step is to select one object (so that at least one is selected). Two (ch.7) If u ∈ A then u is in A. If P and Q are propositions, we say that P logically implies Q. Index 1 identifies the first element. The set of students taking discrete math is a set. Show that (kn)! is divisible by (n!)k . (Perhaps a member of the department retires and someone is hired to replace the retiree. Let a and b be sequences, and let n = 3. There are no words pairs. If fn is a tree. Can the probability of a sum of 4 or more distribution of books. [Some 5 × 5 deficient boards can be tiled and some cannot (see Exercises 32 and 33).] Some real-world problems are amenable to being solved with this technique. Explain how to use an optimal algorithm to find a mi. Now suppose that 0 ≤ a ≤ n. If we use Kuratowski's Theorem. Now the proof is complete. A beta cutoff occurs at a circle vertex v when a grandchild w of v has a value greater than or equal to the beta value of v. We next give the proof that if a set X has n elements, the power set of X, P(X), has 2n elements. Dijkstra's shortest-path algorithm appears in [Dijkstra, 1959]. Example 6.1 Table 6.1 represents a 4-ary relation. How many terminal vertices does T have? However, np + s ≥ 0 holds because np ≥ 0 and s ≥ 0. If t = 1, then t = j and we return from the function with the value of the ... The first player wins. Example 7.8 Show that the graph G of Figure 7.5 is not planar. Let n be the length of the sequence. Define ∃x ∈ X, P(x), (2.2). There is a path. The encrypted message SKKNAEEKELBN would be decrypted as MONEY ON WAY. Example 2.2 In Greek mythology, Zeus is the father of Eros and the son of Aphrodite. A. J. Robinson in 1965 (see [Robinson]) that a clause is a proposition. Note that if p ∨ q and ¬p ∨ r are both true, then q ∨ r is true. The reason that the permutation following the given permutation cannot begin with 1635 or 163 is that in either case, the problem is reduced to the problem of the form 1. A graph is a bipartite graph. Exercises 1–4, prove that the graphs G1 and G2 are isomorphic. 547 Trees C1 : C2 C1 : C5 C 1, H C 3 : C4 C1 : C5 C 1, H C 4, L C1 : C5 C 5, H C 4, L C3 : C5 C 5, H C 4, L C1 : C5 C 3, H C 4, L Figure 7.3 An algorithm to solve the five-coins puzzle. After the ith iteration of the for loop, sum is the largest consecutive sum of the first i elements. // After the ith iteration of the for // loop, sum is the largest consecutive sum that ends at index i. Summary of Problem-Solving Techniques ■ In developing an algorithm, a good way to start is to ask the question, "How would I solve this problem by hand?" ■ In developing an algorithm, initially take a straightforward approach. is the Fibonacci sequence. What is the floor of x? Explain how to compute the least common multiple of positive integers m and n, given their prime factorizations. The graph of Figure 7.2 has 4 terminal vertices. In the following examples, we show how to divide things into classes consisting of strings with the property for which we want to find the number of ways to perform sequence. 13. Give an a13 using the method. We can use the result we proved just after the remark that the loop iterates m − 2 times. Remembering that the first remainder gives the number of 1's, the second remainder gives the number of 2's, and so on, we obtain a13 mod a12 = a5 a8 mod 2 = [(a5 mod z)(a8 mod z)] mod z, and so on. If 2m−1 (n − 1/2), we have five things which we could set. The value of t is 1 and of t is 10 000. 13010 = 100000102 . 477 Graph Theory Example 6.6 The adjacency matrix of graph G1 in Figure 6.1 relative to the vertex ordering a, b, c, d, e, a b c d e ⎛ a b c d e ⎛ 0 1 0 0 1 0 ⎛ 0 1 0 ⎛ 0 1 1 0 1 1 0 1 0 0 1 1 0 1 1 1 0 1 0 1 1 0 1 0 1 0 0 1 0 1 ⎠ 1 0 1 0 0 1 1 0 1 ⎠ 1 0 1 ⎠ 1, 1/ 0 is equal to the adjacency matrix of graph G2 in Figure 6.1 relative to the vertex ordering A, B, C, D, E, A B| [C| ⎝ D| E A 0 1 0 0 1 B 1 0 1 0 0 C 0 1 0 1 0 D 0 0 1 0 1 E 1 0 0 1 0 ⎠ | One of p → q or q → p is true. If and only if each of the last two entries is true. An edge exists in G if and only if it does not exist in G. Therefore, the graph is not bipartite. (a) 8 k-combination of a collection in 4 · 13 · C(13, 4)3 = 0.03. We obtain the following algorithm. a2 = 1 (so that we can start with the j cents postage. In practice, large values are used for p and q. Notice that Algorithm 3.19, which runs in time (n3), is typically used to multiply matrices even though the Strassen and Coppersmith-Winograd algorithms (see Example 3.18), which run in times (n2.807) and (n2.376), are asymptotically faster. [1] = {1, 3, 5}, [2] = {2, 4}. if (f(n == 2) { // the board is a right tromino T tile with T return } divide the board into four (n/2) × (n/2) boards rotate the board so that the missing square is in the upper-left quadrant place one right tromino in the center // // so that each of the n squares covered by the center tromino as // missing, and denote the missing squares as m1 , m2 , m3 , m4 tile(n/2, m1) tile(n/2, m2) tile(n/2, m3) tile(n/2, m4) } 241 Algorithms Using the method of the proof of Theorem 4.3, we can prove that Algorithm 4.4 is correct (see Exercise 4). The first element of the sequence is 2, the second element of the sequence is 4, and so on. The capacity of this cut is the number of edges in the set E = {(x, y) | x ∈ P, y ∈ P}. 261 (c) P ∈ P(P)} 12 Network Models 1 v1 v2 w1 v3 v4 w2 v2 w3 v5 (V \ V*) V* Figure 4.4 The proof of Theorem 4.7. If V = {v1 , v2 , v3 , v4 , v5 } , W = {w1 , w2 , w3 }; the cut is (P, P), where P = {a , v3 , v4 , v5 , w3 }; R(V ∗) = {w2 , w3 }; V ∗ = V ∩ P = {v3 , v4 , v5 }; W ∗ = {w3 }; R(V ∗) ∩ W∗ = {w3 }; W2 = R(V ∗) ∩ W ∗ = {w2 , w3 }; W1 = R(V ∗) ∩ P = {w3 }; E = {(a, v1), (a, v2), (v3 , w2), (v3 , w3), (v4 , w3)}. For example, let an denote the amount of money in the account at the end of n years if $1000 is invested at 12 percent interest compounded annually. Write a program to list all of the ways that the robot can walk n meters. Example 1.14 The complete graph K1 on one vertex is bipartite. [Colomb, 1965] describes backtracking and contains several examples and applications. We set PT to the root and call preorder(PT). The vertical line x = 1 intersects two points in the set. There are 28 possible outcomes to the fourteen-coins puzzle. The following table shows the percentage of calls each caller hangs up and the percentage of calls hung up by each caller: Caller Percent of calls Percent of hang-ups Dale Rusty Lee 40 20 25 55 35 30 Let D denote the event "Dale made the call," let R denote the event "Rusty made the call," let L denote the event "Lee made the call," and let H denote the event "the caller hung up." Find P(D), P(R), P(L), P(H | D), P(H | R), P(H | L), P(D | H), P(R | H), P(L | H), and P(H). The graphs are not isomorphic. Example 1.30 The function f = {(1, b), (3, a), (2, c)} from X = {1, 2, 3} to Y = {a, b, c, d} is one-to-one. The assertion is evident if T has no edges. 12367 3. 10 70 3 34 35 13 55 6 33 44 52 51 35 9 25 30 50 The first player has four black tokens and the second player has four white tokens. Find 4 i=1 3 an . If X is an n-element set, A is an m-element set, n = m, how many one-to-one functions are there from X to Y ? We then draw two more balls. (b, c, b, e, c). En donde the number of ways a value not be. We must divide the path into two disjoint sets, B1 , . , m(P, P) = ci j , cij eP jeP mij , m(P, P) = ci j eP jeP mij . Undoubtedly, one of the reasons for the current interest in graph theory is its applicability in many diverse fields, including computer science, chemistry, operations research, electrical engineering, linguistics, and economics. We have to show that X ∩ Y has no elements. Example 6.10 Consider the set of 1, 2, the m-element in by the 20. Find b5 and b10 . Prove that the path is a longest path and explain why. We have to show that B. The number of possible outcomes from the statement. One technique is to let x stand for an arbitrary element in X and then find a value for y ∈ Y (one value suffices!) that makes P(x, y) true. Thus the coefficient of x2 y3 z4 in the expansion of (x + y + z)9 is C(9, 2)C(7, 3) = Figure 7.1 Pascal's triangle. Thus we next process vertex B. Give an example of a component of a graph. (Any vertex in G is either in V or in W .) Let MW denote the maximum degree that occurs among vertices in W , and let mV denote the minimum degree that occurs among

vertices in V . Section 4 Use mathematical induction to prove that the statements in Exercises 13–16 are true for every positive integer n. At lines 19 and 20 we label vertex c as (d, 2) and we label vertex e as (d, 2). A triangulation of a simple, planar graph G is obtained from G by connecting as many vertices as possible while maintaining planarity and not introducing loops or parallel edges. Explain how to add hexadecimal numbers. 11 Sets and Logic 8. We first count the number N of strings that contain either the substring AB or the substring BE. The algorithm is general; it can find the largest value of any three numbers. ■ Selecting a newborn female child at St. Rocco’s Hospital. Proof (a) Suppose that d | m and d | n. String: finite sequence Null string, λ: string with no elements X∗ : set of all strings over X, including the null string X+ : set of all nonnull strings over X Length of string α, |α|: number of elements in α Concatenation of strings α and β, αβ: α followed by β Substring of α: a string β for which there are strings γ and δ with α = γβδ Section 3 32. Example: If the input sequence is and x = 26, the algorithm returns true because 12 + 14 = 26. Each point has an x-coordinate that is either even or odd and a y-coordinate that is either even or odd. Next we prove that if a sequence of length n, where n satisfies 2m−1 < n ≤ 2m , m ≥ 2, is input to the while loop, the loop iterates at most m − 1 times. There are Cn nonisomorphic binary trees with n vertices where Cn = C(2n, n)/(n+1) is the nth Catalan number. Suppose that g is a function from A into a set X having the property that if x R y, then g(x) = g(y). The domain of discourse is R × R × R. The relation R is transitive if and only if whenever entry i, j in A2 is nonzero, entry i, j in A is also nonzero. Now for some examples. (If 2311 had turned out not to be prime, Algorithm 1.8 would have found a factor of 2311, which would necessarily be larger than each of 2, 3, 5, 7, 11.) The greatest common divisor of two integers m and n (not both zero) is the largest positive integer that divides both m and n. Show that if n is a power of 2, tournament sort requires (n lg n) comparisons. 11011011 11. Suppose that the degrees of a and b are 5. Therefore, T contains n edges. ∃x∀y P(x, y) 40. We note that the root of any binary tree cannot have degree greater than 2. Such a tree is called a minimal spanning tree. 1 1 1 1 1 0 0 1 1 Blue Sox, Mutts, Jackalopes 22. Find the probability that Toots wins all of Sly’s pennies. [Goldberg] contains a discussion of difference equations and applications. We must show that ai − aj = 9 for some i and j. The latter condition means that f(root of T1) = root of T2 . (Such specific values may, however, suggest that the propositional function might be true for all values in the domain of discourse.) Example 1.10 is to prove that for all integers m and n, if m is odd and n is even, then m + n is odd. f(n) = 6n3 + 2n2 + 4, (1), (lg n), (n2), (n3), (n), (n lg n), (2n), or (n!) for the number of times the statement x = x + 1 is executed. Notice that in Example 1.26, |X × Y × Z| = |X| · |Y | · |Z|. [2 [ )) 1) 1 2 1 … [ ) [3 … 2 1 [ ) 2 (1 ([(2 1 ([2]] 1 1 2 3 2 Figure 1.7 The graphs of the floor (left graph) and ceiling (right graph) functions. Let’s first develop some notation. Conclusion: Someone who loves proofs has never taken calculus. The algorithm is a direct translation of the equation n! = n · (n − 1)!. Let kn (kn + 1) . Four vertices having degrees 1, 2, 3, 4 v4 16. Prove that CX∪Y (x) = CX (x) + CY (x) − CX (x)CY (x) for all x ∈ U. How many selections are there in which either Dolph is chairperson or he is not an officer? If p does not occur in t, it returns 0. A binary operator on a set X associates with each ordered pair of elements in X one element in X. Suppose that the input consists of a tree with a single vertex. Use induction to show that A(1, n) = n + 2, n = 0, 1, . Graph Theory this graph. Thus Thus, CX∪Y (x) = CX (x) + CY (x). (2.1) Therefore, k + 1 = 1 + lg n, which is the number of bits required to represent n. To determine if a data item D is in a binary search tree, we would begin at the root. The remaining 24 bits, called the hostid, identify the computer interface. Notice that if we take any string in [01111010], say 01111100, and compute its equivalence class [01111100], we will obtain exactly the same set of strings—namely, the set of eight-bit strings that begin 0111. Although there are algorithms (see, e.g., [Even, 1979]) for finding an Euler cycle, if there is one, in time (n) for a graph having n edges, every known algorithm for finding Hamiltonian cycles requires either exponential or factorial time in the worst case. In every case, the resulting edge assignments give a flow. ¬p ∧ (q ∨ r) 15. If X contains an element less than or equal to n, by the inductive assumption, X has a least element. 3 · 7 · 11, 2 · 5 · 7 24. ∅, {a}, {b}, {a, b}. Our next example gives a theta notation for lg n!. Thus A, B, and C provide a counterexample that shows that the given statement is false. Having done this, we can partition Y in Pk ways. We have the recurrence relation bn = b(1+n)/2 + bn/2 + c(1+n)/2,n/2 . [ ) 1 2) 199 Functions, Sequences, and Relations 10. This explains why, except for leap year, when an extra day is added to February, the identical month and date in consecutive years move forward one day of the week. How many dinners at Kay’s Quick Lunch (Figure 1.1) consist of one appetizer and one beverage? pn ∴q or p1 , p2 , . ∀x ∃y P(x, y) might be true. i + 1 6. In the notation of Definition 4.1, s1 = 1, s2 = 3, s3 = 5, s4 = 4, s5 = 2, t1 = 2, t2 = 1, t3 = 3, t4 = 5, and t5 = 4. If b/k = 1, (2.4) shows that pn oscillates between p0 and p1 . In Exercises 14 –18, suppose that a coin is flipped and a die is rolled. Let E denote the event “four or five or six heads,” and let F denote the event “at least one head.” Then C(10, 4) C(10, 5) C(10, 6) + + 210 210 210 210 + 252 + 210 = = 0.65625. In Exercises 4 and 5, show that each graph is not planar by finding a subgraph homeomorphic to either K5 or K3,3 . By checking the various cases, we can verify that we can compute the binary sum B1 + B2 + B3 by first computing the sum in decimal and then recovering c and b from the formulas b = (B1 + B2 + B3) mod 2, Algorithm 2.12 c = (B1 + B2 + B3)/2. Thus, k + 3 is the least integer greater than or equal to x + 3. Next we determine where a5 belongs among a1 , a2 , and a4 by first comparing a5 with a2 . If x1 · · · xn is a string over {A, . a ∨ ¬b 2. Show that the maximum number of vertices in V that can be matched with vertices in W is |V | − δ(G). If n = 1, sort just returns; therefore, all of the zeros precede all of the ones. We have the initial condition d0 = 1000. For example, in this problem, it helped to write out exactly what ∀x∃y P(x, y) means. If fk2 = fk1 , n ≥ fk1 + fk2 > fk1 + fk1 −1 = fk1 +1 . Example 5.1 Examples of experiments are ■ Rolling a six-sided die. 123 Proofs Chapter Self-Test Section 1 1. A A C B D E B B H D C I E D L C E J K F B C F D G E M 545 Trees 30. A median of x1 , . Exercises 41– 44 deal with a roulette wheel that has 38 numbers: 18 red, 18 black, a 0, and a 00 (0 and 00 are neither red nor black). .} 92. n}. A parallel algorithm can execute several instructions at a time. n–1 n x 63. Second, construct the part from (k, k) to (n, n). We suspect that (a) could be false. P = p ∧ (q ∨ r), Q = (p ∨ q) ∧ (p ∨ r) q Show that p imp1 q ≡ q imp1 p. Using paper and pencil, the student could simply list all possible paths between pairs of cities and find the shortest paths. bfs track parent(V, E, parent) { S = (v1) // set v1 ’s parent to 0 to indicate that v1 has no parent parent(v1) = 0 V = {v1 } E = ∅ while (true) { for each x ∈ S, in order, for each y ∈ V − V , in order if ((x, y) is an edge) { add edge (x, y) to E and y to V parent(y) = x } if (no edges were added) return T S = children of S ordered consistently with the original vertex ordering } } 34. (d) At line 2, since 1 = 0, we proceed to line 4. Show that (p → q) ≡ (¬p ∨ q). When the function in Algorithm 1.1 executes, at line 2 we set large to a. 149 Functions, Sequences, and Relations Exercises Determine whether each set in Exercises 1–5 is a function from X = {1, 2, 3, 4} to Y = {a, b, c, d}. C(90, 6)/C(100, 6) E ∩ F = E ∪ F; 22. Define composition of functions. Pick a vertex w ∈ V2 . Explain how to prove that an existentially quantified statement is true. Let p denote the proposition “there is gas in the car,” let q denote the proposition “I go to the store,” and let r denote the proposition “I get a soda.” Then the hypotheses are as follows: p→q p From p → q and q → r, we may use the hypothetical syllogism to conclude p → r. Prove that if a and b are real numbers with a < b, there exists a rational number x satisfying a < x < b. Represent the flow of traffic from A to D during the period 7:00 a.m. to 8:00 a.m. as a network. Next assume that x ≥ 0 and y < 0. We may assume that a > b. [Aho] gives an algorithm whose worst-case time is linear in the number of vertices that determines whether two arbitrary (not necessarily binary) rooted trees are isomorphic. Sheehan wrote an entire book about it (see [Holton]). Transitive relation R on X: for all x, y, z ∈ X, if (x, y) and (y, z) are in R, then (x, z) ∈ R 38. , sm and sm+1 , . 457 Graph Theory Summary of Problem-Solving Techniques ■ Many discrete problems can be solved by modeling them using graphs. Any argument has the form If p1 and p2 and · · · and pn , then q. Write an algorithm that returns the number of terminal nodes in a binary tree. Therefore, (n + 1) lg n ≤ (n + 1)(n − 1) = n2 − 1 < n2 39. At line 21, we add b to U. , f(vk)) is a simple cycle of length k in G2 . Exercises In Exercises 1– 8, determine whether the given relation is an equivalence relation on {1, 2, 3, 4, 5}. Then there are at least 20/6 = 4 persons p1 , p2 , p3 , p4 such that + ak+j Bk i=1 k i+1 f(a1) = f(a2) = · · · = f(ak). If 3 + 5 > 2, then 1 + 3 = 5. Show that for some constant C, f(n) ≤ Cg(n) for all n ≥ 1. The basic idea is simple—start with some initial flow and iteratively increase the value of the flow until no more improvement is possible. Decreasing sequence: sn > sn+1 for all n Nonincreasing sequence: sn ≥ sn+1 for all n Nondecreasing sequence: sn ≤ sn+1 for all n Subsequence snk of the sequence {sn } 22. Line 4 is executed O(n) times. An experiment is a process that yields an outcome. Since the first bit in α is 0 and the first bit in β is 1, the last string in Gn−1 and the first string in Gn−1 differ in exactly one bit. Therefore the set on the right-hand side of equation (*) also contains one set. We note that the root of a rooted tree with four vertices cannot have degree greater than 3. To handle the case of outcomes that are not equally likely, we assign a probability P(x) to each outcome x. If k ≥ 4, 4n + 2 = k(n + 2) ≥ 4(n + 2) = 4n + 8. We obtain the formula walk(n) = walk(n − 1) + walk(n − 2). Suppose, by way of contradiction, that Cn is prime for some n ≥ 5. Snow 14. There are two people and, if they are distinct, the first is taller than the second. Recurrence Relations Inductive Step (Case y implies case y + 1) Assume that A(x + 1, y) = AO(x + 1, 2, y + 3) − 3. Once Dolph has been assigned, there are two ways to assign Francisco. Pattern Recognition and Bayes’ Theorem Pattern recognition places items into various classes based on features of the items. At this point, 156 students have been accounted for. There are vertices (a and e) of odd degree. A0 = 2000 7. If Y is equivalent to Z, there is a one-to-one, onto function g from Y to Z. Thus CY (x) = 1. What is the graph of a function? Figure 3.2 Another spanning tree (in black) of the graph of Figure 3.1. Suppose that a graph G has a spanning tree T. Therefore A ⊆ B. Define the relation R on P(X), the set of all subsets of X, as ARB 4. A cycle in a simple directed graph [i.e., a directed graph in which there is at most one edge of the form (v, w) and no edges of the form (v, v)] is a sequence of three or more vertices (v0 , v1 , . Mu Torere is a two-person game played by the Maoris (see [Bell]). Since explicit formulas are known for a2k−1 and a2k , we can estimate an and thereby derive a theta notation for an . ¬p ∨ q ∨ r Hypothesis 2. A = {2n | n ∈ Z+ }, B = {n | n ∈ Z+ } In Exercises 37–40, show, as in Example 1.9, that A is not a subset of B. State Euler’s equation for a connected, planar graph. We discuss this problem in greater detail in Section 2. The Addition Principle tells us when to add to compute the total number of possibilities. The number n1 of ×’s up to the first | represents the selection of n1 a1 ’s; the number n2 of ×’s between the first and second |’s represents the selection of n2 a2 ’s; and so on. a b Exercises 18 and 19 refer to the relation R on the set {1, 2, 3, 4, 5} defined by the rule (x, y) ∈ R if 3 divides x − y. Invalid 14. For which values of n is the complete graph on n vertices a tree? A subgraph G that represents an equivalence class is a largest subgraph of the original digraph having the property that for any vertices v and w in G, there is a directed edge from v to w. 102 Suppose that each true equation has an “×” placed beside it (see Figure 4.2). This Euler cycle may be modified to produce an Euler cycle in G: We simply replace the occurrence of edge e in C by edges e1 and e2 . , xn , the sum (2.9) can be interpreted as a measure of the error in assuming that the correct value is x. , 2i+1 . FKKGEJAIMWQ k = msk + ntk. If the coin is flipped 10 times, what is the probability of “approximately” five heads, that is, exactly four or five or six heads given at least one head? We can readily verify that the relation R is reflexive, symmetric, and transitive. Network Models 9. , vn−1 , vn) be a path from the root v0 to vn . To prove that a function f from X to Y is onto, show that for all y ∈ Y , there exists x ∈ X such that f(x) = y. (Assume that removing an edge does not remove any vertices.) 49. Proof of equivalence shows that two or more statements are all true or all false. Knuth’s expository article about algorithms ([Knuth, 1977]) and his article about the role of algorithms in the mathematical sciences ([Knuth, 1985]) are also recommended. Therefore c = d. To complete the selection, we must choose five additional balls. Using the definition of f , we have α1 β1 = α2 β2 . Given two graphs G1 and G2 , suppose that there is a one-toone, onto function f from the vertices of G1 to the vertices of G2 and a one-to-one, onto function g from the edges of G1 to the edges of G2 , so that if an edge e is incident on v and w in G1 , the edge g(e) is incident on f(v) and f(w) in G2 . No Yes No No Yes No No Yes Yes No No No No Yes No (6, 5, 2, 4, 3, 2, 1) (6, 5, 2, 4) (2, 6, 5, 2, 4, 3, 2) (5, 6, 2, 5) (7) We next reexamine the problem introduced in Section 1 of finding a cycle in a graph that traverses each edge exactly one time. Suppose that person 1 gets order 2. Given an arrow diagram, the relation is not symmetric if there are two distinct vertices x and y with a directed edge from x to y but no directed edge from y to x. Write a program that adds hexadecimal numbers. ⎛ ⎞ (abca) 1 0 0 0 1 1 0 0 (acda) ⎜ 0 1 0 0 1 0 0 1 ⎟ (acdb) ⎝ 0 0 1 0 0 1 0 1 ⎠ (bcdeb) 0 0 0 1 0 1 1 1 30. s satisfies ns mod (p − 1)(q − 1) = 1. No 28. How many ordered pairs (bi , ci) are there? , f(N)/g(N)}. Find a formula for the sequence d defined by 22. Definition 1.1 Let n and d be integers, d = 0. The sample space consists of all 36 possible outcomes with each outcome assigned the value 1/36 (see Figure 6.2). f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 2 41. If the coins do not balance, we proceed as in the solution given for Exercise 4 (for the 12-coins puzzle). [Any such selection can be ordered to produce (3.3).] By Theorem 3.5, the total number of selections possible is C(k + n − 1, k). An alternative to breadth-first search is depth-first search, which proceeds to successive levels in a tree at the earliest possible opportunity. PSYCH COMPSCI Figure 1.6 A Venn diagram of three sets CALC, PSYCH, and COMPSCI. We can use Theorem 3.9 to show that Algorithm 2.4 produces a maximal flow. If we select a from the first factor, b from the second factor, and a from the third factor and multiply, we obtain the term aba. If we expand Dn = (n − 1)(Dn−1 + Dn−2), we obtain Dn = nDn−1 − Dn−1 + (n − 1)Dn−2 . Four edges; four vertices having degrees 1, 2, 3, 4 v7 15. Satisfied with the chances now, the person always carries a bomb on a plane so that the probability of someone else carrying a bomb, and thus there being two bombs on the plane, is 0.000000000001—small enough to be safe. Write recursive and nonrecursive programs to compute the Fibonacci sequence. 32. From the definition of “minimum,” if d1 ≤ d2 , then d = min{d1 , d2 } = d1 and d = d1 ≤ d2 . Show that there are at least two available items in the list exactly nine items apart. Recursively invoke this algorithm with input n − 1. 394 where Sn,k denotes a Stirling number of the second kind. Give an example of something in the real world that can be modeled by a directed graph. The string with no elements is called the null string and is denoted λ. i=3 6 11. For example, the sum s for the ISBN 978-1-59448-950-1 is s = 9 + 3 · 7 + 8 + 3 · 1 + 5 + 3 · 9 + 4 + 3 · 4 + 8 + 3 · 9 + 5 + 3 · 0 = 129. Define n-cube. Yes 22. The network might be a transportation network through which commodities flow, a pipeline network through which oil flows, a computer network through which data flow, or any number of other possibilities. Example 1.10 Ackermann’s Function Ackermann’s function can be defined by the recurrence relations A(m, 0) = A(m − 1, 1), A(m, n) = A(m − 1, A(m, n − 1)), m = 1, 2, . A ∩ (B ∪ C) 13. Let T be a full binary tree. List all rise/fall permutations of 1, 2, 3. Let the 35 students be the pigeons and the 24 letters of the alphabet be the pigeonholes. Now the probability of getting a sum of 10 is no longer 1/12 because we know that one of the outcomes shown shaded in Figure 6.2 has occurred. (n lg n) 28. 546 funnyorder(PT) { if (PT == null) return process PT r = right child of PT funnyorder(r) l = left child of PT funnyorder(l) } In Exercises 32 and 33, every expression involves only the operands A, B, . The end position will be the same as in the original puzzle— the disks will be stacked in order from largest (bottom) to smallest. Show that set equivalence is an equivalence relation. Answer parts (a)–(c) for the sequence defined by the rules: 1. We first assign positions to the n1 items of type 1, which can be done in C(n, n1) ways. But the five-coins puzzle has 10 outcomes: C1 , L, C1 , H, C2 , L, C2 , H, C3 , L, C3 , H, C4 , L, C4 , H, C5 , L, C5 , H. P(F) P(F) To complete the proof of Bayes’ Theorem, we need to show that P(F) = n P(F | Ci)P(Ci). Find z1 . We take a row from Table 6.1 and a row from Table 6.2 and if ID Number = PID, we combine the rows. By definition, any set is a subset of the universal set, so ∅ ⊆ U. The relation R defined on T by iRj if i = j or task i must be done before task j orders the tasks. Show that if n is a power of 2, say n = 2k , then k lg(n/2i) = (lg2 n). The Basis Step is n = 1. Example 1.24 If X = {1, 2, 3} and Y = {a, b}, then X × Y = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)} Y × X = {(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)} X × X = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} Y × Y = {(a, a), (a, b), (b, a), (b, b)}. In Exercises 56–58, determine whether each relation R defined on the collection of all nonempty subsets of real numbers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order. 216 Algorithms Algorithm 2.4 Shuffle This algorithm shuffles the values in the sequence a1 , . Whether the sequence of statements is finite or infinite, the steps needed for the proof by mathematical induction are the same. We must show that for all positive integers n1 and n2 , if f(n1) = f(n2), then n1 = n2 . Suppose that we arrange the bags in increasing order of the number of coins that they contain. 0 20. , P6 . Summing over all k, we find that the total number of paths is n copies of the second element in the set, and so on. The function f is not one-to-one, but it is onto. , n. Show that if a sequence a is smoothed by c, the resulting sequence is smoothed by c, and so on k times; then, the sequence that results can be obtained by one smoothing of a by the binomial smoother of size k + 1. Since g is onto, there is an edge e in G1 with g(e) = E. As our first example of an algorithm written in pseudocode, we rewrite the first algorithm in this section, which finds the maximum of three numbers. The problem is to stack the cubes, one on top of the other, so that whether the stack is viewed from front, back, left, or right, one sees all four colors. For example, if H is the event “head on first toss,” and T is the event “tail on second toss,” we expect that events H and T are independent. , Xn : |X1 ∪ X2 ∪ · · · ∪ Xn | = |Xi | − 1≤i≤n + |Xi ∩ Xj | 1≤i n. Therefore, max{x, y} + min{x, y} = x + y. 3 6 4 E F G H I J K L M ALPHA CUTOFF Figure 9.8 Evaluating vertex A using a two-level, depth-first search with alpha-beta pruning. If E and F are independent events, are E and F independent? Let G be a directed graph and let v be a vertex in G. The complement of X is U − X, where U is a given universal set. Find the level of every vertex in the adjacent tree rooted at c. This universally quantified statement is true provided that the conditional proposition if p(x1 , x2 , . Example 5.21 Given that for every positive integer n, n2 ≥ n is true, we may use universal instantiation to conclude that 542 ≥ 54 since 54 is a positive integer (i.e., a member of the domain of discourse). Game tree Minimax procedure n-level search Evaluation function Alpha-beta pruning Alpha value Alpha cutoff Beta value Beta cutoff Chapter Self-Test Section 1 1. As a final example, aabbba is in L, for if we take α = aabb and β = ba, then α ∈ L and β ∈ L; by the second rule, aabbba = αβ ∈ L. Definition 6.3 Let E be an event. We state the method of Example 1.16 as Theorem 1.17. The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is reflexive because for each element x ∈ X, (x, x) ∈ R; specifically, (1, 1), (2, 2), (3, 3), and (4, 4) are each in R. Thus what is needed is a spanning tree the sum of whose weights is a minimum. By the inductive assumption, within the last n − 1 elements all of the zeros precede all of the ones. alpha beta prune(PT, PT type, PT level, n, E, ab val) { if (PT level == n) { contents(PT) = E(PT) return } if (PT type == box) { contents(PT) = −∞ for each child C of PT { alpha beta prune(C, circle, PT level + 1, n, E, content(PT)) c val = contents(C) if (c val ≥ ab val) { contents(PT) = ab val return } 586 if (c val > contents(PT)) contents(PT) = c val Input: The root PT of a game tree; the type PT type of PT (box or circle); the level PT level of PT; the maximum level n to which the search is to be conducted; an evaluation function E; and a number ab val (which is either the alpha- or beta-value of the parent of PT). We first swap ai and aj , where i = 1 and j = rand(1, 5). 210 /310 6. Write a program that finds a spanning tree in a graph. Problem-Solving Tips To model a given situation as a graph, you must first decide what the vertices represent. If the first domino is placed as shown, there are an−1 ways to cover the 2 × (n − 1) board that remains. 100’s place (10 2) 10’s place (10 1) 1000’s place (10 3) 1’s place (10 0) 3 8 5 4 Symbol 3 Symbol 0 Symbol 2 Symbol 1 Figure 2.1 The decimal number system. Describe [f]. Definition 1.1 states: A tree T is a simple graph satisfying the following: If v and w are vertices in T, there is a unique simple path from v to w. Sentence (d) is either true or false (but not both), but no one knows which at this time. 82. The chairperson can be selected in six ways. Suppose that in this problem the path is required to begin at vertex a and end at vertex e. are valid arguments, the argument p1 , p2 , . Setting tn−4 = −tn−3 qn−2 + sn−3 and sn−4 = tn−3 , we obtain rn−1 = tn−4 rn−3 + sn−4 rn−4 . By Theorem 5.4, a tree in X1 has n − 1 internal vertices and 2n − 1 total vertices. What are the equivalence classes? Alice can be assigned an office in three ways, the highest remaining office can be filled in five ways, and the last office can be filled in four ways. This optimal strategy is contained in the game tree: The first player always moves to a position that maximizes the value of the children. Summary of Problem-Solving Techniques ■ Look for a similar problem and imitate its solution. Write a program that finds the composition R◦S of relations R and S. Example 5.19 Suppose that the domain of discourse of the propositional function P is {1, 2, 3, 4}. Exercises In Exercises 1–3, find the r-combination that will be generated by Algorithm 4.9 with n = 7 after the r-combination given. Basis Step 71 − 1 = 6 is divisible by 6. In words: Every real number has a cube root. Proof The vertices of T consist of the vertices that are children (of some parent) and the vertices that are not children (of any parent). Then z = pq = 713 and φ = (p − 1)(q − 1) = 660. In general, alpha-beta pruning allows us to bypass many vertices in a game tree yet still find the value of a vertex. Decrypt 411 using s = 569 as in Example 4.2. In Exercises 7–11, assume that we choose primes p = 17, q = 23, and n = 31. Such a graph cannot contain a homeomorphic copy of K3,3 or K5 . Since vertex g now has degree 2, we cannot include edge (c, g) or (g, d). For example, in the Huffman coding tree of Figure 1.10, moving from a vertex to a left child corresponds to using the bit 1, and moving from a vertex to a right child corresponds to using the bit 0. ; assign the nth element of X to one of A, B − A, X − B. We just showed that a tree is a connected, acyclic graph. There is exactly one route that first meets the top edge at the first position from the left. Since 2h−1 < 2h , (5.2) is established for this case. In proving p ↔ q, we are proving that p and q are logically equivalent, that is, p and q are either both true or both false. Then we order the children of the root with the rightmost child first and use the alpha-beta procedure to obtain Trees 8 8 8 9 8 12 8 8 9 12 9 4 7 1 13 Chapter Self-Test 1. The vertices represent processors. It turns out that if |S| ≤ |R(S)| for all subsets S of V , then G has a complete matching. We divide the subsets of n n j=1 i=1 = + +n = 2n By the inductive assumption, 1 = n. If (v1 , . P(F) Thus if E and F are independent events, P(E) = P(E | F) = P(E ∩ F) P(F) or P(E ∩ F) = P(E)P(F). State the max flow, min cut theorem. Instead of proving (p1 ∨ p2 ∨ · · · ∨ pn) → q, (2.1) (p1 → q) ∧ (p2 → q) ∧ · · · ∧ (pn → q). in octal? Use mathematical induction to show that 2 2 −fn−1 f2n = fn+1 2 and f2n+1 = fn2 +fn+1 for all n ≥ 2. r 0 + r 1 + · · · r n = 35. This maximum value (1) is assigned to the box. Since n = 10 > 0, the body of the while loop executes. Let n be a nonnegative integer. Since the 11 outcomes shaded are equally likely, the probability of getting a sum of 10 given that at least one die is 5 is 1/11. [Problem 4.] Here we are given a value, ta, and asked to prove that it is the least upper bound of the given set, which here we denote as tX. All but {a, b} are proper subsets. each of T1 and T2 is nonempty, after which we check that the left subtrees of T1 and T2 are isomorphic and that the right subtrees of T1 and T2 are isomorphic. This is equal to C(k + t − 1, k), the number of ways to select the positions for the ×’s; hence there are C(k + t − 1, t − 1) = C(k + t − 1, k) unordered, k-element selections from X, repetitions allowed. State the Binomial Theorem. Find all substrings of the string aabaabb. Sets and Logic If we let P(x) denote the propositional function “x is an integer” and Q(x) denote the propositional function “x is rational,” the argument becomes ∀x ∈√ R (P(x) → Q(x)) ¬Q(2) √ ∴ ¬P(2) √ √ Since 2 √ ∈ R, we may use universal instantiation to conclude P(2) → Q(π). , xn : ordering of r elements of x1 , . We show how the preceding algorithm executes for some specific values of a, b, and c. Suppose that the first player can win in nim. Using Algorithm 3.19 to compute A2 requires time (n3). return 6. Find a property of G1 that G2 does not have but that G2 would have if G1 and G2 were isomorphic. For all n, p(n) = ak nk + ak−1 nk−1 + · · · + a1 n + a0 ≥ ak nk = C2 nk , where C2 = ak . 36 Since 11 outcomes belong to F (see Figure 6.2), P(F) = 11 . How many outcomes have neither die showing 2? Again, you’re trying to derive an inequality so you can replace terms in f(n) with other terms if the result is smaller (again, see Example 3.3). (The main diagonal of a square matrix consists of the entries on a line from the upper left to the lower right.) The relation R is reflexive if and only if (x, x) ∈ R for all x ∈ X. i=1 53. Therefore CX (x) + CY (x) = 1 + 1 = 2. Since 13567 is the smallest string that begins 135 and represents a 5-combination of X, the answer is 13567. ∃x∃y T3 (x, y) 23. Use the ordinary algorithm for adding decimal numbers to add binary numbers—except replace the decimal addition table by the binary addition table. Since this problem is in a chapter, which deals with graphs, it would probably be a good idea to try to model the problem as a graph. A committee composed of Morgan, Tyler, Max, and Leslie is to select a president and secretary. Explain how the project operator works and give an example. 182 if A ∪ Y = B ∪ Y. By a similar kind of argument (see Exercise 15), we can show that the graph K5 of Figure 7.3 is not planar. For example, if you conclude that x ∈ X or x ∈ Y because it is known that x ∈ X ∪ Y, write “Since x ∈ X ∪ Y , x ∈ X or x ∈ Y ,” or perhaps even “Since x ∈ X ∪ Y , by the definition of union x ∈ X or x ∈ Y ” if, like Richard Nixon, you want to be perfectly clear. Show that there are at least two unavailable items in the list either three or six items apart. The contestants’ names are listed on the left. No! It is possible to concatenate different strings and produce the same string. If n = 1, the only item in the sequence is s1 and the algorithm correctly returns it. (Actually, we can shorten this list considerably; see Theorem 1.7.) 260 Introduction to Number Theory Example 1.6 By inspection, we find that none of 2, 3, 4, 5, . For example, Z− denotes the set of negative integers, namely −1, −2, −3, . Notice that f , e, and v satisfy the equation 2 D C B 6 5 A 4 3 Figure 7.2 A connected, planar graph with f = 4 faces (A, B, C, D), e = 8 edges, and v = 6 vertices; f = e − v + 2. One technique is to argue that P(x, y) is true using the symbols x and y to stand for arbitrary elements in X and Y . e1 ⎛1 v1 v2 ⎜ v3 ⎜ ⎜ v4 ⎜ ⎜ v5 ⎜ ⎝ v6 v7 1 0 0 0 0 0 e2 0 1 1 0 0 0 0 e3 0 0 1 1 0 0 0 v4 0 1 1 0 1 0 0 v5 0 0 0 1 0 1 1 v6 1 1 0 0 1 0 1 v7 ⎞ 0 1 ⎟ 0 ⎟ ⎟ 0 ⎟ ⎟ 1 ⎟ ⎠ 1 0 e4 0 1 0 1 0 0 0 e5 0 1 0 0 0 0 1 e6 0 1 0 0 0 1 0 e7 1 0 0 0 0 1 0 e8 0 0 0 0 1 0 1 e9 0 0 0 0 1 1 0 e10 0 0 0 1 1 0 0 e11 ⎞ 0 0 ⎟ 0 ⎟ ⎟ 0 ⎟ ⎟ 0 ⎟ ⎠ 1 1 19. , 6, n mod d is not equal to zero. Trace Algorithm 4.2 for n = 4. A maximal matching for the graph of Figure 4.1 is shown with black lines. 1 2 38. Is τ nonincreasing? Breadth-first search 572 20. What is this notation called? Find the value of the tic-tac-toe position using the evaluation function of Example 9.1. 3 6 8 2 1 7 3 2 4 8 36. Therefore, the overall time to test whether a relation on an n-element set is transitive, using Algorithm 3.19 to compute A2 , is (n3). Suppose also that |X| ≥ 2. Let p denote the proposition “the computer has one gigabyte of memory,” let q denote the proposition “the computer can run ‘Blast ’em,’” and let r denote the proposition “the sonics will be impressive.” The argument can be written symbolically as p→q q→r ∴p→r Therefore, the argument uses the hypothetical syllogism rule of inference. What is the best-case time of an algorithm? Since n mod d = 3738 mod 2 equals 0, the algorithm returns d = 2 to indicate that n = 3738 is composite and 2 is a divisor of 3738. This statement is true because there is at least one positive integer x (namely x = 1) for which x ≤ y is true for every positive integer y. We must select either three or four defective discs. , |A| = n, and then using the Binomial Theorem. We must show that the statement is true for n + 1. Let V be a set of n vertices and let s be a “dissimilarity function” on V ×V (see Example 8.1.7). w1 b 6 A 4 a w2 2 3 2 z c 3 w3 3 d 4 B Figure 1.4 The network of Figure 1.3 with a designated source and sink. The audience will go to sleep if the chairperson gives the lecture. Again, by the definition of intersection, since x ∈ / X ∩ Z and x ∈ X, 78 Proofs x∈ / Z. Which of the integers in Exercises 1–8 are prime? The beta value of a vertex is dependent on the current state of the search and changes as the search progresses. To thwart such attacks, implementors of RSA have taken steps to alter the observed time to decrypt messages. We have proved that R ∩ S is transitive. The remainder r equals zero if and only if d divides n. Since b is zero, t could be assigned any value; we choose t = 0. In representing an integer, reading from the right, the first symbol represents the number of 1’s, the next symbol the number of 8’s, the next symbol the number of 82 ’s, and so on. For example, the graph of Figure 1.13 is bipartite since each edge is incident on one vertex in V1 = {v1 , v2 , v3 } and one vertex in V2 = {v4 , v5 }. Show that if G has a Hamiltonian cycle, V1 and V2 have the same number of elements. A function assigns to each member of a set X exactly one member of a set Y . ∴ q 6. The following algorithm sorts five items using at most seven comparisons in the worst case. What is a counterexample? In fact, Q is a proper subset of R because, √ √ for example, 2 ∈ R, but 2 ∈ Q. In the notation of Definition 4.1, p = 3, q = 4, s1 = 1, s2 = 3, s3 = 2, t1 = 1, t2 = 3, t3 = 2, and t4 = 4. Section 3 9. , sn , where the si are distinct. The process by which circle seeks the minimum of its children and box seeks the maximum of its children is called the minimax procedure. Example 4.7 Find the string that follows 13467 when we list the 5-combinations of X = {1, 2, 3, 4, 5, 6, 7}. Let us try to find K3,3 in the graph G of Figure 7.6. We first note that the vertices a, b, f , and e each have degree 4. Find the least common multiple of each pair of integers in Exercises 13–24. Let V be the set of computers and W be the set of disk drives. Since R is not transitive, there exist (a, b), (b, c) ∈ R, but (a, c) ∈ / R. (3.2) The recurrence relation (3.2) is typical of those that result from divide-andconquer algorithms. E1 or E2 (or both) 1. Consider the input 10, 4, 2 and key = 10. , xn when n is even. Prove or disprove: If A, B, and C are sets satisfying A C = B C, then A = B. , s4 is 8 13 20 27 If s5 is 16, after it is inserted, s1 , . Therefore, p ∧ q → r is true (regardless of whether r is true or false). Example 5.8 The matrix of the relation R = {(a, a), (b, b), (c, c), (d, d), (a, c), (c, b)} on {a, b, c, d}, relative to the ordering a, b, c, d, is ⎛ ⎞ 1 0 1 0 ⎜0 1 0 0⎟ ⎜ A=⎜ ⎝0 1 1 0⎠. We have the initial conditions d0 = 200, d1 = 220. {2} ∈ P({1, 2}) 83. i=2 while (i < n) { i = i2 x=x+1 } 27. What is the digraph of a binary relation? At this point we will summarize and justify the techniques used to solve the preceding recurrence relation. Draw the complement of the graph G1 of Exercise 7. 5 d 7 e 4 3 6 4 c a 6 8 7 b 5 17. If K5 is planar, e ≤ 3v − 6 becomes 10 ≤ 3 · 5 − 6 = 9. 181 Functions, Sequences, and Relations In the digraph of an equivalence relation, an equivalence class is a largest subgraph of the original digraph having the property that for any vertices v and w in G, there is a directed edge from v to w. There is an edge between teams for each game played. We must then prove that the inequality is true for n + 1; that is, we must prove that (n + 1)! ≥ 2n (4.11) is true. If we substitute this expression for an−1 into (2.1), we obtain an = an−1 ↓ = an−2 + 3 = an−2 + 2 · 3. Trees Example 1.4 If we designate e as the root in the tree T of Figure 1.5, we obtain the rooted tree T shown in Figure 1.5. The vertices a, b, c, d, e, f, g, h, i, j are on (respectively) levels 2, 1, 2, 1, 0, 1, 1, 2, 2, 3. (b) If v and w are vertices in T , there is a unique simple path from v to w (definition of tree). Write f as a set of ordered pairs and draw the arrow diagram of f . Also, provide an answer to the query. For each string in Exercises 4–6, explain (as in Example 4.15) exactly how Algorithm 4.14 generates the next permutation. Derive a recurrence relation for the number of multiplications needed to evaluate an n×n determinant by the cofactor method. Suppose first that the root of T has only one child. Therefore, T is a tree. A sequence s is decreasing if sn > sn+1 for all n for which n and n + 1 are in the domain of the sequence. Much effort has been invested in finding good values for a linear congruential method. (2.2) we prove As we will show, proof by cases is justified because the two statements are equivalent. Since d = n, d < n. There are C(13, 4)C(13, 4)C(13, 4)C(13, 1) hands that contain four spades, four hearts, four diamonds, and one club. k=0 Proof The proof precedes the statement of the theorem. We obtain the labels shown in Figure 4.4. You should verify that the next iteration of the algorithm produces the labeling shown in Figure 4.5 and that at the termination of the algorithm, z is labeled 5, indicating that the length of a shortest path from a to z is 5. Place a check by the root of each subtree that is pruned. Therefore lg n = m. Taking n = 3 in (1.3), we obtain a3 = a2 + 3. 6 5 6. Now n mod d = 9 mod 3 equals 0, so the algorithm returns d = 3 to indicate that n = 9 is composite and 3 is a divisor of 9. In the California Daily 3 game, a contestant must select three numbers among 0 to 9. Since 1 = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 3P(2) + P(2) + P(2) + P(2) + P(2) + P(2) = 8P(2), † This section can be omitted without loss of continuity. The cases n = 1, 2, . buyer [Name] United Supplies, ABC Unlimited, JCN Electronics, Danny’s, Underhanded Sales, DePaul University 11. Then (a, b) ∈ R. Thus the outline of our proof becomes d = min{d1 , d2 } and x ≤ d (Hypotheses) Subproof: Show that d ≤ d1 and d ≤ d2 . We will prove that T has n − 1 edges by induction on n. Section 5 Exercises 17–20 refer to the relations R1 = {(1, x), (2, x), (2, y), (3, y)}, R2 = {(x, a), (x, b), (y, a), (y, c)}. Is v decreasing? The first string 0000 determines the equivalence class consisting of all eight-bit strings that begin 0000; the second string 0001 determines the equivalence class consisting of all eight-bit strings that begin 0001; and so on. Backtracking Section 4 22. It is assumed that if a disk is on a peg, only a disk of smaller diameter can be placed on top of the first disk. In this example, we count the number of Class A addresses. Subtract the formula for Sn−1 from the formula for Sn and use the result to derive the recurrence relation Sn = 2Sn−1 − Sn−2 + Sn−3 . By construction, at the termination of Algorithm 4.3, the resulting graph, Tn−1 , is a connected, acyclic subgraph of the given graph G containing all the vertices of G; hence Tn−1 is a spanning tree of G. The statement is false if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is false. It is also apparent from the arrow diagram (see Figure 1.3) that this set is not a function because there are two arrows from 1. The second technique was to compare the prime factorizations of a and b. Consider the numbers a1 , a2 , ..., a45 (8.3) and a1 + 9, a2 + 9, ..., a45 + 9. if n is odd and greater than 1 if n is even and greater than 1 and initial condition u1 = 1. i=1 Prove that n k=1 ak bk = n sk (bk − bk+1) + sn bn+1 . (c) The weight Cij of the directed edge (i, j), called the capacity of (i, j), is a nonnegative number. For each pair of vertices v = (p1 , p2 , p3) and w = (q1 , q2 , q3), we set s(v, w) = |p1 − q1 | + |p2 − q2 | + |p3 − q3 |. Thus if person 1 gets order 2, the only possibility is 2, 3, 1. B ∩ U 8. 147 Functions, Sequences, and Relations This decomposition technique is important in differential calculus since there are rules for differentiating simple functions such as g, h, and w and also rules about how to differentiate the composition of functions. We noted in the preceding paragraph that x > y, x, y ∈ R, has this property. Show that h([x]) = g(x) defines a function from the set of equivalence classes of A into X. When a grandchild w of v has a value of at most x, the subtree whose root is the parent of w is deleted. Conclusion: The Council does not approve the funds, or the Olympic Games are canceled. If we modify the usual method of multiplying base 10 integers for base 2, it can be shown (see Exercise 65) that the time required to multiply a and b is O(lg a lg b). sin(x/2) This is the same as showing that [after multiplying by the term sin(x/2)] x nx x + cos(n + 1)x sin 2 2 2 x (n + 1)x . Dijkstra’s shortestpath algorithm for weighted graphs can be considered as a generalization of breadth-first search (see Exercise 25). We replace three 3’s by one 7 and two 1’s to obtain a solution S. Definition 3.10 Suppose that the propositions P and Q are made up of the propositions p1 , . Example 4.10 combination(r, n) { for i = 1 to r si = i println(s1 , . We note that a set S of points in the plane defines a function precisely when each vertical line intersects at most one point of S. Find the probability of obtaining a bridge hand consisting only of red cards, that is, no spades and no clubs. , si is a loop invariant. h a b c g i j k d e f l G1 G2 Figure 6.4 Nonisomorphic graphs. } 19. Similarly at line 7, when no tree is input to preorder, again nothing is processed. For example if m = 15 and n = 4, then k1 = 7 and k2 = 2. Does the conclusion to Example 8.5 necessarily follow if there are more than six pictures? Comments The fact that every nonempty set of real numbers that is bounded above has a least upper bound is called the completeness property of the real numbers. Since A ∪ B = B, x ∈ B. 2n ≥ n2 , n = 4, 5, . [Requires calculus] Assume the formula for differentiating products: dg df d(fg) =f +g . We see that upper bounds e, f , and g for a set X (shown in color) look like X e f g What would it mean that a is not an upper bound for a set X? To this end, let x be any real number whatsoever. What is the probability that it is a jack? Now suppose that we want to count unordered selections of k elements, repetitions allowed, from among t items. Explain a method of verifying that X is not a subset of Y . If n = 3 we proceed to line 4 since n = 0. Therefore A ∪ B = B. (lg n) s2 = ‘G’. Use Exercises 58 and 60. Among those that do not have the HIV virus, approximately 2 percent test positive on the ELISA test. {1} 74. Show that G has a complete matching if and only if δ(G) = 0. If X is equivalent to Y , there is a one-to-one, onto function f from X to Y . Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is D4 × D4 . 3 · 4 · 3 44. Since MORE is greater than MANUSCRIPTS, we go to the right child. By the generalized De Morgan’s laws for logic, this latter statement is equivalent to 15. Finally, we compute the value of the root by maximizing over the children. Arguing as in the proofs of Theorems 8.3 and 8.7, we can show that there are five nonisomorphic binary trees with three vertices. Implement Algorithm 2.3, insertion sort, as a program. A problem in this class can be solved using a divide-and-conquer technique in which the problem is decomposed into problems of the same type as the original problem. The algorithm continues in this manner until it swaps an−1 and arand(n−1,n) . An equivalence relation on a set X partitions X into subsets. The sorting problem is easily described: Given n items x1 , ..., xn , arrange them in nondecreasing (or nonincreasing) order. If we let k denote the number of times we execute the body of the while loop, the number of times we execute x = x + 1 is at most n n n n + + + · · · + k−1 . List all rise/fall permutations of 1, 2, 3, 4, 5. These examples illustrate the Multiplication Principle. Two classes 1 2 3 5 4 37. In either case, |x| ≥ x; so the proof is complete. Euclidean algorithm 27. For example, if one algorithm takes n steps to solve a problem and another algorithm takes n2 steps to solve the same problem, we would surely prefer the first algorithm, assuming that the space requirements are acceptable. Yes 39. At line 2, since i > j (1 > 5) is false, we proceed to line 4, where we set k to 3. The conjunction of p and q, denoted p ∧ q, is the proposition p and q. This is almost the problem of n − 1 persons all getting the wrong orders. Assume that the theorem is true for n. The reason for the term “partial order” is that in general some elements in X may be incomparable. We define a relation R on the set of all permutations of X by the rule: p1 Rp2 if p2 is obtained from p1 by permuting the order of the objects of type 1 (but not changing their location) and/or permuting the order of the objects of type 2 (but not changing their location) . After the value of i is set, the while loop compares ti · · · ti+m−1 and p1 · · · pm . Prove that A is the matrix of R1 ∩ R2 . Therefore, the graph of Figure 3.5 does not contain a Hamiltonian cycle. Any matching is contained in a maximal matching. Suppose that we select the two edges shown as solid lines in Figure 8.7. Now consider the problem of picking two edges incident on vertex R. {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)} 13. List the elements in each set. Define X is a proper subset of Y . If n is composite, the algorithm returns a divisor d satisfying 2 ≤ d ≤ n. At line 5, since key (‘Z’) is not equal to s5 (‘X’), we proceed to line 7. ¬(p ∨ q) ≡ ¬p ∧ ¬q, ¬(p ∧ q) ≡ ¬p ∨ ¬q Section 3 p ∴p∨q 11. We use the method of Algorithm 3.6 to produce a tree T . C1 C2 C3 C4 C5 C6 K 3, 3 Figure 7.1 Cities connected by expressways. Define a relation R on RR , the set of functions from R to R, by f R g if f(0) = g(0). Trees 23. Suppose that G has two minimal spanning trees T1 and T2 . Answer the questions in Exercises 7–15 for the following tree. Since s = s mod φ, there exists q such that s = qφ + s. For example, the author of this book has Erd˝ os number five. Edges of the form (v, v1) and (v, v2), where v has degree 2 and v1 = v2 5. Give an algorithm to construct a binary search tree. Example 1.9 The integer n = −21 is odd because there exists an integer k (namely k = −11) such that n = 2k + 1; that is, −21 = 2 · − 11 + 1. A ∩ U = ∅ 90. satisfies the recurrence relation Fn = n−1 Inductive Step Assume that tk = 1 for k < n. (1.1) The first term is 5 because of instruction 1. The inductive step has been verified and the proof is complete. U 4. Logic focuses on the relationship among statements as opposed to the content of any particular statement. 333 Counting Methods and the Pigeonhole Principle Theorem 3.2 Suppose that a sequence S of n items has n1 identical objects of type 1, n2 identical objects of type 2, . 51. Show that there is a de Bruijn sequence for every n = 1, 2, . Parts (b) and (c) follow from the fact that the number of vertices in V matched is equal to the value of the corresponding flow. The values P(x) need not all be the same. Example 4.5 A robot can take steps of 1 meter or 2 meters. e4 0 1 0 1 0 0 0 e5 0 1 0 0 1 0 0 a e6 0 1 0 0 0 1 0 e7 0 0 0 0 1 1 0 e8 ⎞ 0 0 ⎟ 0 ⎟ ⎟ 0 ⎟ ⎟ 0 ⎟ ⎠ 1 1 b e c 2. a b Using (7.1), we find that 2e ≥ 4(e − v + 2). Any values greater than 4 serve as additional upper bounds. In Example 1.6 we showed that there are 3n ordered pairs (A, B) satisfying A ⊆ B ⊆ X, where X is an n-element set. We have proved equation (1.3). In practice, other constraints are imposed such as the proximity of various components that must be interconnected and restrictions on the ratios of width to height of the resulting rectangle. We conclude by discussing recursive algorithms—algorithms that refer to themselves. max{x, y} + min{x, y} = y + x = x + y. It is denoted X × Y. , sn and the length n of the sequence s1 , s2 , . Initially, all vertices have temporary labels. First assume that x ≥ 0 and y ≥ 0. So suppose that f(α1 , β1) = f(α2 , β2). i=1 In particular, the sum of the degrees of all the vertices in a graph is even. Now L(x) ≤ L(u) + w(u, x) ≤ length of P < L(v). Now consider an edge (j, i), where j ∈ P, i ∈ P. Problem-Solving Tips To prove that the universally quantified statement ∀x P(x) is true, show that for every x in the domain of discourse, the proposition P(x) is true. Thus there exists x with x ∈ A and x ∈ / B. 155 69. C(4, 3) = 4 28. Prove that the following are equivalent for sets A, B, and C: (a) A ∩ B = ∅ (b) B ⊆ A (c) A B = A ∪ B, where is the symmetric difference operator. Therefore, by definition (a) there exists x ∈ X such that a − ε < x. The first description of the RSA encryption system was in Martin Gardner’s February 1977 Scientific American column (see [Gardner, 1977]). We count the number of strings in which no I appears before any L and then subtract from the total number of strings. Assume that all months and dates are equally likely, and ignore February 29 birthdays. After introducing some terminology, we will give several examples of computing probabilities. By following instructions 1 and 2, we can compute any term in the sequence. 46. 1 2 n n+1 q 3. The search is said to be an n-level search if we limit the search to n levels below the given vertex. , pn are equivalent, the usual method is to prove (p1 → p2) ∧ (p2 → p3) ∧ · · · ∧ (pn−1 → pn) ∧ (pn → p1). In Exercises 57–60, let X = {1, 2} and Y = {a, b, c}. Starting at a, we obtain (a, c, z), but the shortest path from path from a to z, we would select the edge (a, c) and then the edge (c, z). 67942, 4209 10. Figure 3.11 A 4 × 4 chessboard and the graph GK4 . That is, Cn = 1 + 0.5(n − 1). The following table shows the results. Suppose that a connected graph G has exactly two vertices v and w of odd degree. Show, by an example, that b is not nondecreasing. If every lower bound b for X satisfies b ≤ a, we call a a greatest lower bound for X. f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 1 Section 7 Review 1. We then recursively invoke binary search (line 11) to continue the search. By the second form of the Pigeonhole Principle, two of these numbers are the same. In words, the Basis Step is to prove that the propositional function S(n) is true for the smallest value n0 in the domain of discourse. We close this section with examples that illustrate both counting principles. If such a value is found, we continue with the next available position. If we add one ai coin to this optimal solution, we obtain an optimal solution for n that uses an am coin. To see this, note that GKn is bipartite. Here n = 3, b = 2, and c3 = 1, c2 = 1, c1 = 0, c0 = 1. Prove the involution law for sets. Graph Theory Richard Johnsonbaugh 433 9. 568 Trees Section Review Exercises 1. Exercises 18–34 concern the Fibonacci sequence {fn }. r: New Atlantic builds a new stadium. Surely this result is not that interesting in its own right, but it can be used to prove other results. Let us define the worst-case time needed to execute the search procedure as the height h of the tree T ∗ . At line 2, since i > j (4 > 4) is false, we proceed to line 4, where we set k to 4. C = {1, 2, 3}, D = {2, 3, 4}, A = {2, 3}, B = C ∩ D 23. Problem-Solving Tips In the Inductive Step of the Strong Form of Mathematical Induction, your goal is to prove case n. Certain growth functions occur so often that they are given special names, as shown in Table 3.3. The functions in Table 3.3, with the exception of (nk), are arranged so that if (f(n)) is above (g(n)), then f(n) ≤ g(n) for all but finitely many positive integers n. 24 34. if (si > si+1) 11. How many unordered, k-element selections are there from a t-element set, repetitions allowed? (c) The children of Zeus are Apollo, Athena, Hermes, and Heracles. A connected, planar graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. Analysis of algorithms Worst-case time of an algorithm Best-case time of an algorithm Average-case time of an algorithm Big oh notation: f(n) = O(g(n)) Omega notation: f(n) = (g(n)) Theta notation: f(n) = (g(n)) Section 4 22. For any vertices v and w, there is a unique simple path from v to w. Thus the value that the algorithm returns is G, which by definition is the greatest common divisor of the input values. For d = 2, . The numbers C(n, r) are known as binomial coefficients because they appear in the expansion (7.2) of the binomial a + b raised to a power. Similarly, we label vertex d as (a, 5) and add d to U. Therefore, k, the number of times the statement x = x + 1 is executed, satisfies lg n < k ≤ 1 + lg n. Is x increasing? The function in Algorithm 1.1 has eight numbered lines. Let x ∈ A ∪ B. If x ∈ X, then 3x2 − x − 2 = 0. The codomain of each function is also the set of all real numbers. 43. 1024 19. {(x, y) | x and y have the same parents} 14. For example, the equation B = {x | x is a positive, even integer} (1.2) describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. This implies that g0 (n) = O(f0 (n)). A (deductive) argument consists of hypotheses together with a conclusion. Therefore, a tree cannot contain a cycle. Show that 2e ≥ 6v. v1 v v2 Figure 8.5 Vertex v has degree 2. If we define f(α, β) = αβ, where α and β are strings over X, then f is a binary operator on X∗ . (p ∨ ¬r) ∧ ¬((q ∨ r) ∨ ¬(r ∨ p)) 50. If x < 0, then |x| = −x. We discuss why the game correctly determines the truth value of a quantified propositional function. Given an arrow diagram, the relation is antisymmetric if between any two distinct vertices there is at most one directed edge. It is a surprising fact that Alfred Hitchcock never won an Academy Award for directing. 117 Proofs Inductive Step We assume that ck < 2k for all k, 1 ≤ k < n, and prove that cn < 2n, n > 1. Example 2.1 tells us that p is even. Finally, we use induction to show that we can make postage of n cents or more using only p-cent and q-cent stamps. Find k=1 n ≥ 1. If the coin is flipped 10 times, what is the probability of exactly five heads given at least one head? As of 2007, the NCAA men’s basketball tournament was a 65-team single-elimination tournament. A ∪ B 2. G2 32. Recall that we can test whether a relation R on an n-element set is transitive by squaring its adjacency matrix, say A, and then comparing A2 with A. Step through the subsets listed in 1 and output each whose sum is m. This is a Hamiltonian cycle and the argument shows that it is unique. Exercise 22 30. How many ordered pairs (A, B) satisfy A ⊆ B ⊆ X? Therefore, X is not a subset of Z. Converse: p → q; if |4| < 3, then −3 < 4 < 3; true. Since G1 and G2 are simple graphs, the entries in the adjacency matrices are either 1 (to indicate that vertices are adjacent) or 0 (to indicate that vertices are not adjacent). x∈S The first condition guarantees that the probability of an outcome is nonnegative and at most 1, and the second condition guarantees that the sum of all the probabilities is 1—that is, that some outcome will occur when the experiment is performed. Input: Output: r, sk , sk+1 , . What can you say about two vertices in a rooted tree that have a descendant in common? Are G1 and G2 isomorphic? Example 2.5 8. There are n ways to choose the first item, n − 1 ways to choose the second item, and so on. Write a program that constructs all n-vertex binary trees. The first person’s birthday can occur on any one of 365 days. Thus the “insert” part of the algorithm is: Beginning at the right of the sorted subsequence, move an element one index to the right if it is greater than the element to insert. Region 4 represents B − A, the elements in B but not in A. bn = n + (n − 1)(n − 2)(n − 3)(n − 4)(n − 5). The Basis Step is i = 1, which is true by inspection. Merge sort uses (n lg n) comparisons in the worst case and is, by Theorem 7.3, optimal. Now g ◦ f is a one-to-one, onto function from X to Z. Show that 6 has no inverse modulo 15. We have shown that if a graph has exactly two vertices v and w of odd degree, there is a path with no repeated edges containing all the edges and vertices from v to w. Explain how the join operator works and give an example. The size of a vertex cover V is the number of vertices in V . From Dear Abby: All men do not cheat on their wives. Because of equation (1), we can then make (n + 1)cents postage n + 1 = (np p + nq q) + (sp + tq) = (np + s)p + (nq + t)q using np + s p-cent stamps and nq + t q-cent stamps. Prove that Cn is prime if and only if n = 2 or n = 3. Give an example of a relation that is not reflexive. Let d = |c|. Prove that there exists i such that si ≤ A. Give the order of each linear homogeneous recurrence relation with constant coefficients. Prove that f is onto Y if and only if whenever g is a function from Y onto any set Z, g ◦ f is onto Z. Exercises Exercises 1–4 concern n teams that play a single-elimination tournament. What is the probability of all girls? Now (2k + 1)2 4k2 + 4k + 1 1 n2 = = = k2 + k + . What does a complete matching represent? Find 93. Also, n − d(q + 1) = r − d < r. Sequence: function whose domain is a set of consecutive integers 16. There is no hurricane and it is not raining. We fix n piles each with a fixed number of tokens. Definition 3.9 Example 3.10 A relation R on a set X is symmetric if for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R. Carefully chosen notation can help in solving a problem. (Once the first four bits are assigned values, the last four bits are determined.) 38. We replace the original problem, compute gcd(a, b), with the problem, compute gcd(b, r). Instructions 1 and 2 do not give an explicit formula for the nth term of the sequence in the sense of providing a formula that we can “plug n into” to obtain the value of the nth term, but by computing term by term we can eventually compute any term of the sequence. You may find the next Mersenne prime! An existence proof of (2.4) that exhibits an element a of the domain of discourse that makes P(a) true is called a constructive proof. p → r Sets and Logic 46. A smoothing of the (infinite) sequence a1 , a2 , . Another type of existence proof indirectly proves (e.g., using proof by contradiction) that a value of x exists that makes the statement true without specifying any particular value of x for which the statement is true. x becomes a2 , and n becomes 15. An understanding of logic can also be useful in clarifying ordinary writing. The next definition makes these ideas precise. Add a p-cent stamp to make postage for k. X × Y 58. We also say that g is an asymptotic tight bound for f . C(52, 13) Section Review Exercises 1. To try to separate the vertices into two disjoint sets, pick a start vertex v. Property (8.3) assures us that, after orienting the front and back sides, we can successfully orient the left and right sides. For example, (1, u) ∈ R2 ◦ R1 because (1, 2) ∈ R1 and (2, u) ∈ R2 . Binary relation from X to Y : set of ordered pairs (x, y), x ∈ X, y ∈ Y 33. In this case, the board is a tromino T . Then cube 1 has a white back face. Now f(g(x)) = f(y) = z. Now suppose that n is input to Algorithm 4.2. Since n = 0, when we execute the function in Algorithm 4.2 we proceed to line 4. A universally quantified proposition generalizes the proposition P 1 ∧ P 2 ∧ · · · ∧ Pn (5.3) in the sense that the individual propositions P1 , P2 , . How many ways are there to select six comics? , a0 . Therefore, the permutation following the given permutation is 164235. Our second proof uses laws of sets. De Morgan’s laws for sets: (A ∪ B) = A ∩ B, (A ∩ B) = A∪B 24. 24. One or the other conditions suffices; you need not (and may not be able to) show both conditions. The graph consisting of two vertices and no edges has neither a Hamiltonian cycle nor an Euler cycle because it is not connected. Does 1101010 represent a number in binary? Prove that for every n ∈ Z, if n2 is odd, then n is odd. Equation (3.6) may be rewritten as 4 = 110 − 53 · 2. If we define a relation R on a set of graphs by the rule G1 R G2 if G1 and G2 are homeomorphic, R is an equivalence relation. 400 18. Now the sum is m + n = (2k1 + 1) + (2k2) = 2(k1 + k2) + 1. By Definition 1.1, m = dq1 (1.1) n = dq2 (1.2) for some integer q1 and for some integer q2 . There is gas in the car. Since the tree that is obtained has i internal vertices, there are 2i + 1 vertices total (Theorem 5.4). An exam has 12 problems. Notice that R and S are transitive. Exercises 21–30 refer to the propositions p, q, and r; p is true, q is false, and r’s status is unknown at this time. Therefore, P(n, r) = C(n, r)r!. , sn arranged in nondecreasing order 1. Explain what it means for two events to be “mutually exclusive.” 3. 15. Some heads and some tails were obtained. The hypothesis expresses a sufficient condition. The algorithm begins by setting result to 1 and x to a mod z = 5 mod 21 = 5. Notice in Example 3.2 (which refers to Figure 3.5) that if we eliminate one edge incident at v2 and one edge incident at v4 , these edges are distinct. The answer to the exercise will be: Total number of strings − N or 5! − N. (Tn is defined in the previous exercise.) 64. We shall assume the existence of a function rand(i, j), which returns a random integer between the integers i and j, inclusive. Is the following reasoning correct? 74. At this point, the computation is + 1 10011011 1011011 10 Continuing in this way, we obtain + Example 2.11 10011011 1011011 11110110 The addition problem of Example 2.10, in decimal, is + 155 91 246 We turn the method of Example 2.10 into an algorithm. (a) R B 1 3 2 G 4. Compute a table of characters and frequencies by sampling some text. But the discussion in this section showed that determining the bad coin from among four coins requires at least three weighings in the worst case. Example: If the sequence is 12 11 12 23 and key is 12, the algorithm returns the value 3. Unless specified otherwise, the sets E and V are assumed to be finite and V is assumed to be nonempty. Then p1 ∨ p2 ∨ · · · ∨ pn is true. Definition 2.8 Let G = (V, E) be a graph. In particular, the root shows the initial configuration of the game. Third Form: Let X, Y , and f be as in the last example. (a) Under what conditions does the maximum number of comparisons occur in Algorithm 3.5? (b) Under what conditions does the minimum number of comparisons occur in Algorithm 3.5? } 22. Does the graph contain an Euler cycle? Since m mod b = 2 mod 2 = 0, c2 is set to 0. We formally state this method of constructing a binary search tree as Algorithm 5.10. The Ko¨ nigsberg bridge problem is now reduced to finding a cycle in the graph of Figure 2.8 that includes all of the edges and all of the vertices. A similarity graph has a dissimilarity function s where s(v, w) measures the dissimilarity of vertices v and w. 331, 993 7. We will see that the methods of finding spanning trees may be applied to other problems as well. A string β is a substring of the string α if there are strings γ and δ with α = γβδ. r : It is hot. We return 1! · 2 = 2. Notice that G is connected. } 7. Have your program print the adjacency matrix, the number of edges, the number of loops, and the degree of each vertex. Example 1.17 Let A = {1, 3, 5}. Thus the coefficient of a5 b4 is 126. 127. If g is one-to-one, then f ◦ g is one-to-one. The number √ 2 is not rational. Six coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. Trees 35. For all integers m and n, if m is odd and n is even, then m + n is odd. On the other hand, the department might not hire anyone.) Surely we would not consider the dean’s statement to be false. In Exercises 1–9, T1 (x, y) is the propositional function “x is taller than y.” Write each proposition in Exercises 1–4 in words. Thus x ∈ X − Y and x ∈ Y − X. Simple systems such as that in Example 4.1 are easily broken since certain letters (e.g., E in English) and letter combinations (e.g., ER in English) appear more frequently than others. Every nonempty set of nonnegative integers has a least element. If we let A = {r, n, s, f } and E = {c, b, t}, the Cartesian product A × E lists the 12 possible dinners consisting of one appetizer and one entree. The graph of Figure 2.1 A A , where all entries of the submatrices A and A are 0. Therefore, 1 + 2 + · · · + n = (n2). True 30. i=0 39. Therefore, A1 = A2 . If the Skyscrapers win, I’ll eat my hat. The average-case time for input of size n of an algorithm is the average time needed to execute the algorithm over some finite set of inputs all of size n. p q p∧q T T F F T F T F T F T F F F 28. Let R = {(1, 2)} and S = {(2, 3)}. Example 1.47 If X is a set of propositions, ∧, ∨, →, and ↔ are binary operators on X. A ∪ A = A, A ∩ A = A 47. Vertices a and b belong to different components— say, a ∈ U and b ∈ V . We return to the top of the while loop. Prove that cn < 4n2 for all n ≥ 1. In Figure 5.2, contestant 7 has a first-round bye. Thus, the number of multiplications is at most (lg n), an improvement over the straightforward algorithm that uses (n) multiplications. ¬c ∨ d ∴ b∨d 98 Proofs Applying (3.1) to expressions 1 and 2, we derive b ∨ c. It follows that p3 is even. Proofs Example 2.1 p q r p→q p ∧ ¬q r ∧ ¬r (p ∧ ¬q) → (r ∧ ¬r) T T T T F F F T T T F F T T F T F F T F T F T F T T F F T T T T F F T T F F F F F F F F F F F F T T F F T T T T We will give a proof by contradiction of the following statement: For every n ∈ Z, if n2 is even, then n is even. Would you rather be the first or second player? Edges are labeled x, y to indicate capacity x and flow y. Lacrosse is not wildly popular in California. B ∩ (C − A) 14. The symbol ∃ is called an existential quantifier. Algorithm 3.2 Binary Search This algorithm looks for a value in a nondecreasing sequence and returns the index of the value if it is found or 0 if it is not found. CX (x) + CY (x) = 1 + 1 = 2. It does not have a Hamiltonian cycle. 551 Trees Section Review Exercises 1. †A starred 294 f(x) = nx mod m. Theorem 4.5 For input consisting of an n-vertex, simple, connected, weighted graph, Dijkstra’s algorithm (Algorithm 4.1) has worst-case run time (n2). Suppose by way of contradiction that it is possible. Let T and T be two spanning trees of a connected graph G. Write an algorithm that returns the index of the first item that is less than its predecessor in the sequence s1 , . Input: s (the sequence s1 , . For example, the problem of computing 5! is reduced to computing 4!; the problem of computing 4! is reduced to computing 3!; and so on. OLD NEVER PROGRAMMERS DIE THEY JUST THEIR T LOSE MEMORIES Figure 5.4 A binary search tree. We show that R ∪ S need not be transitive. Newman, was reprinted as [Newman]. Next, we add the edge (a, x), with minimal x, to our tree. Example 2.1 3. For strings of the same length over {1, 2, . Example 6.6 postorder(PT) { if (PT == null) return l = left child of PT postorder(l) r = right child of PT postorder(r) process PT } In what order are the vertices of the binary tree of Figure 6.3 processed if postorder traversal is used? Assume that each pair of lines meets in a point, but that no three lines meet in a point. We could then simply add a 2-cent stamp to make n-cents postage. If P(di , dj) is true for every pair di , dj , the condition ¬P(di , dj) in the if statement is always false. A reasonable definition of the execution time is the number of iterations of the while loop. Connect processors 1 and 2, 2 and 3, 2 and 4, 3 and 4. Definition 2.1 Let T be a tree with root v0 . g: None Repeat this procedure with the vertices on level 2: d: Include (d, f). 183 Functions, Sequences, and Relations 48. The proof of this theorem is illustrated in Figure 3.1. 34 91 mod 57 (1 modulus operation) 57, 34 requires 4 modulus operations (to make a total of 5) 57 f6 and 34 f5 (by inductive assumption) 91 57 .1 34 57 34 f6 f5 f7 Figure 3.1 The proof of Theorem 3.5. The pair 91, 57, which requires n + 1 = 5 modulus operations, is input to the Euclidean algorithm. Both input sequences must be sorted. i 526 j k l 12. (a) R is reflexive because any eight-bit string has the same number of zeros as itself. Dijkstra, which efficiently solves this problem, is the topic of this section. What is the probability that the guess is correct? We first prove If n is composite, then n has a divisor d satisfying 2 ≤ d ≤ √ n. ■ As in mathematical induction, extend a solution of a smaller problem to a larger problem. Analysis of algorithms refers to the process of deriving estimates for the time and space needed to execute algorithms. Notice that the relation R of Example 5.4 is symmetric and that the matrix of R is symmetric about the main diagonal. Given the matrix of an equivalence relation R on X, how can we easily find the equivalence class containing the element x ∈ X? , sn and the length n of the sequence The maximum value in the sequence find max(s, n) { if (n == 1) return s1 x = find max(s, n − 1) if (x > sn) return x else return sn } We prove that the algorithm is correct using induction on n. If you are asked to find an equivalence relation, you can either find the equivalence relation directly or construct a partition and then use the associated equivalence relation. A 31. Give an example of increasing positive functions f and g f(n) = O(g(n)) 1 or otherwise, prove that g(n) = O(f(n)). In general, the symbol in position n (with the rightmost symbol being in position 0) represents the number of 2n ’s. For each number pair a, b in Exercises 1–10, find integers s and t such that sa + tb = gcd(a, b). τ(R1 ∪ R2) = τ(R1) ∪ τ(R2) 60. Collision for a hash function H: H(x) = H(y) 5. 223 17. (a) Write the first four terms of the sequence. If we let Y be the set obtained from X by removing x, Y has n elements. We can turn it into this problem if we tell persons 2, 3, . Proofs of Equivalence Some theorems are of the form p if and only if q. Section 3 Review 1. (i, a) ∈ R1 and (a, k) ∈ R2 or 2. A serial computer executes one instruction at a time. Hint: Write the Inclusion-Exclusion Principle for two finite sets as |A ∪ B| = |A| + |B| − |A ∩ B| and let A = X and B = Y ∪ Z. Finding medical records in a hospital is another searching problem. = 4. , n} onto {1, . Preorder is ABCDEFGHIJ. , and step t can then be done in nt ways, then the number of different possible activities is n1 · n2 · · · nt . The one-card suit can be chosen in 4 ways, and the card itself can be chosen in 13 ways. To enter Utopia, you must possess a driver’s license or a passport. Example 3.11 De Morgan’s Laws for Logic We will verify the first of De Morgan’s laws ¬(p ∨ q) ≡ ¬p ∧ ¬q, ¬(p ∧ q) ≡ ¬p ∨ ¬q, and leave the second as an exercise (see Exercise 76). Now person 2 will not get order 1 because person 2 thinks it’s order 2. {{1}, {2}, {3, 4}} 17. A graph G has a spanning tree if and only if G is connected. Inductive Step Assume that ak ≤ 3k + 2 when k < n. For which values of m and n is the complete bipartite graph on m and n vertices a tree? Every vertex has degree 4. Because recurrence relations are closely related to recursive algorithms, recurrence relations arise naturally in the analysis of recursive algorithms. What was the population in 1970? In Exercises 88 and 89, a six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. Also, baa = α2 β2 if we set α2 = ba and β2 = a. ∀x∀y P(x, y) 87. Show how the algorithm constructs the 3-cube from the 2-cube. The relation is symmetric because whenever (x, y) is in R, (y, x) is also in R. 12 0 0 # 0 1 0 1 0 8 $ % 1 0 2 3 Figure 1.15 The final tree of Figure 1.13 with each frequency replaced by a character having that frequency. Let t(n) be the total number of times that i is incremented and j is decremented in the following pseudocode, where a1 , a2 , . How many different possibilities are available to the consumer? Thirteen persons have first names Dennis, Evita, and Ferdinand and last names Oh, Pietro, Quine, and 368 10. j∈P i∈P In Figure 3.1, the value 5 of the flow is less than the capacity 8 of the cut. These values are stored in added parameters named s and t. Abstractly, we define a relation to be a set of ordered pairs. v1 v1 v1 e1 e2 v2 v3 e1 e3 v2 v4 v1 e1 e2 e11 v6 v1 e1 e3 v2 v3 v4 v 1 e1 v 2 e3 v3 v4 v1 v2 v3 v4 7. A = {1, 2}, B = {x | x3 − 2x2 − x + 2 = 0} 27. 52!/(13!)4 42. Theorem 4.3 Dijkstra’s shortest-path algorithm (Algorithm 4.1) correctly finds the length of a shortest path from a to z. The output is the next (in lexicographic order) permutation. x + 9 = 15. 134. ■ List the permutations that begin with s2 . In a description such as (1.2), we check to see whether the element x has the property listed. Equation (*) now becomes 1. We have proved that A ∩ B = A. Find P(B). The cardinality of X (i.e., the number of elements in X) 5. Big clock city, the famous Waterbury clock. 451 Graph Theory Figure 2.11 Dominoes. Determine whether the rooted trees are isomorphic. x2 , To prove that a function f from X to Y is not one-to-one, find x1 , x2 ∈ X, x1 = such that f(x1) = f(x2). k=1 = fn−1 fn+1 + (−1) can be expressed as the sum of distinct Fibonacci numbers, no two of which are consecutive. If x is not a terminal vertex, x is an internal (or branch) vertex. Now an = a(n−1)/2 + a(n+1)/2 + 2 ≤ (3/2)(n − 1) −2 2 Recurrence Relations In case n is even, we have bn = 2bn/2 + cn/2,n/2 ; so (3/2)(n + 1) + −2 +2 2 3n 3 3(n − 1) −2+2= − = 2 2 2 3n −2 . To show if −d < x < d then |x| < d, we also use proof by cases. A single vertex is a “cycle” of length 0. We assign pigeon x to pigeonhole f(x). Exercise 9, Section 1 16. (3.1) Statement (3.1) states that on the condition that the Mathematics Department gets an additional $60,000, then the Mathematics Department will hire one new faculty member. 11 430 28. To reduce the asymptotic time, we need to take a hard look at the pseudocode to see where it can be improved. In words, this is the complete sentence: Thus x is in X. Then there are at least |W1 | edges in E of type III. R = X × X − R. Such a representation can be used by a computer to analyze a relation. , i − 1 are also in the range of f . If the pennies are both heads or both tails, Toots wins. 473 Graph Theory Notice that in a graph without loops each column has two 1’s and that the sum of a row gives the degree of the vertex identified with that row. 11 505 Graph Theory For the 4-cube, we have e = 32 and v = 16, so Euler’s formula becomes 26. The second player always wins. Which of Exercises 71–73 must also be true? FROM THE IN-LAWS In this chapter we discuss network models, which make use of directed graphs. If tj is 0, there is no edge from j to k, so there are sj tj = 0 paths of length n + 1 from i to k, where the last edge is (j, k). You can go to the Super Bowl unless you can’t afford the ticket. Input: Output: 1. If g(n) = O(f(n)), then for every C > 0, |g(n)| > C|f(n)|. Describe an intersection operation on a relational database. if (c > large) // if c is larger than large, update large 6. [Akl; Leighton; Lester; Lewis; Miller; and Quinn] discuss parallel computers and algorithms for parallel computers. This time is called the best-case time for inputs of size n. Find n ≥ 1. Is b increasing? are otherwise identical in appearance. Of the 79 students taking calculus, 45 have now been accounted for. What is the literal meaning? , 6. Therefore m + n is even. Symmetric 11. Let S = {v1 , . The steps of the algorithm are stated sufficiently precisely so that the algorithm

could be written in a programming language and executed by a computer. We have labeled the edges e1 , . Therefore statement (b) is false. To obtain a designated source and sink, we can obtain an equivalent transport network by tying together the sources into a supersource and tying together the sinks into a supersink (see Figure 1.4). The expression nk of Definition 2.11 tells us which terms of (2.4) to choose to obtain subsequence (2.3); thus, n1 = 3, n2 = 4. ∴ I don't study hard. (3.2) The algorithm then repeats using the values b and r, b > r. Input: n Output: walk(n) walk(n) { if (n == 1 ∨ n == 2) return n return walk(n − 1) + walk(n − 2) } 242 Algorithms Using the method of the proof of Theorem 4.3, we can prove that Algorithm 4.6 is correct (see Exercise 7). (c) If d | m, then d | mn. In this example, n0 in inequality (5.1) is equal to 2 and (5.1) becomes 2 ≤ k < n. What is the disjunction of p and q? Therefore, p → q is true, q → p is false. We can find the minimum-length path by listing all possible paths from a to e that pass through every vertex exactly one time and choose the shortest one (see Table 1.1). If an algorithm requires t(n) units of time to terminate in the worst case for an input of size n and t(n) = O(g(n)), we say that the worst-case time required by the algorithm is of order at most g(n) or that the worst-case time required by the algorithm is O(g(n)). for all n ≥ m, where Hk , the kth harmonic number, is defined as k=1 n2n−1 = 1 m+1 i=1 n + 1 2i 1 = , C(n, i) 2n i+1 n−1 i=0 + for all n ∈ Z . What is wrong with the following argument, which supposedly counts the number of partitions of a 10-element set into eight (nonempty) subsets? To prove that the existentially quantified statement ∃x P(x) is false, show that for every x in the domain of discourse, the proposition P(x) is false. The advantages of pseudocode over ordinary text are that pseudocode has more precision, structure, and universality. Write a program to determine whether a relation is a function from a set X to a set Y . Trees 4. n>1 We have c3 = c1 + 3 = (c0 + 3) + 3 = 6 ≤ 2 · 3. Find n ≥ 2. A tree of height 6 or more must have seven or more vertices. Define d is a divisor of n. Line (∗) results from the equation = amk−1 ≤ an and an ≤ amk = c + kd (a − b)(ak−1 b0 + ak−2 b1 + · · · + a1 bk−2 + a0 bk−1) = ak−bk with a = 3 and b = 2. Six vertices each of degree 3 23. We show that we can tile a 2k+1 × 2k+1 deficient board. Therefore, f is onto. Conclude that |X1 | = Cn−1 for all n ≥ 2. We must prove that A(x + 1, y + 1) = AO(x + 1, 2, y + 4) − 3. If G is acyclic, by Theorem 2.3, G is a tree. Define a relation R on X as s1 R s2 if some substring of s1 of length 2 is equal to some substring of s2 of length 2. q: New Atlantic gets the Olympic Games. Let T be a spanning tree for a graph G. (a) Represent the Instant Insanity puzzle by a graph. a a 1 b g c f e 7 6 i 5 3 9 10 c 5 G2 4 4 G2 1 b f 8 7 G1 2. Problem-Solving Corner Problem Let G be a directed, bipartite graph with disjoint vertex sets V and W in which the edges are directed from vertices in V to vertices in W . Can a knight move around a chessboard and return to its origin vertices nal position making every move exactly once? Thus there is an optimal solution for n − ai that uses an am coin. Example 6.22 Detecting the HIV Virus The enzyme-linked immunosorbent assay (ELISA) test is used to detect antibodies in blood and can indicate the presence of the HIV virus. Give a proof using mathematical induction that your algorithm is correct. The order of quantifiers is important; changing the order can change the meaning. What is the level of a vertex in a rooted tree? Since MORE is less than NOT, we go to the left child. Example 7.4 In the graph G of Figure 7.4, the edges (v, v1) and (v, v2) are in series. , k. Therefore 0 ≥ 6. Thus the answer is C(6, 1)C(7, 3) + C(7, 4). If at least one die shows a 5, one of the shaded outcomes occurs. Prove the Inclusion-Exclusion Principle for three finite sets: |X∪Y ∪Z| = |X|+|Y |+|Z|−|X∩Y |−|X∩Z|−|Y ∩Z|+|X∩Y ∩Z|. The next five digits identify the manufacturer, the next five digits identify the product, and the last digit is a check digit. Basis Step (n = 4) We can make four cents postage by using two 2-cent stamps. There exists x and there exists y such that P(x, y). Write a nonrecursive algorithm to compute n!. r, revpost(T2), revpost(T1). Write a program to determine whether a relation is antisymmetric. No. There are vertices of odd degree. Specifically, this road inspector must travel all of these roads and file reports on road conditions, visibility of lines on the roads, status of traffic signs, and so on. We consider two cases: a ≤ b and a > b. .] until the process reaches the straightforward problem of computing 0!. The recursive call occurs at line 12, where we invoke this algorithm with input of size n − 1. (5.7) Comparing (5.6) and (5.7), we see that the ambiguity results from whether the negation applies to Q(x) (the first interpretation) or to the entire statement ∀x(P(x) → Q(x)) (the second interpretation). We next define what it means for the best-case, worst-case, or average-case time of an algorithm to be of order at most g(n). Trees Hint: Consider a preorder traversal of the binary tree in which a one means that an edge is present, and a zero means that an edge is absent. What must the graph look like? , sn , a string α, k, and n A list of all r-combinations of {sk , sk+1 , . In this case, there are four comparisons at lines 1 and 3. Simple graphs G1 and G2 are isomorphic if and only if there is a one-to-one, onto function f from the vertex set of G1 to the vertex set of G2 that preserves the adjacency relation in the sense that vertices vi and vj are adjacent in G1 if and only if the vertices f(vi) and f(vj) are adjacent in G2 . 2 n 1+ 1 1 + · · · + = (lg n). In this example, the sum is 2 because there are two paths (a, b, c), (a, d, c) of length 2 from a to c. ■ For statements involving negation, De Morgan's laws can be very helpful. Then {{a}, {a, b}} = {{c}, {c, d}}. Prove that your values do give an optimal solution. When n is even, the sequence is divided evenly and the next sequence processed by the loop has length n/2. By using the formula bn+1 − an+1 ai bn−i = b−a n g(n) = O(f(n)). In 1 3 2 1 2 1 3 2 3 2 1 3 1 3 2 1 shows that the only possibility for the missing square is a corner. 1+ 1 1 + · · · + = (lg n). Because fact = i! is an invariant, at this point, fact = n!. Proof by cases is useful if the hypotheses naturally break down into parts. Therefore, if e and n are nonnegative integers, the properties "has e edges" and "has n vertices" are invariants. Now suppose that if n satisfies 2k−1 ≤ n < 2k , the statement x = x + 1 is executed k times. Because n/2 < n, it follows that n/2 < n. We call the sequence {snk } a subsequence of {sn }. Since there are n entries on the main diagonal, there are n2 + n n2 − n +n= 2 2 Counting Methods and the Pigeonhole Principle entries on or above the main diagonal. The two remaining edges may be added in one of two ways [see Figure 8.9(c) and (d)]. Exercises In Exercises 1–5, find the minimal spanning tree given by Algorithm 4.3 for each graph. 1 + lg n hi 16i i=0 5. 3, 2 4, 0 c z 3, 3 5, 2 z d a Figure 2.6 After increasing the flow of Figure 2.5 by 1. Algorithm 2.16 Exponentiation By Repeated Squaring This algorithm computes an using repeated squaring. Solving this last recurrence relation by iteration, we obtain Dn = (−1)n + nDn−1 = (−1)n + n[(−1)n−1 + (n − 1)Dn−2] = (−1)n + n(−1)n−1 + n(n − 1)[(−1)n−2 + (n − 2)Dn−3] .. 3 ➜ Analysis of Algorithms A computer program, even though derived from a correct algorithm, might be useless for certain types of input because the time needed to run the program or the space needed to hold the data, program variables, and so on, is too great. a = 143, n = 100, z = 230 62. Execution resumes in part (b) at line 4 after computing 2! (= 2). What is a binary operator? Sometimes, the lower bound is unattainable. Therefore, n is in X. Example 3.1 The expression a ∨ b ∨ ¬c ∨ d is a clause since the terms a, b, ¬c, and d are separated by or's and each term is a variable or the negation of a variable. 594 Network Models Definition 1.6 Let F be a flow in a network G. , 10. Example 2.23 proved that there are Cr such paths. Thus the number of routes is equal to C(n + m, n). Professor T. Therefore, at least three weighings are required in the worst case to identify the bad coin and determine whether it is heavy or light. The relation {(1, 2), (2, 1)} is not a transitive relation on {1, 2}. Using symmetry, we have (x, y) and (y, x) both in R. If S ⊊ V , we let R(S) = {w ∈ W | v ∈ S and (v, w) is an edge in G}. It returns the smallest index i such that p occurs in t starting at index i. Prove that the following are equivalent: (a) A ⊆ B (b) A ∩ B = A (c) A ∪ B = B. Since m > 1, there exists x such that f(x) = nx mod m = 1. ∀x∃y P(x, y) 76. As an example, the formula 18. Sometimes the sum and product notations are modified to denote sums and products indexed over arbitrary sets of integers. Then either i is 429 Recurrence Relations set to 3 or j is set to 2. Now c1 r n−1 + c2 r n−2 = r n−2 (c1 r + c2) = r n−2 r 2 = r n ; thus the sequence r n, n = 0, 1, . Each step can be done in two ways. We show that if GKn has a Hamiltonian cycle, n is even. Full binary tree; four internal vertices; five terminal vertices 10. We define a dissimilarity function s as follows. The truth value of the proposition p ↔ q is defined by the following truth table: p q p↔q T T F F T F T F T F F T It is traditional in mathematical definitions to use "if" to mean "if and only if." Consider, for example, the definition of set equality: If sets X and Y have the same elements, then X and Y are equal. To put the answer in graph terms, there is no path from vertex Gre to vertex Gre in Figure 1.2 that traverses every edge exactly once. A good route always leaves (0, 0) by moving right to (1, 0) and it always arrives at (k, k) by moving up from (k, k − 1). The value of each terminal vertex is written under the vertex. Explain how to convert from decimal to binary. , nk 6 (n − 7) 7 7 are distinct. Rewrite the sum 84. It then follows that si+1 is the largest value in the subsequence s1 , . 23 41. Reflect each vertex and edge in the x-axis. ∃x∃y(x2 + y2 = 9) 48. If n is not the sum of two primes, output "no" and stop. Example 3.7 Write the conditional proposition, If Jerry receives a scholarship, then he will go to college, and its converse symbolically and in words. As in postfix, no parentheses are needed and no conventions are necessary regarding the order of operations. Example 3.26 The composition of the relations R1 = {(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)} and R2 = {(2, u), (4, s), (4, t), (6, t), (8, u)} is R2 ◦ R1 = {(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)}. Example 5.2 When an element of a sequence is defined in terms of some of its predecessors, the Strong Form of Mathematical Induction is sometimes useful to prove a property of the sequence. 9450 represents a number in either decimal or hexadecimal. The Inductive Step assumes that the invariant is true and then proves that if the condition that controls looping is true (so that the loop body is executed again), the invariant is true after the loop body executes. A companion to the greatest common divisor is the least common multiple. SALESPERSONS 4. The only subset of the empty set is the empty set itself; thus, |P (X)| = 1 = 20 = 2n . Find (a) αβ (b) βα (c) |α| (d) |αβα| Section 3 In Exercises 9 and 10, determine whether the relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order. By Theorem 3.5, this can be done in C(23 + 4 − 1, 4 − 1) = C(26, 3) = 2600 ways. When n > 2, b(n+1)/2 comparisons are required for the first recursive call and bn/2 comparisons are required for the second recursive call. How many eight-bit strings begin with 0 and end with 101? Exercise 1; a0 = 2 13. We have shown that if d and n are integers, d > 0, there exist integers q and r satisfying n = dq + r 0 ≤ r < d. Problem-Solving Tips The matrix of a relation R is another way to represent or specify a relation from X to Y . An acyclic graph with eight vertices has seven edges. Algorithm 3.10 Solving the Four-Queens Problem Using Backtracking This algorithm uses backtracking to search for an arrangement of four tokens on a 4 × 4 grid so that no two tokens are on the same row, column, or diagonal. ∀x (P(x) ∧ Q(x)) 42. Let the words sink in. We must show that for all n, if equation n is true, then equation n + 1 is also true. Use Exercises 26 and 27 and Exercise 25, Section 5.3, to derive a contradiction. d, the desired conclusion. Since x ∈ X ∩ Y , x ∈ X and x ∈ Y . (p ∧ (q ∨ r)) → (r ∨ (q ∨ p)) 55. To determine whether one of these formulas may be relevant to a particular problem, first be sure that the problem asks for unordered selections. Suppose that the probability of a person having a headache is 0.01, that the probability of a person having a fever given that the person has a headache is 0.4, and that the probability of a person having a fever is 0.02. Trace Algorithm 2.3 for the input 34 20 144 55. Basis Step (n = 0) Since G is connected, if G has no edges, G consists of a single vertex. Definition 1.1 A recurrence relation for the sequence a0 , a1 , . Since n = 5 > 0, the body of the while loop executes. Thus R ⊇ τ(R). Since the first quantifier is ∀, Farley goes first and chooses a value for x. Suppose that there is such a graph with vertices a, b, c, d, e, f . , k − 1, are distinct, since f is one-to-one.] 20. We note that Sn is contained within Sn+1 , in the sense that Sn+1 = 1 + 2 + · · · + n + (n + 1) = Sn + (n + 1). (No one has yet found a simple formula for the number of polyominoes of order s.) Numerous problems using polyominoes have been devised (see [Martin]). In order for the given statement to be true, we must find a propositional function P(x, y) satisfying (1). c2 = 2, c3 = 3, c4 = 8, c5 = 9. Let G be a connected, weighted graph, let v be a vertex in G, and let e be an edge of minimum weight incident on v. Each vertex shows a particular position in the game. 95 Proofs 4. X ⊂ Y , X is a proper subset of Y : X ⊊ Y and X = Y 10. The resulting sequence is called a subsequence of the original sequence. Thus the midpoint of the pair pi and pj has integer coordinates. Rewrite the sum n f(α) = string consisting of the first three characters of α. Thus, we may assume that m > 1 and n > 1. What is a collision resolution policy? visits the nodes in the reverse order of postorder. Give an example (different from those of Example 1.2) of a definition in the system of real numbers. If one penny is a head and the other a tail, Sly wins. Show that the n-cube is planar if n ≤ 3 and not planar if n > 3. lg n denotes log2 n (the logarithm of n to base 2). Section 5 1. If i < n is true (so that the for loop body executes again), i becomes i+1. It then tiles the four subboards, and, by the inductive assumption, these subboards are correctly tiled. The set D2 consists of four students: Dale, who is 6 feet tall; Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. If a hypothesis is not a clause, it must be replaced by an equivalent expression that is either a clause or the and of clauses. (b) List the equivalence classes of X. Thus, in the worst case, Algorithm 1.8 runs in exponential time in the input size s. Tell you what? Proofs In Exercises 12–17, using induction, verify the inequality. {(x, y) | x and y have the same first name} 12. Now (n + 1)! = (n + 1)(n!) ≥ (n + 1)2n−1 ≥ 2·2 n−1 by (4.10) since n + 1 ≥ 2 =2 . (For Exercise 10) True 19. This method can be used to establish a general formula. i! j! (n − i − j)! 0≤i+j≤n 25. For example, 2, 3, and 11 are prime numbers. Definition 4.1 528 Let G be a weighted graph. , sj , i, and j si , . Show that if c is a common divisor of a and b, then c divides g. For the supply and demand functions of Figure 1.5, the price approaches that given by the intersection of the supply and demand curves. Sets and Logic Let p denote the proposition "the Chargers get a good linebacker," let q denote the proposition "the Chargers can beat the Broncos," let r denote the proposition "the Chargers can beat the Jets," and let s denote the proposition "the Chargers can beat the Dolphins." Then the hypotheses are: p→q q→r q→s p. To show that the universally quantified statement ∀x P(x) is false, it is sufficient to find one value x in the domain of discourse for which the proposition P(x) is false. To prove that ∃x∀y P(x, y) is false, where the domain of discourse is X × Y , you must show that for all x ∈ X, there is at least one y ∈ Y such that P(x, y) is false. Write a recurrence relation that describes the worst-case time required by merge sort. 20 48. i=3 23. Prove that 3 2 is irrational. For the sequence a defined by an = 2n + 2, find (a) a6 (b) (c) 3 i=1 3 i=1 ai ai (n − i)r i i=1 replacing the index i by k, where i = k + 2. 220, 1400 17. Show that S0 = 1, S1 = 2, S2 = 6, and S3 = 22. The given statement says that the sum of any two positive real numbers is positive, so we need two universal quantifiers. 4! contain the substring AE and 4! contain the substring EA; therefore, the total number is 2 · 4!. A county health inspector told a restaurant offering four-egg quiches that, because research by the FDA (Food and Drug Administration) shows that one in four eggs is contaminated with salmonella bacteria, the restaurant should use only three eggs in each quiche. There are three possible ways to obtain the sum of 10—(4, 6), (5, 5), (6, 4)—that is, the size of the event "obtaining a sum of 10" is 3. The converse can be proved similarly. 1007 47. The least integer greater than or equal to x is the unique integer k satisfying k − 1 < x ≤ k. It follows that cn = 2cn−1 + 3n−1 − cn−1 = cn−1 + 3n−1 . We then have 2i persons arranged in a circle, and, beginning with 1, we eliminate the second person, then the fourth person, and so on. As examples: Distance 1 2 3 4 Sequence of Steps Number of Ways to Walk 1 1, 1 or 2 1, 1, 1 or 1, 2 or 2, 1 1, 1, 1, 1 or 1, 1, 2 or 1, 2, 1 or 2, 1, 1 or 1 2 3 5 2, 2 Let walk(n) denote the number of ways the robot can walk n meters. At line 10 we set U = {a}. {1}, {1, 3}, {1, 4}, {1, 3, 4} 24. Since n = 2lg n , nn+1 = (2lg n)n+1 = 2(n+1) lg n . However, we can convert s to the proper value by setting s = s mod φ. , zk }, and assume that each set element is listed exactly one time per set (see Figure 6.1). Trees also arise in theoretical problems such as the optimal time for sorting (see Section 7). Find positive integers a1 and a2 such that a1 > 2a2 > 1, a2 does not divide a1 , and the algorithm, with available denominations 1, a1 , a2 , does not always produce the fewest number of stamps to make a given amount of postage. , in this order, and terminate at hn , where hole hi corresponds to vertex vi . , n by considering how many have n in the second, fourth, . Draw a binary tree with exactly two left children and one right child. (a) Use the formulas 2. Order is taken into account. At line 14, we swap sm and sk . What is a decision tree? (3.6) It then sets a = 53 and b = 4. Dividing by d, we obtain r n =q+ . The preceding example illustrates the Principle of Mathematical Induction. Suppose that person 1 gets order 3. Similarly, if the robot begins by taking a 2-meter step, a distance of n − 2 meters remains and, in this case, the remainder of the walk can be completed in walk(n − 2) ways. By the inductive assumption cn/2 = ck < 2k = 2n/2. Example 1.2 The graph of Figure 1.1 is a transport network. Push photo button. Let the 15 individual socks be the pigeons and let the 14 types of pairs be the pigeonholes. Tell whether the relation of Exercise 10 is reflexive, symmetric, transitive, antisymmetric, a partial order, and/or an equivalence relation. Let A1 be the matrix of R1 and let A2 be the matrix of R2 . What is an adjacency matrix? 13, 132 2 3 3 23. (a) 3, 5, 8, 12 (b) a1 = 3 (c) an = an−1 + n 2. {x} ∈ {x} 79. Since a maximal flow gives a maximal matching, Algorithm 2.4 applied to a matching network produces a maximal matching. Algorithms 5. A sequence s is nonincreasing if sn ≥ sn+1 for all n. (a) Replacing n by 0 in definition 2.1, we obtain s0 = 20 + 4 · 30 = 5. Let α ∈ L and suppose that |α| = n > 0. This accounts for m − 1 tests of the form sk < key. b c c d b a e 1. A proposition is typically expressed as a declarative sentence (as opposed to a question, command, etc.). an = −8an−1 − 16an−2 + 3n √ 46. (i2 − 2ij + j 2) i2 − 2 n n i=1 j=1 ij + n n i=1 j=1 j2 2. Now R1 ⊆ R1 ∪ D. Prove or disprove: A ∪ (B C) = (A ∪ B) (A ∪ C) for all sets A, B, and C. , nk are positive integers satisfying n1 < n2 < · · · < nk . At line 2, the condition i ≥ 1 is again true. This result is known as the Fundamental Theorem of Arithmetic or the unique factorization theorem. Then, assuming that S(k) is true for all k satisfying n0 ≤ k < n, (5.1) we prove that S(n) is true. Inverse f −1 of a one-to-one, onto function f : {(y, x) | (x, y) ∈ f } 12. What is the interpretation of ∃x∀yP(x, y)? (3.6) k − 1 < lg n ≤ k. Now 7n+1 − 1 = 7 · 7n − 1 = 7n − 1 + 6 · 7n . We sometimes denote a function f from X to Y as f : X → Y . This formula holds for any pair of numbers as we will show in Theorem 1.25. A fundamental result about full binary trees is our next theorem. 16. Population growth Section 3 10. In fact, since p and q are both greater than 1, one of s or t is positive and the other is negative. f(x) = sin 2x 25. A ∪ U = U, A ∩ ∅ = ∅ U A 35. x a G G Figure 7.9 The proof of Theorem 7.9 for the case that G has no cycles. Assume that a1 = 1. g h i a b c d e b d a c f e g h f i 587 Trees 13. See Algorithm 3.3. 1 = 16 − (21 − 16 · 1) · 3 = 16 · 4 − 21 · 3. v1 v2 v4 v3 v5 v8 v7 v6 v9 Figure 1.14 A graph that is not bipartite. What is a similarity graph? The domain is contained in the horizontal axis and the codomain is contained in the vertical axis. , sr) // print the ith combination } } We will show how Algorithm 4.9 generates the 5-combination of {1, 2, 3, 4, 5, 6, 7} that follows 23467. Combining P(2)√→ Q(π) and ¬Q(π), we may use modus tollens (see Figure 4.1) to conclude ¬P(2). Suppose that n is in the fourth position. , sn }. Explain how to compute an using repeated squaring. To transmit a = 572 to the holder of public key 713, 29, the sender computes c = an mod z = 57229 mod 713 = 113 and sends 113. The to-be-sent messages were temporarily stored on a disk. b3 = 3, b4 = 4 47. define an equivalence relation on the set of real-valued functions on {1, 2, . Find a Hamiltonian cycle in the 3-cube. 8!/(3!2!) 10. Let p : |4| < 3 and q : −3 < 4 < 3. If X1 , . Since each outcome is assigned the value 1/36, the probability of getting a sum of 10 is 1/12. Find P(B | A), P(B | D), and P(B | N). Proof Let x ∈ X. 23 P(E)P(F) = (1 − P(E))(1 − P(F)) = 1 − P(E) − P(F) + P(E)P(F) 14. Prove that X ⊊ X ∪ Y for all sets X and Y . Let f and g be functions from the nonnegative real numbers to the nonnegative real numbers defined by the equations n2 f(x) = 2x, g(x) = x2 . We assume that T is connected and that the removal of any edge (but no vertices) from T disconnects T. (d, c, b) 11. Arguing as in Example 1.4, we find that there are 27 7-bit strings. If d and n are integers, d > 0, there exist unique integers q (quotient) and r (remainder) satisfying n = dq + r, 0 ≤ r < d. Are the events "there are children of both sexes" and "there is at most one girl" independent? We let dn be the number of moves an optimal solution requires. Since each of the five bits can be selected in two ways, by the Multiplication Principle, there are 2 · 2 · 2 · 2 · 2 = 25 = 32 eight-bit strings that begin 101. Then G is connected if and only if T has n vertices. Step 7—move disk 3 from peg 1 to peg 3. Ultimately, a terminal vertex having value 1 is reached where the first player wins the game. Find a formula for CXY . ■ Review the Problem-Solving Tips sections in this chapter. R1 = {(x, y) | x divides y}; R1 is from X to Y ; R2 = {(y, z) | y > z}; R2 is from Y to Z; ordering of X and Y : 2, 3, 4, 5; ordering of Z: 1, 2, 3, 4 18. Economist Robert J. A large number of solvable problems have an as yet undetermined status; they are thought to be intractable, but none of them has been proved to be intractable. At lines 1 and 2, we initialize the flow to 0 in each edge (see Figure 2.7). , n and v0 = vn . Prove that there exists i such that si ≥ A. Find 5 i=1 99. Solving this system for A, B, C, we obtain A=B= 1 , 2 C = 0. The solution, S with one am coin removed, is optimal for n − am . Depth-first search can be used as a searching algorithm, in which case it is called backtracking. (b) A is an equivalence class of R1 ∩ R2 if and only if there are equivalence classes A1 of R1 and A2 of R2 such that A = A1 ∩ A2 . Notice that if the binary representation of the positive integer n is n = 1 · 2k + bk−1 2k−1 + · · · + b0 20 , then 2k ≤ n and n = 1 · 2k + bk−1 2k−1 + · · · + b0 20 ≤ 1 · 2k + 1 · 2k−1 + · · · + 1 · 20 = 2k+1 − 1 < 2k+1 . Algorithm 4.2 Computing n Factorial This recursive algorithm computes n!. The graph is an undirected, simple graph. 2 Taking the logarithm to the base 3/2, we obtain n + 1 < log3/2 m. Problem-Solving Tips To prove a1 + a2 + · · · + an = F(n), for all n ≥ 1, where F(n) is the formula for the sum, first verify the equation for n = 1 a1 = F(1) (Basis Step). How many different car license plates can be constructed if the licenses contain three letters followed by two digits if repetitions are allowed? We assume further that there is a time lag as the supply reacts to changes. For example, the following graph is colored with three colors. , vn), where v0 = v and vn = w. Example 3.7 If we replace each integer 1, 2, . In Example 3.8, we showed that gcd(n, φ) = 1 and that s n + t φ = 1, where s = −67 and t = 27. We consider two cases: X contains an element less than or equal to n, and X does not contain an element less than or equal to n. Symmetric relation R on X: for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R 36. – A 11. It follows from Theorem 4.5 that Algorithm 4.1 is optimal. The assertion can be restated as: If x > 0 and y > 0, then x + y > 0. and The inductive step is finished and the proof is complete. Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse. If both roots of t 2 − c1 t − c 2 = 0 (2.17) are equal to r, then there exist constants b and d such that an = br n + dnr n , n = 0, 1, . First, since we are looking only for the maximum sum, there is no need to record all of the sums; we will store only the maximum sum that ends at index i. Algorithms typically contain loops, in which case, deriving t(n) requires counting the number of iterations of the loops. We illustrate with another example. Problem-Solving Tips An equivalence relation is a relation that is reflexive, symmetric and transitive. We conclude that lg n! = O(n lg n). In a minimum-weight Hamiltonian cycle, every vertex must have degree 2. Show that the maximum number of edges in an n-vertex dag is n(n − 1)/2. We cannot use the Addition Principle and add |X| and |Y | to compute |X ∪ Y | because the Addition Principle requires X and Y to be disjoint. Since a (free) tree is a simple graph, trees T1 and T2 are isomorphic if and only if there is a one-to-one, onto function f from the vertex set of T1 to the vertex set of T2 that preserves the adjacency relation; that is, vertices vi and vj are adjacent in T1 if and only if the vertices f(vi) and f(vj) are adjacent in T2 . How do you disprove a universally quantified statement? Now mn = 2mk +nk pq. How many are taking none of the three subjects? At lines 7 and 8, a becomes b and b becomes r. Notice that for this choice of nq , nq ≥ −t also holds (so that we can still make n+1 cents postage, too). In how many ways can we select six books? We now extend the logical system of Sections 2 and 3 so that we can handle statements that include "for every" and "for some." Definition 5.4 Let P be a propositional function with domain of discourse D. In (2), from the assumption that T is connected but the removal of any edge (but no vertices) from T disconnects T, we must deduce that T is a tree. , si so that s1 , . The time required by an algorithm is the number of steps to termination. q = 5, r = 2 19. We write f(n) = (g(n)) and say that f(n) is of order at least g(n) or f(n) is omega of g(n) if there exists a positive constant C2 such that |f(n)| ≥ C2 |g(n)| for all but finitely many positive integers n. [It is true that if m and m + n are even integers, then n is even (see Exercise 12).] Section Review Exercises † 1. Therefore, f(n) = (n lg n). Therefore, n − kn ≤ T(n) = (kn + rn − 1)2kn + 1 < 2kn 2kn + 1 √ √ ≤ 2 2n2 2n + 1 √ = O(4 n). Since R is reflexive, y R y. 98. ≤ k2k + k2k = 2 · 2k k = 2n lg n. cn = 2 + i=1 ci , n ≥ 2; c1 = 1 38. ■ It may help to solve the parts of a problem in a different order than that given in the problem statement. Since X is a nonempty set of nonnegative integers, by the Well-Ordering Property, X has a smallest element, which we denote r. What is an alternative name of the prefix form of an expression? Set s = s mod φ. Cube 1 is to have red and green opposing left and right faces. Condition (4.7) is sometimes called the Basis Step and condition (4.8) is sometimes called the Inductive Step. paths in graphs. Which of Exercises 74–76 must also be true? For example, if you are going to prove that X = Y , write "We will prove that X = Y " before launching into this part of the proof. Continue in this way until you reach the root. If we modify the four-coins puzzle by requiring only that we identify the bad coin (without determining whether it is heavy or light), we can solve the puzzle in two weighings in the worst case (see Exercise 1). In the RSA public-key cryptosystem, how does one encrypt a and send it to the holder of public key z, n? Write an algorithm whose input is a permutation of {1, 2, . Thus the total number of comparisons is 4 + 6(k − 1) + 2 + 4 = 6k + 4. Perform the computation n i i=0 bi 2 in decimal. There is an edge from team ti to team tj if ti beat tj at least one time. The complete graph on five vertices K5 7. 2 + 4 + · · · + 2n + 2(n + 1) = n(n + 1) + 2(n + 1) = (n + 1)(n + 2) 2n(n + 1)(2n + 1) 14. X∗ is the set of all strings over X. Explain how to prove that a universally quantified statement is false. What is a right child in a binary tree? For every x and for every y, P(x, y). Write a program that tests whether two logical expressions in p, q, and r are logically equivalent. Counting Methods and the Pigeonhole Principle 24. abab, The length of a string α is the number of elements in α. Thus the matrix of the relation R2 ◦ R1 , relative to the previously chosen orderings 1, 2, 3 and x, y, z, is ⎛x 1 1 2 ⎝0 3 1 y z⎠ 1 0 1 1⎠ . Example 3.9 Four-Queens Problem The four-queens problem is to place four tokens on a 4 × 4 grid so that no two tokens are on the same row, column, or diagonal. The nth term is n plus the previous term. Examples: There is one way to parenthesize a1 ∗ a2 , namely, (a1 ∗ a2). We assume that at time n, the quantity qn sold at price pn is given by the equation pn = a − bqn , where a and b are positive parameters. By assumption, either x ∈ A or x ∈ B. ax . In particular, if {X1 , X2 , . When is this quantified expression true? By considering for i1 = 1 to n for i2 = 1 to i1 println(i1 , i2) and Example 3.9, deduce 36. Example 6.13 Using the generalized De Morgan's laws for logic, we find that the negation of ∀x∃y P(x, y) is ¬(∀x∃y P(x, y)) ≡ ∃x¬(∃y P(x, y)) ≡ ∃x∀y ¬P(x, y). 1 simple, the degrees of c, d, e, and f are each at least 2; thus there is no such graph. In Exercises 60–65, represent the proposition symbolically by letting p : You heard the "Flying Pigs" rock concert. We previously gave a proof of this result using mathematical induction. , where {fn } is the Fibonacci sequence. Increasing sequence: sn < sn+1 for all n 196 18. r : You take multi-vitamins. Give an example of independent events. the user could enter "Alfred Hitchcock" Herrmann OR Waxman -tv (2.1) [For those who have not studied discrete mathematics, clicking on Advanced on the Google home page yields a page in which the user can fill in boxes to achieve the same result as (2.1).] Problem-Solving Tips Although there may be a shorter way to determine the truth values of a proposition P formed by combining propositions p1 , . satisfy the same recurrence relation, s1 = f2 , and s2 = f3 , from which it follows that sn = fn+1 for n ≥ 1. For example, for the relation PLAYER given in Table 6.1, PLAYER [Position = c] will select the tuples (22012, Johnsonbaugh, c, 22), Example 6.4 (84341, Cage, c, 30). [For example, (2, 3) ∈ R, but (3, 2) ∈ / R.] The relation R is reflexive and transitive. Figure 1.4 with flows Fa,w1 = 2, FAz = 0, FcB = 4, Fdc = 2. , n} onto if k ≤ 18. P(2) = P(4) = P(6) = 1/12. If d ≤ n, then n has a divisor d (namely d = d) satisfying 2 ≤ d ≤ n. (2.10) Often in mathematics, when trying to solve a more difficult instance of some problem, we begin with an expression that solved a simpler version. The union of R1 and R2 is the n-ary relation R1 ∪ R2 . At line 7, key < sk ('G' < 'C') is false, so at line 10 we set i to 2. e, f, g, j; j 17. Select a theta notation from among (1), (n), (n2), (n3), (2n), or (n!) for the number of times the line x = x + 1 is executed. Since p is true and q is false, p ∨ q is true. Since r1 − r2 = 0, we can solve for d. List the order in which the vertices are processed using postorder traversal. 34. r : You pass the course. The analysis begins with the terminal vertices. Suppose that x R y. Write a program that tests whether A is decreasing. What is inorder traversal? Now Inductive Step Suppose that n > 11 and assume that if k < n, k is odd, k > 5, and 3 divides k2 − 1, then a k × k deficient board can be tiled with trominoes. Some Common Errors In Example 1.10, we pointed out that it is an error to use the same notation for two possibly distinct quantities. In this section and the next, we introduce some general methods of proof. Notes The first half of [Knuth, 1977] introduces the concept of an algorithm and various mathematical topics, including mathematical induction. Determine the values of the flows. Sally Ruth Sam Math Physics Econ a c d b 17. Today is Monday or it is raining. Tell whether the path (v2 , v3 , v4 , v2 , v6 , v1 , v2) in the graph is a simple path, a cycle, a simple cycle, or none of these. Consider ∀x∀y P(x, y). Similarly, if S = {(2, 3)}, S is transitive. Vertex v is labeled (predecessor(v), val(v)). What is the probability of not getting a 5? Arguing as in the previous paragraph, we find that there are 666 multiples of 3 and 5 between 1 and 10,000. We will illustrate the general method of solving linear homogeneous recurrence relations with constant coefficients by finding an explicit formula for the sequence defined by the recurrence relation an = 5an−1 − 6an−2 (2.9) and initial conditions a0 = 7, a1 = 16. Find the children of d and of e. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is D4 × D4 . Since i = 0 and power = 1, ci ∗ power = 1 ∗ 1 = 1. The relation R from the set X of planets to the set Y of integers defined by (x, y) ∈ R if x is in position y from the sun (nearest the sun being in position 1, second nearest the sun being in position 2, and so on) 9. , pn−1 → pn , pn → p1 . Thus C is not a Hamiltonian cycle. Trace Algorithm 2.3 for the input 64 77 15 3. For the example of Figure 4.1, the relation obtained could be described as "is the same color as." The next theorem shows that such a relation is always reflexive, symmetric, and transitive. Conversely, if S is a partition of X and we define xRy to mean that for some S ∈ S, both x and y belong to S, then R is an equivalence relation. State the complement laws for sets. Then the algorithm requires h comparisons in the worst case, so h = f(n). Show that if a flow exists in G, a minimal flow exists in G with value max{m(P, P) − C(P, P) | (P, P) is a cut in G}. Establish the recurrence relation bn = bn/2 + b(n+1)/2 + 2 b(n−1)/2 + 2 bn/2 + 1 if n is odd if n is even. (sr may have the maximum value n, sr−1 may have the maximum value n − 1, etc.) Then ti = si , for i = 1, . U A B 2 3 4 1 Figure 1.3 A Venn diagram. Show that there are at least two available items in the list exactly four items apart. Then there exists k such that n = 2k + 1. 2)130 remainder = 0 1's bit 2)65 remainder = 1 2's bit 2)32 remainder = 0 4's bit 2)16 remainder = 0 8's bit 2)8 remainder = 0 16's bit 2)4 remainder = 0 32's bit 2)2 remainder = 0 64's bit 2)1 remainder = 1 128's bit 0 We may stop when the quotient is 0. Let f be the function from X = {0, 1, 2, 3, 4, 5} to X defined by f(x) = 4x mod 6. Example 2.13 Find an explicit formula for the Fibonacci sequence. (d) The descendants of Kronos are Zeus, Poseidon, Hades, Ares, Apollo, Athena, Hermes, and Heracles. If F is a flow in a network G and (P, P) is a cut in G and the capacity of (P, P) exceeds the value of the flow, F , then the cut (P, P) is not minimal and the flow F is not maximal. Let an denote the number of ways to exactly cover this board by 1 × 2 dominoes. We are given that ∀x∃y P(x, y) is true, which says, in words, for every x, there exists at least one y for which P(x, y) is true. We call D the domain of discourse of P. 1101 + 101100 + 11011011 In Exercises 26–31, express each hexadecimal number in decimal. Now only Counting Methods and the Pigeonhole Principle processors 3 and 4 are directly connected to the same number of processors. Comments It's important to verify that an alleged partition is truly a partition before using the Addition Principle. 77 Proofs Discussion The outline of the proof is X, Y , and Z are sets. For a fixed number S, we insert an edge between vertices v and w if s(v, w) < S. In how many ways can 10 balls be selected if at least one red ball, at least two blue balls, and at least three green balls must be selected? Since (1 + 1/n)n+1 = 4, when n = 1, 4≥ 1+ 1 n n+1 = n+1 n n+1 n+1 n n+1 8. By the Multiplication Principle, there are (27 − 1)(224 − 2) = 127 · 16,777,214 = 2,130,706,178 Class A Internet addresses. (b) Show that an ≤ an−1 + 2, n ≥ 3. Everybody loves somebody. Answer 74–81 for the sequence w defined by wn = 74. Given a tree that defines a Huffman code, such as Figure 1.10, any bit string [e.g., (1.1)] can be uniquely decoded even though the characters are represented by variable-length bit strings. n i=m ai ? (4.3) Argument (4.3) is said to be valid if the conclusion follows from the hypotheses; that is, if p1 and p2 and · · · and pn are true, then q must also be true. If A = {1, 3, 4, 5, 6, 7}, B = {x | x is an even integer}, C = {2, 3, 4, 5, 6}, find (A ∩ B) − C. What is set notation? But, by definition, this algorithm requires bn−1 comparisons for input of size n − 1. {(b, c), (c, b), (d, d)} 17. X ⊆ Y , X is a subset of Y : every element in X is also in Y 9. For which values of m and n does the graph of Exercise 37, Section 2, contain a Hamiltonian cycle? † Exercise 596 numbers in color indicate that a hint or solution appears at the end of this chapter. Chapter Self-Test 1. Therefore, the sequence U defined by Un = 5 · 2n + 2 · 3n satisfies the recurrence relation (2.9) and the initial conditions (2.10). Since i = 3 and power = 8, ci ∗ power = 1 ∗ 8 = 8. As an example, we describe a randomized algorithm that shuffles a sequence of numbers. Prove that (A B) A = B for all sets A and B. 1 k 266 Introduction to Number Theory Proof Let l = lcm(m, n). , m − 1; set tm = sm + 1; and set tm+1 · · · tr = (sm + 2)(sm + 3) · · ·. Acyclic; four edges, six vertices 24. Now x2 = p2 /q2 is rational, which is a contradiction. See [Akl; Leighton; Lester; Lewis; Miller; and Quinn] for sorting using parallel machines. Then f(g(x1)) = f(g(x2)). Algorithms 13. If p and q are propositions, the conditional proposition is the proposition if p then q. Section 5 Review 1. Output: r, n A list of all r-combinations of {1, 2, . A link consists of two nodes sharing files. Similarly, if the prime pi is not a factor of n, we let bi = 0.) Then 1 ,b1) max(a2 ,b2) k ,bk) lcm(m, n) = pmax(a p2 · · · pmax(a . Show that n! = O(nn). Example 5.5 Incidence Matrix To obtain the incidence matrix of the graph in Figure 5.4, we label the rows with the vertices and the columns with the edges (in some arbitrary order). We then change the label L on each vertex in H1 to 0L and we change the label L on each vertex in H2 to 1L. 4n−1 for all n ≥ 1. Since 4! = 24, there are 24 possible outcomes to the problem of sorting four items (when the items are distinct). Let Sn denote the number of 1's in the binary representation of n. Find the 32. 73. The following table gives the values of this sum for n = 1, 2, 3, 4. Next, move the kn disks on peg 1 to peg 4 by invoking the optimal three-peg algorithm (see Example 1.8) and using only pegs 1, 3, and 4. Explain how to convert from decimal to hexadecimal. Construct an optimal Huffman code for the set of letters in the table. Compute φ. (Less formally, R is symmetric if and only if A is symmetric about the main diagonal.) The reason is that R is symmetric if and only if whenever (x, y) is in R, (y, x) is also in R. How many strings do not contain the letter A if repetitions are not allowed? Example 1.2 Melissa Virus In the late 1990s, a computer virus named Melissa wreaked havoc by overwhelming system resources. 5 31. If not, we next check whether p occurs at index 3 in t, and so on. The inverse relation generalizes the inverse function. Prove that for all real numbers x, y, and z, if x + y + z ≥ 3, then either x ≥ 1 or y ≥ 1 or z ≥ 1. Thus n2 is odd, which contradicts the hypothesis n2 is even. We are given text t and we want to find the first occurrence of pattern p in t or determine that p does not occur in t. If c ∈ Y , set s3 = 1; if c ∈ Y , set s3 = 0. Example 2.13 Hexadecimal Addition Add the hexadecimal numbers 84F and 42EA. We must show that X contains a largest element. Give a formula for the probability of E given F . This is a variant of the n-queens problem. Suppose that (x, y), (y, x) ∈ R−1 . {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)} 1. Thus n2 ≤ 40/3 and n ≤ 3. Under My Documents are Fax, My Data Sources, My Pictures, and others. This contradiction shows that T is acyclic. a 3 c b 5 4 7 5 6 d 7 p: p1 , . i=1 67. If f(n) = (g(n)), then 2f(n) = (2g(n)). Determine whether the following argument is valid. Also, write a sequence of operations that could be used to answer the queries. Then, at lines 8 and 9 we label vertex a as (−, ∞) (see Figure 2.8). Suppose that S(1) is true and, for all n ≥ 1, if S(n) is true, then S(n + 1) is true. A problem that does not have a worst-case polynomial-time algorithm is said to be intractable. (n2) 16. bn = small = small left 39. Recurrence Relations 71. See Exercise 30 for a similar graph model. We compute a5 using the formula a5 mod z = aa4 mod z = [(a mod z)(a4 mod z)] mod z. Table 4.1 summarizes this process. Give an algorithm to execute an inorder traversal. What is an existence proof? Give an example of an equivalence class for your equivalence relation of Exercise 1. The variable n is incremented and becomes 1. We may select numbers to be placed in the first three positions in C(n − 1, 3) ways. v9 e8 v8 v 10 19. We will discuss the hexadecimal system and leave the octal system to the exercises (see Exercises 42–47). 4 − 1 = 3 19. Graph Theory c 2009 by From Discrete Mathematics, Seventh Edition, Richard Johnsonbaugh. We prove only ∅ = U. An alpha cutoff occurs at a box vertex when a grandchild w of v has a value less than or equal to the alpha value of v. 263 Introduction to Number Theory Theorem 1.12 The number of primes is infinite. Since b = 0, we proceed to line 6, where we set r to a mod b = 72 mod 36 = 0. Adjacency matrix 45. e1 is incident on v1 and v2 . f(x) = 4x3 − 5, X = set of real numbers 5. The statement is true because, for every real number x, it is true that the square of x is positive or zero. Notice that the digraph of this relation (see Figure 3.1) has this property. Then a ≥ fn+2 and b ≥ fn+1 , where {fn } denotes the Fibonacci sequence. Since we need two edges incident on R, our initial selection of edges incident on B must be revised. Finding a Solution Suppose there are n persons. What is an isolated vertex? ■ Begin with 1. Let P(x, y) be the propositional function x > y with domain of discourse R ×R. Example 1.5 Frequently in manufacturing, it is necessary to bore many holes in sheets of metal (see Figure 1.6). Then (x, y) ∈ R and (x, y) ∈ S. Definition 2.11 Example 2.12 Let {sn } be a sequence defined for n = m, m + 1, . n = 80 In Exercises 59–61, trace Algorithm 2.19 for the given values of a, n, and z. , sn , n Output: s1 , . ⎛ 1 2 a 1 0 b⎢ 1 1⎥ c⎢0 1 d⎝ 0 0 e 0 0 1. What is 162 n i=m ai ? 31. Therefore l = pc11 · · · pckk for some c1 , . Notice that preorder traversal may be obtained by following the route shown in Figure 6.5, and that reverse postorder traversal may be obtained by following the route shown in Figure 6.6. If data are stored in a binary search tree, as described in Section 5, inorder traversal will process the data in order, since the sequence left/root/right agrees with the ordering of the data in the tree. State three forms of the Pigeonhole Principle. (n) 7. Describe the cobweb in economics. (8.1) ■ Each cube should be represented by an edge exactly once in each graph. The graph of Exercise 9 does not have an Euler cycle because it is not connected. Prove that 2m + 5n2 = 20 has no solution in positive integers. Recurrence Relations At the end of n − 1 years, the amount is An−1 . A system in which the roads do cross is illustrated in Figure 7.1. If you try drawing a system in which the roads do not cross, you will soon be convinced that it cannot be done. Let us look at the definition of "minimum." If d1 ≤ d2 , then d = min{d1 , d2 } = d1 and d = d1 ≤ d2 . What is a function from X to Y ? Thus the total number of selections is C(5, 3)C(95, 1) + C(5, 4). This procedure continues until insertion sort inserts sn into the sorted subsequence s1 , . The sequence b, c (2.3) is a subsequence of the sequence t1 = a, t2 = a, t3 = b, t4 = c, t5 = q. Give an example of an (n + 1)-element subset X of {1, 2, . Similarly, an existentially quantified proposition generalizes the proposition P 1 ∨ P 2 ∨ · · · ∨ Pn (5.5) in the sense that the individual propositions P1 , P2 , . Write an algorithm that prints the characters and their codes given a Huffman coding tree (see Example 1.8). 5, 3 d 3, 2 e b 4, 0 c 5. That is, we compare the word to be added with the word at v and move to the left child of v if the word to be added is less than the word at v; otherwise, we move to the right child of v. The last equality is true because of the definition of the Fibonacci numbers: fn = fn−1 + fn−2 for all n ≥ 3. time to search in an n-vertex balanced binary search tree is O(lg n). 6h. We repeat this process. p → q 43. List the elements of the event E2 and E3 . Notice that since a2 , . Draw the two trees that would be created after the preceding tree when tournament sort is applied. a∨b 2. Therefore, b2 = 1. We have proved the Inductive Step. We return to the top of the for loop. Pascal's triangle: C(n + 1, k) = C(n, k − 1) + C(n, k) Section 5 Section 8 15. The converse is also true; every connected, acyclic graph is a tree. Now 5 and 8 combine to give a contradiction, and the proof is complete. The set D4 consists of three students: Pat, Sandy, and Gale, each of whom is 5 feet 11 inches tall. We have Fij = Fji , j∈V i∈V j∈V since each double sum is i∈V Fe , e∈E where E is the set of edges. In both cases, we see that pathologies result. Could examples in Section 6 other than Example 6.6 have been used to solve problems (a) and (b) in this Problem-Solving Corner? True or false? Since a > b, we move to line 5. In Exercises 95 and 96, give an example of a unary operator [different from f(x) = x, for all x] on the given set. If p is not in t, at least one comparison must be performed for each i. We continue until an explicit formula is obtained. Define f(n) = (g(n)). Since L(a) is zero, L(a) is the length of a shortest path from a to a. (3.1) (b) How many solutions in integers are there to (3.1) satisfying x1 > 0, x2 > 1, x3 > 2, x4 ≥ 0? Show that existential generalization (see Figure 5.1) is valid. †A starred 514 Code A B C D E 1 00 01 10 0 which, he argues, store the text in less space than that used by an optimal Huffman code. Therefore, fact = i! is an invariant for the while loop. Give an example of the use of each form of the Pigeonhole Principle. (x = x and y = y), or (y = y and x = 0 and x = 1), or (y = y and x = 1 and x = 0), or (x = x and y = 0 and y = 1), or (x = x and y = 1 and y = 0). If mk is the highest power of 2 that divides m, then m = 2mk p, where p is odd. Write a program that accepts as input any of the following: ■ ■ ■ a listing of edges of a graph given as pairs of positive integers the adjacency matrix the incidence matrix and outputs the other two. Switch to one of the unchosen, unopened doors. Assume that pow = ai−1 . For example, the sequence u can also be denoted {um }∞ m=0 . Example 1.22 The function f = {(1, b), (3, a), (2, c)} from X = {1, 2, 3} to Y = {a, b, c, d} is one-to-one. 49 Sets and Logic Section Review Exercises 9. 164 yi 4 101. = (n + 1) lg Dividing by n + 1 gives the desired inequality. 2 Show that nn+1 = O(2n). 280 Introduction to Number Theory Theorem 2.17 If a, b, and z are positive integers, ab mod z = [(a mod z)(b mod z)] mod z. Decrypt the message JDQHLHIF AU using the key of Example 4.1. 298 5. We might generate random arrangements of sequences to use as input to test or time a sorting program. More precisely, we require a total ordering of the tasks t1 , t2 , t3 , t4 , t5 such that if ti R tj , then i = j or ti precedes tj in the list. From p → r and p, we may use modus ponens to conclude r. x + y = x + y 78. A television poll of 151 persons found that 68 watched "Law and Disorder"; 61 watched "25"; 52 watched "The Tenors"; 16 watched both "Law and Disorder" and "25"; 25 watched both "Law and Disorder" and "The Tenors"; 19 watched both "25" and "The Tenors"; and 26 watched none of these shows. If n = 2, each person throws a pie at the other and there are no survivors. Four microprocessors are randomly selected from a lot of 100 microprocessors among which 10 are defective. Show that a tree has either one or two centers. Find A1 , A2 , and A3 . In either case, x ∈ Z. A string contains AB and BE if and only if it contains ABE and the number of such strings is 3!. 10 Section 1 55. Combining these rules, we can differentiate more complicated functions. By induction, t ≤ 2h−1 . Proof Let an be the number of comparisons required by Algorithm 3.8 to sort n items in the worst case. If p is a person, define a function f by letting f(p) be the room in which person p is located. Either there is a hurricane or it is raining, but there is no hurricane. In this section we consider orderings of sequences containing repetitions and unordered selections in which repetitions are allowed. Introduction to Number Theory Richard Johnsonbaugh 259 6. Since X ⊊ Y and x ∈ X, x ∈ Y . Algorithm 3.15 Input: s1 , . The role of logic in reasoning about computer programs is discussed by [Gries]. We may assume that n ≥ am−1 + am . Now use Exercise 13 to deduce a contradiction. Suppose that n is even. By dividing the routes into classes based on when, after leaving the lower-left corner, the route first meets the diagonal line from the lower-left corner to the upper-right corner, derive the recurrence relation 1 Cn = C(2(n + 1), n + 1) − Ck C(2(n − k), n − k). A similar analysis shows that the price tends to that given by the intersection of the supply and demand curves (Figure 1.5) when b < k; the increasing price-swings case (Figure 1.7) occurs when b > k (see Exercises 38 and 39). P(J | B) = = P(B | J)P(J) P(B | J)P(J) + P(B | T)P(T) + P(B | R)P(R) (0.05)(0.25) (0.05)(0.25) + (0.03)(0.30) + (0.02)(0.45) = 0.409836065. I have it at 42 exposures, but I wanna get it to 47 because. The domain of the function a is X = {1, 2, . A03 30. ■ If one proof technique seems not to be working, try another. Theorem 3.4 A graph G has a spanning tree if and only if G is connected. Let x ∈ X ∩ Y . What is the error in the following reasoning? Then, by definition, ρ(R) = R ∪ D, where R is any relation on X. Implement Algorithm 1.8, testing whether a positive integer is prime, as a program. Notice that If Maria visits France, then she goes to the Eiffel Tower, is not an equivalent formulation. Prove that among a group of six students, at least two received the same grade on the final exam. 6 9 18 Conditional Probability Suppose that we roll two fair dice. ∀x P(x) 16. There are C(n, r) r-combinations. 7643 44. ⎢ ⎢⎝0 0 0 0 2⎠ 1 1 1 2 0 ⎛a b c d e ⎞ 0 1 0 1 0 ⎢1 0 1 0 1⎥ ⎥ ⎢0 1 0 1 1⎥ . What is an existentially quantified statement? 60 Assume that ∃x∀y P(x, y) is false and that the domain of discourse is nonempty. Swap d and r. Examples: 0111 R 1010 (because both 0111 and 1010 contain 01). Example 5.3 The matrix of the relation R from {2, 3, 4} to {5, 6, 7, 8}, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by xRy if x divides y is ⎛5 6 7 8 ⎞ 2 0 1 0 1 3 ⎝0 1 0 0 ⎠ . Quantifiers true for at least one y, doesn't it seem unlikely that it would follow that P(x, y) is true for every y? C(52, 13) 55. Now x2 = p2 /q2 . ∀x¬(∃y P(x, y)) 71. h n d e i j k o p 26. Repeat if z ∈ / T. Prove that the n-cube has a simple cycle of length m if and only if m ≥ 4 and m is even. Section 5 Review Section 4 p→q p ∴q 1. (2.3) We show that proving (2.3) shows that p1 , p2 , . The value of the root is 3. 1 In designing printed circuits it is desirable to have as few lines cross as possible; thus the designer of printed circuits faces the problem of planarity. We must prove that it takes n multiplications to compute the product a1 a2 · · · an if parentheses are inserted in any manner whatsoever. τ(σ(ρ(R1))) = {(x, y) | x, y ∈ {1, 2, 3, 4}} 12. Lexicographic order generalizes ordinary dictionary order. [Vilenkin] contains many worked-out combinatorial examples. If the set S is an infinite set, the algorithm will not terminate, so it lacks the finiteness and output properties. Output: A minimal spanning tree. Trees ■ When trying to construct a proof, consider using closely related definitions and theorems. Let R be a reflexive relation on X satisfying: for all x, y, z ∈ X, if x R y and y R z, then z R x. Let an be the minimum number of links required by n nodes so that all files are known to all nodes. (d) T is connected and has n − 1 edges. Taking x = −2/3, we have x ∈ X but x∈ / Z. Let X denote the set of sequences with finite domain. Just check the web link. What is a database management system? Basis Step cos x = cos x + · · · + cos nx + cos(n + 1)x cos[(x/2)(n + 1)] sin(nx/2) = + cos(n + 1)x. There is a loop at every vertex in Figure 3.1. Figure 3.1 The digraph of the relation of Example 3.4. Example 3.5 The relation R on X = {a, b, c, d} given by the digraph of Figure 3.2 is R = {(a, a), (b, c), (c, b), (d, d)}. Deduce that the number of ways to divide a convex (n + 2)sided polygon into triangles by drawing n − 1 nonintersecting lines through the corners is Cn , n ≥ 1. 2 n y 49. A member of E is one of the three types: Type I: (a, v), v ∈ V ; Type II: (v, w), v ∈ V, w ∈ W; Type III: (w, z), w ∈ W. We have found a prime, namely 2311, larger than each of 2, 3, 5, 7, 11. Prove that there exists i, 1 < i < n, such that si = 0. If x ∈ U, then x ∈ A ∪ U. 3 40. and thus x ∈ A. 6c. Show that σ(R) is symmetric. T1 as in Exercise 1 w1 w2 w3 w4 w6 w5 3. Explain how a tree can be used to represent an expression. This form of the definition can often be used to prove that a function is one-to-one. 18/38 44. In this case, the code found a value in the domain of discourse, namely di , for which P(di) is true. (We can divide both 4 and 6 by 2.) The fraction 3/8 is in lowest terms because the greatest common divisor of 3 and 8 is 1. Example 1.43 If f(x) = log3 x and g(x) = x4 , f(g(x)) = log3 (x4), Example 1.44 g(f(x)) = (log3 x)4 . Then S(n) is true for every positive integer n. Finally, we insert a3 in its proper place. – A * + A B C – D prefix: ∗ + AB − CD postfix: AB + CD − ∗ C + D E prefix: − ∗ A ∗ BC/C + DE usual infix: A ∗ B ∗ C − C/(D + E) parened infix: ((A ∗ (B ∗ C)) − (C/(D + E))) 16. Take x = y = 0. Give an example similar to Example 1.5 of a tree that is used to specify hierarchical relationships. To find s and t, we work back, beginning with the last equation [equation (3.7)] in which r = 0. Basis Step (i = 1) The first time we arrive at line 9, because of the initialization steps (lines 2–4), L(a) is zero and all other L-values are ∞. (b) Find an initial condition for the sequence. The minimum-queens problem asks for the minimum number of queens that can attack all of the squares of an n×n board (i.e., the minimum number of queens such that each row, column, and diagonal contains at least one queen). The resulting flow will then be a maximal flow. Now Fai = Fji − Fij i j∈P = i j∈P Fji + j∈P i∈P = Fji − Fji − j∈P i∈P Fij − j∈P i∈P j∈P i∈P j∈P i∈P Example 3.8 i Fij ≤ j∈P i∈P Fij j∈P i∈P Fji ≤ Cji . 6k. ui , for 2 ≤ i ≤ 7, is equal to one. We offer $1000 cash or 0 percent interest for two years. n! = n(n − 1) · · · 2 · 1 ≤ n · n · · · n = nn Section 3 Review 1. After the swap we have 77 64 96 i Since 15 < 44, 44 must move one position to the right 15 44 We next swap ai and aj , where i = 3 and j = rand(3, 5) = 5. If some off-diagonal entry in the matrix Y is zero, what can you say about the graph G? A relation R on a set X is antisymmetric if for all x, y ∈ X, if (x, y) ∈ R and (y, x) ∈ R, then x = y. Example 2.2 We will give a proof by contradiction of the following statement: For all real numbers x and y, if x + y ≥ 2, then either x ≥ 1 or y ≥ 1. √ If n has a divisor d satisfying 2 ≤ d ≤ n, by Definition 1.4 n is composite. Multiplying by t and noting that t > 0, we have z = tx ≤ ta. This technique is time-consuming and was shown mainly to illustrate exactly what is meant by common divisors and the greatest common divisor. [In fact, (2.1) is a simple cycle since no vertices are repeated except for v0 and w0 .] Thus there is a unique simple path from any vertex to any other vertex in T. For example, if n = 1, we obtain the proposition P(1) : 1 is an odd integer (which is true). After two months, the pair alive in the beginning becomes productive and adds one additional pair. Set bn = an /n! to obtain bn = −2bn−1 + 3bn−2 . Next, consider the statement ∃x∀y(x + y = 0). , f(N)/g(N)} ≤ C. p: I take hotel management. In the rooted tree structure, each vertex represents a file or a folder. [For K5] 6. Therefore X = Y and f is one-to-one. It is helpful at this point to introduce some terminology. We call such a convention operator precedence. Thus the average-case time of Algorithm 3.17 is (n). 296 Introduction to Number Theory Example 4.1 If a key is defined as character: ABCDEFGHIJKLMNOPQRSTUVWXYZ replaced by: EIJFUAXVHWP GSRKOBTQYDMLZNC the message SEND MONEY would be encrypted as QARUESKRAN. The choice between the two unchosen, unopened doors is made randomly. Since p is true, p ∧ (q → r) is true. Therefore, there are at least k values, a1 , . Bit Decimal number system Binary number system Computer representation of integers: when represented in binary, the positive integer n requires 1 + lg n bits Hexadecimal number system Base of a number system Convert binary to decimal Convert decimal to binary Convert hexadecimal to decimal Convert decimal to hexadecimal Add binary numbers Add hexadecimal numbers Compute an using repeated squaring ab mod z = [(a mod z)(b mod z)] mod z Compute an mod z using repeated squaring Section 3 26. Suppose that we want to prove a statement for all n ≥ 3 involving cn . Repetitions in a string can be specified by superscripts. Finally assume that x < 0 and y < 0. (b) We give an example to show that statement (b) can be false while the given statement is true. A similar argument shows that the entries on the diagonals in A1 and A2 are also equal. Figure 1.5 21. The x's denote the elements in E1 , the y's denote the elements in E2 , and the z's denote the elements in E1 ∩ E2 . defined by the equations c1 = 0, cn = cn/2 + n2 for all n > 1. are explicitly given values for a finite number of the terms of the sequence. 94 1 is the first element, Proofs 26. We state the Inclusion-Exclusion Principle for two sets as Theorem 1.12. By the inductive assumption, the value returned, sum(n − 1), is equal to 2 + · · · + 2(n − 1). 4, 4 (−1,) a z c 2, 2 (a, 1) 3, 2 4, 4 (−1,) d b c 2, 2 4, 2 e 2, 2 Figure 2.10 After increasing the flow on path (a, d, e, z) by 2 and the fourth and final labeling. An argument like that in Example 2.1 shows that p is even. If E1 and E2 are events, P(E1 ∪ E2) = P(E1) + P(E2) − P(E1 ∩ E2). What is the probability of all girls given that there is at least one girl? Does 40. an = 3 · 5n + (−2)n 8. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is D2 × D2 . If we multiply the first equation by r1 and subtract, we obtain d(r1 − r2) = r1 C0 − C1 . Show that if a tree has two centers they are adjacent. (5.1) Proof We will prove the equivalent inequality t ≤ 2h (5.2) by induction on h. (b) f(r1) = r2 . Prove that bn = ((lg n)2). The quotient 7 becomes the new value of n. TABLE 6.1 ■ PLAYER ID Number 22012 93831 58199 84341 01180 26710 61049 39826 Name Johnsonbaugh Glover Battey Cage Homer Score Johnsonbaugh Singleton Position Age c of p c 1b p of 2b 22 24 18 30 37 22 30 31 We can also express an n-ary relation as a collection of n-tuples. To this end, we let p1 , p2 , . In computer science, trees are useful in organizing and relating data in a database (see Example 1.7). Tiling with polyominoes is the subject of the book by [Martin]. 18. Since m = 2, the condition m > 0 is true; so we execute the body of the while loop. Suppose that a box vertex v is known to have a value of at least x. Let ai be the number of processors to which processor i is directly connected. Prove that Algorithm 3.7 is correct. 29. Give an example of a network with exactly two maximal flows, where each Fij is a nonnegative integer. Explain how to use an arrow diagram to determine whether a function is one-to-one. (In general, in the base N number system, N distinct symbols, representing 0, 1, 2, . Let X be a nonempty set of nonnegative integers. Byte needs to store text made up of the characters A, B, C, D, E, which occur with the following frequencies: E N P L 14. We could define the codomain also to be the set of all the reciprocals that can be computed and displayed. Each of these recipients, in turn, sent e-mail to 50 addresses. Find 3. A smoothing sequence b0 , . By Exercise 18, Tournament Sort requires k comparisons to find the second-largest element. Find lcm(2 · 52 · 72 · 134 , 74 · 132 · 17). Therefore, to determine if a positive integer n is composite, it suffices to test whether any of the integers 2, 3, . Rewrite each propositional function in Exercises 21–27 using only negation, disjunction, and conjunction. The least upper bound of our previous set X = {1, 2, 3, 4} is 4. (2) In (1), from the assumption that T is a tree, we must deduce that T is connected but the removal of any edge (but no vertices) from T disconnects T. As another example, if an algorithm consists of a single loop whose body executes in at most C steps, for some constant C, we might count the number of iterations of the loop. (2) si < key is false, but in the algorithm, the relation si < key evaluates to true. Alice can be assigned an office in three ways, Dolph can be assigned an office in two ways, and the last office can be filled in four ways. Section 6 In Exercises 21–24, write a sequence of operations to answer the query. The height of a rooted tree is the maximum level number that occurs. However, if Farley can choose a value 56 Sets and Logic for x so that every value you choose for y makes P(x, y) false, then you will always lose the game and the propositional function is false. Therefore, f(n) ≤ Cg(n) for all n. In this sense, t(n) grows like 60n2 . For the code on the package of taco shells, we would have s = 3(0 + 4 + 0 + 0 + 8 + 0) + 5 + 4 + 0 + 0 + 0 = 45. The set (1.2) is a function from X = {1, 2, 3} to Y = {a, b, c}. Many discrete problems can be solved by modeling them using graphs. {(8840, Hammer), (9921, Pliers), (452, Paint), (2207, Carpet)} 4. Within a proof, a proof of an auxiliary result is called a subproof. (2n + 1)2 is an odd integer. 631245 17. Theorem 6.5 Let E be an event. If we regard the single-elimination tournament of Figure 1.1 as a graph (see Figure 1.2), we obtain a tree. If a function from X to Y is not onto, some element in Y in its arrow diagram will fail to have an arrow pointing to it (see Figures 1.9 and 1.10). d a b c d e f 2. z = pq = 59 · 101 = 5959 15. Then, for all n, p(n) = ak nk + ak−1 nk−1 + · · · + a1 n + a0 ≤ ak nk + ak−1 nk + · · · + a1 nk + a0 nk = (ak + ak−1 + · · · + a1 + a0)nk = C1 nk . Prove your answer. In the preceding figure, e is a least upper bound for the set X. What is the probability that the wheel lands on 0 or 00? For example, if S consists of numbers, we can use ordinary ordering defined on numbers, and if S consists 536 Trees of strings of alphabetic characters, we can use lexicographic order. u3 = u10 = u5 = u16 = u8 = u4 = u2 = 1. 113. Proof To show that (a)–(d) are equivalent, we will prove four results: If (a), then (b); if (b), then (c); if (c), then (d); and if (d), then (a). large = s1 4. Binary search 13. Let S = {1}, T = {1, 3}, U = {a}, and V = {a, c}. Example 1.50 If X is a set of propositions, ¬ is a unary operator on X. a-1; b-1; c-1; d-1; e-2; f -3; g-3; h-4; i-2; j-3; k-0 11. Therefore, edges (g, d), (d, e), (e, f), and (f, k) are in H. List the members of P ({a, b, c, d}). Show, by an example, that Algorithm 3.6 can produce identical spanning trees for a connected graph G from two distinct vertex orderings of G. Example 1.37 For the function f = {(1, a), (2, c), (3, b)}, we have f −1 = {(a, 1), (c, 2), (b, 3)}. We present several examples that illustrate the use of the Strong Form of Mathematical Induction. At line 6, we compute r = a mod b. We have succeeded in moving n disks from peg 1 to peg 3. 299 Introduction to Number Theory Section 2 5. Discrete Mathematics Johnsonbaugh Seventh Edition ISBN 978-1-29202-261-9 9 781292 022611 Discrete Mathematics Richard Johnsonbaugh Seventh Edition Discrete Mathematics Richard Johnsonbaugh Seventh Edition Pearson Education Limited Edinburgh Gate Harlow Essex CM20 2JE England and Associated Companies throughout the world Visit us on the World Wide Web at: www.pearsoned.co.uk © Pearson Education Limited 2014 All rights reserved. The resulting formula can now be proved using mathematical induction. Suppose that the conclusion is false. Since the product of two n × n matrices contains n2 terms, any algorithm that multiplies two n × n matrices requires time at least (n2). Choosing the edges labeled 1 and 4 leads to the graph of Figure 8.11. 3 Sets and Logic According to the criteria in the paragraph immediately preceding Example 1.2, we must show that for every x, if x ∈ A, then x ∈ B, (1.3) if x ∈ B, then x ∈ A. To prove that a relation R on a set X is not symmetric, find x, y ∈ X such that (x, y) ∈ R and (y, x) ∈ R. If, on the other hand, a universal set is specified as U = {1, 3, 5, 7, 9}, then A = {7, 9}. The computation (1.6) can be carried out for an arbitrary value of n to obtain An = (1.12)An−1 .. Since 3 divides 15 and 3 divides 6sq, 3 divides 1. Suppose that there exist i and j such that si = there exists k such that sk < A. Prove that an = fn+1 for all n ≥ 1. Therefore, a complete matching exists. max = 0 sum = 0 for i = 1 to n { if (sum + si > 0) sum = sum + si else sum = 0 if (sum > max) max = sum } return max } Since this algorithm has a single for loop that runs from 1 to n, max sum3's time is (n). Now n = 2k − 2 = 2(k − 1) is even. The statement is true because it is possible to find at least one real number x for which the proposition x 2 = x2 + 1 5 41 Sets and Logic is true. This function is one-to-one because each element in Y has at most one arrow pointing to it. Computing the right-hand side in decimal, we find that 1011012 = 1 · 32 + 0 · 16 + 1 · 8 + 1 · 4 + 0 · 2 + 1 · 1 = 32 + 8 + 4 + 1 = 4510 . Therefore, x ∈ [d] and [c] ⊆ [d]. 2 Let an denote the number of comparisons required by the algorithm in the worst case. If n = 2, we obtain the proposition P(2) : 2 is an odd integer (which is false). Recurrence relations are useful in certain counting problems. Similarly, we write 12 in the region representing CALC ∩ PSYCH ∩ COMPSCI and 16 in the region representing CALC ∩ PSYCH ∩ COMPSCI. In symbols, a relation R is symmetric if ∀x∀y[(x, y) ∈ R] → [(y, x) ∈ R]. Since there are two choices for the winner of each game, the tournament can unfold in 2n−1 ways. If an n-bit string begins with 11 and does not contain the pattern 111, then the third bit must be 0. Define the sequence t1 , t2 , . It was not until 1973 that 1,000,000 decimal digits of π were computed; so, p is false. In addition, duplicates are eliminated (see Example 6.4). For example, in Table 6.1, we can take the attribute ID Number as a key. The graphs are not isomorphic since they do not have the same number of vertices. If there is one disk, move it and stop. In the multi-state Big Game, to win the grand prize the contestant must match five distinct numbers, in any order, among the numbers 1 through 50, and one Big Money Ball number between 1 and 36, all randomly drawn by a lottery representative. x ∈ X 6. In particular, we can assume the truth of the statement for k = n − 2. If G is a connected graph and T is a spanning tree for G, there is an ordering of the vertices of G such that Algorithm 3.6 produces T as a spanning tree. z = pq = 17 · 23 = 391 Therefore, (ntk) mod m = (k − msk) mod m = k mod m = k. An isomorphism is f(v1) = w1 , f(v2) = w3 , f(v3) = w4 , f(v5) = w7 , f(v6) = w6 , f(v7) = w5 . We have obtained the desired contradiction. 80. 456 n−2 1 d(G) = max{dist(v, w) | v and w are vertices in G}. State the existential instantiation rule of inference. A counterexample is x = 2, y = 0. This equation may be solved by iteration. , 76, 78, 79. Finiteness—the algorithm terminates; that is, it stops after finitely many instructions have been executed. 1 34. The algorithm first tries i = 1, then i = 2, and so on. Show how to determine the number of bad coins in at most n − 1 weighings. We call the proofs of auxiliary results subproofs. Show that the height h of an n-vertex balanced binary tree satisfies h = O(lg n). Then f(α, λ) = αλ = α. As another example, if q = 6, the remainder would be n − dq = 74 − 13 · 6 = − 4, which is negative. If a relatively small graph has a Hamiltonian cycle, trial and error will discover one. Basis Step (n = 1) dx = 1 1x1−1 dx Inductive Step fki . Therefore, D = τ(D) ⊆ τ(R1 ∪ D). By Theorem 1.22, gcd(82320, 950796) = 2max(4,2) · 3max(1,2) · 5max(1,0) · 7max(3,4) · 11max(0,1) = 24 · 32 · 51 · 74 · 111 = 19015920. Arguing as in Example 1.7, we find that |X| = 26 , |Y | = 25 , and |X∩Y | = 23 . Similarly, if 3n2 ≤ 40, then n2 ≤ 40/3 and n can be at most 3. (c) Since a sequence is a function, a sequence is a set of ordered pairs. Let's see how the inductive step should work and then see what basis steps we need. 7 < 10 if and only if (4 < 2 and 6 is not less than 6). f is a commutative, binary operator. Plug in the value for ak and you have solved the recurrence relation. 1 + 2 + ··· + n = n(n + 1) . 6j. Find the inverse (as a set of ordered pairs) of each relation in Exercises 1–16. Example 1.10 In how many ways can we select two books from different subjects among five distinct computer science books, three distinct mathematics books, and two distinct art books? , n + p − 1 for some n that we get to choose. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is D3 × D3 . Tree; four internal vertices; six terminal vertices 28. How can the student be sure that the program works properly for large input—almost surely the kind of input on which the instructor will test the program? (In Example 1.11, we gave a direct proof of this statement using the definition of set equality.) In each of Exercises 31–43, if the statement is true, prove it; otherwise, give a counterexample. ¬p ∨ q 2. Given an arrow diagram, the relation is not transitive if there are three distinct vertices x, y, z and directed edges from x to y and from y to z, but no directed edge from x to z. Similarly, the first string in Gn−1 and the last string in Gn−1 differ in exactly one bit. Since lines 9 and 12 are nested within a 469 Graph Theory while loop, which takes time O(n), the total time for lines 9 and 12 is O(n2). 2 Notice that this formula gives fn in terms of one predecessor rather than two predecessors as in the original definition. Since Ri is reflexive, x1 R1 x1 and x2 R2 x2 . In Theorem 8.3 we showed that there are three nonisomorphic free trees having five vertices. d 5. R = {(1, 1)}. By definition, the invocation at line 11 will require a total of am invocations, where m is the size of the sequence that is input at line 11. B A 12. The number of function calls in the program TABLE 1.2 ■ C++ programs that implement the same algorithm. List the members of the event "4 appears on at least one die." 8. Dividing by t n−2 , we obtain the equivalent equation t 2 − 5t + 6 = 0. 349CC + 922D 39. Basis Step (n = 1) c1 = 0 ≤ 0 = 1 lg 1 Inductive Step cn = 2cn/2 + n ≤ 2n/2 lgn/2 + n ≤ 2(n/2) lg(n/2) + n = n(lg n − 1) + n = n lg n 133 Proofs 20. Theorem 3.7 shows that in the worst case, n − 1 comparisons are needed to merge two sequences the sum of whose lengths is n. Suppose that we want to prove a statement for all n ≥ 4 involving cn . How can one determine whether graphs are isomorphic from their adjacency matrices? Substitute for an−1 , which will yield an expression involving an−2 , to obtain an = . A = A 38. Since d is a positive integer, 1 − d ≤ 0. Write an algorithm that finds the lengths of the shortest paths from a given vertex to every other vertex in a connected, weighted graph G. Inductive Step Now we assume that if X contains an element less than or equal to n, then X has a least element. The idea is that distinct secret keys require distinct amounts of time to decrypt messages and, by using this timing information, an unauthorized person might be able to unveil the secret key and thus decrypt the message. Derive this result by considering the cases |A| = 0, |A| = 1, . 1 1 1 + + · · · + < log n. After k iterations, we obtain the sequence defined by k−1 i=0 n n + 2 2i 2n+1 Section 8 Review k=1 aj = n−1 n + 2 n + 2 2i + n+1 2n i+1 n n+2 + C(n, i)−1 + C(n + 1, n + 1)−1 n+1 i=1 7. A B 574 Section 7 25. Trace 4. 6e. Our final examples show that induction is not limited to proving formulas for sums and verifying inequalities. We add b and d to U so that U = {b, d}. We divide the proof into cases because the definition of absolute value is itself divided into cases x ≥ 0 and x < 0 (see Example 1.2). But x ∈ / ∅, so x ∈ A. Given two sets X and Y , there are various set operations involving X and

Y that can produce a new set. Give an example of an equivalence relation. The greatest power of 2 less than or equal to 100,000 is 216 . How can the logic game be used to determine the truth value of a quantified expression? Each prospective recipient chooses two primes p and q and computes z = pq. is nondecreasing. Since |S| = |S |, S is optimal. We can sometimes obtain a contradiction by showing that edges that must be in the Hamiltonian cycle form a cycle of length less than n (see Example 3.3). The argument that Z ⊆ Y is the same as that for Y ⊆ Z with the roles of Y and Z reversed. Therefore, gcd(a, b) = gcd(b, r). It is straightforward to verify that a connected graph with one or two vertices, each of which has even degree, has an Euler cycle (see Exercise 69); thus we assume that the graph has at least three vertices. 50. Show that the graph has no Hamiltonian cycle. one way the tournament can unfold is Scientists 14. A large value of s(v, w) indicates dissimilarity, while a small value indicates similarity. The restriction is to ensure that G is actually a graph. There are 13 clockwise spirals (marked with white thread) and 8 counterclockwise spirals (marked with dark thread). Because pow = ai−1 is an invariant, at this point pow = an . 614 (b) Use Algorithm 2.4 to find a maximal matching. Their names, together with their order codes, are entered into a spreadsheet: The records are to be sorted in alphabetical order, but an error occurs when the names are sorted but the order codes are unchanged: As a result, no one receives the correct compact discs. Sometimes, however, we do want to take order into account. Write a backtracking algorithm that determines whether k queens can attack all squares of an n × n board. If there is a unique edge e associated with the ordered pair (v, w) of vertices, we write e = (v, w), which denotes an edge from v to w. Since |β| < n and |γ| < n, i=2 by the inductive hypothesis β and γ each have equal numbers n−1 of a's and b's. If x1 < x2 < x3 < x4 , a median is any value between x2 and x3 , including x2 and x3 . As shown, we begin at the root and place coin C1 in the left pan and coin C2 in the right pan. Draw a 2-cube. Is the statement of Exercise 17 a propositional function? n = 209 4. The straightforward way to compute this power is to repeatedly multiply by a a · a· · · a, n a's which uses n − 1 multiplications. An inventory consists of a list of 100 items, each marked "available" or "unavailable." There are 55 available items. If Rn is the number of ways of spending all the money, derive a recurrence relation for the sequence R1 , R2 , . Therefore, fishing is a popular sport or lacrosse is wildly popular in California. R = {(a, w), (a, y), (c, y), (d, w), (d, x), (d, y), (d, z)} 11. Everyone who can hit the ball a long way can make a lot of money. What is a cut in a network? To obtain a lower bound, we might imitate the preceding argument and replace each integer 1, 2, . 22–29]). Exercises 30–33 concern an unprepared student who takes a 10-question true–false quiz and guesses at the answer to every question. Input: Output: s1 , . Having made this selection, we can choose three terms for the y's in C(7, 3) ways. List the members of the event "the die shows an odd number." 4. We must show that f is one-to-one and onto X∗ . We can also quickly determine whether a relation R on a set X is symmetric by examining the matrix A of R (relative to some ordering). f(x) = 46. We have counted the elements in X − Y (eight-bit strings that start 10 but do not end 011) once and the elements in Y − X (eight-bit strings that end 011 but do not start 10) once, but we have counted the elements in X ∩ Y (eight-bit strings that start 10 and end 011) twice (see Figure 1.3). In order for (b) to be false, we must have for every x, there exists at least one value of y such that P(x, y) is false. Our next example shows that we can add hexadecimal numbers in the same way that we add decimal or binary numbers. p 12. 6 ➜ Isomorphisms of Graphs The following instructions are given to two persons who cannot see each other's paper: "Draw and label five vertices a, b, c, d, and e. In words, a relation R is not antisymmetric if there exist x and y, x = y, such that (x, y) and (y, x) are both in R. (A ∪ B) = A ∩ B, (A ∩ B) = A ∪ B 39. Let Pn denote the number of ways to divide a convex (n + 2)sided polygon, n ≥ 1, into triangles by drawing n − 1 lines through the corners that do not intersect in the interior of the polygon. Now examine the cases n = 0, 1, 2, 3, 4. ∀x(x2 > x) 51. In inequality (5.1), n0 ≤ k ensures that k is in the domain of discourse {n0 , n0 + 1, n0 + 2, . Exercise 8, Section 1 11. What is an initial condition? It is not transitive because, for example, (b, c) and (c, b) are in R, but (b, b) is not in R.] Given an equivalence relation on a set X, we can partition X by grouping related members of X. Then there exists x ∈ (X − Y) ∩ (Y − X). 56. It then recursively sorts each half and merges the halves to produce sorted output. (We will explain shortly how s1 , . First Form: If 20 persons (pigeons) go into six rooms (pigeonholes), then some room contains at least two persons. Some elements in X∗ are λ, a, b, b20 a5 ba. Prove that for all sets A, B, and C, C ⊆ A and C ⊆ B if and only if C ⊆ A ∩ B. (A polygon is convex if any line joining two points in the polygon lies wholly in the polygon.) For example, there are five ways to divide a convex pentagon into triangles by drawing two nonintersecting lines through the corners: 42. What is a full binary tree? Exercises Exercises 1–4 refer to the sequence s1 = 'C', s2 = 'G', s3 = 'J', s4 = 'M', s5 = 'X'. If A is the matrix of a relation R, R is transitive if and only if whenever entry i, j in A2 is nonzero, entry i, j in A is also nonzero. (c) Finds the children of v. i=1 62. Eventually, there is a single winner at the root. Recall that to prove from the definition of set equality that these sets are equal, we must show that for all x, if x ∈ X ∩ (Y − Z), then x ∈ (X ∩ Y) − (X ∩ Z) and if x ∈ (X ∩ Y) − (X ∩ Z), then x ∈ X ∩ (Y − Z). We let L(v) denote the label of vertex v. Theorem 1.3 Let m, n, and d be integers. What is an improperly oriented edge with respect to a path? (c) What is the interpretation of a complete matching? Therefore, if M is a mesh p vertices by q vertices, where p ≤ 2i and q ≤ 2j , the (i + j)-cube contains a subgraph isomorphic to M. Determine the truth value of each statement. ∃x ¬P(x) 26. We have proved equation (1.4). Then there is a knight's tour on the n × n board if and only if GKn has a Hamiltonian cycle. The first player always wins. (n2 15. C(10 + 3 − 1, 10) 18. (b) At line 2, since 3 = 0, we proceed to line 4. Since n is odd, there exists an integer k such that n = 2k + 1. Two dice are rolled. A sequence s is nondecreasing if sn ≤ sn+1 for all n. We are given that |F ∩ B ∩ M| = 10, |F ∩ B| = 36, |F ∩ M| = 20, |B ∩ M| = 18, |F | = 65, |B| = 76, and |M| = 63. The case n even is treated similarly. We must have c = d; otherwise the set on 127 Proofs the right-hand side of equation (*) would contain two distinct sets. State which graphs in Exercises 17–23 are bipartite graphs. 8.3 = 8, 9.1 = 10, −8.7 = −9, −11.3 = −11, 6 = 6, −8 = −8 The floor of x "rounds x down" while the ceiling of x "rounds x up." We will use the floor and ceiling functions throughout the book. Since the second quantifier is ∀, Farley goes second. Definition 2.6 16 The truth value of the proposition p ∨ q, called the inclusive-or of p and q, is defined by the truth table p q p∨q T T F F T F T F T T T F Sets and Logic Example 2.7 If p : A millennium is 100 years, q : A millennium is 1000 years, then p is false, q is true, and the disjunction, p ∨ q: A millennium is 100 years or a millennium is 1000 years, is true. In words, a is not an upper bound for a set X if there exists x in X such that x > a. In the conditional proposition p → q, p is a sufficient condition. Example 1.19 In 2007, the U.S. first-class postage rate for retail flat mail up to 13 ounces was 80 cents for the first ounce or fraction thereof and 17 cents for each additional ounce or fraction thereof.† The postage P(w) as a function of weight w is given by the equation P(w) P(w) = 80 + 17 w − 1 , ( ] The expression w−1 counts the number of additional ounces beyond 1, with a fraction counting as one additional ounce. Example 1.7 The set of integers Z is a subset of the set of rational numbers Q. We shall count the number of such functions having exactly j 1's. 365 Counting Methods and the Pigeonhole Principle Figure 8.1 n = 6 pigeons in k = 4 pigeonholes. S1 = × × (n − 1)n × 2 n(n + 1) Sn = × 2 (n + 1)(n + 2) Sn+1 = ? In the RSA system, messages are represented as numbers. Since MORE is less than NECESSARILY, we attempt to go to the left child. Definition 3.23 Let R be a relation from X to Y . Hint: n/2 ≥ (n − 1)/2 for all n. Example 2.23 If α = aabab and β = a3 b4 a32 , then |α| = 5 and |β| = 39. , O2n−1 . Prove that P(X) ∪ P(Y) ⊆ P(X ∪ Y) for all sets X and Y . 10732 46. During one time unit, all processors in the n-cube may execute an instruction simultaneously and then communicate with an adjacent processor. We have noted that 4 is the least 96 upper bound for X. By solving the recurrence relation, we can determine the time needed by the algorithm. Thus there are 7!P(8, 5) such arrangements. Has a vertex inside some simple cycle 4 z 25. Thus, y = 2x . Prove that Algorithm 4.4 is correct. 371 Counting Methods and the Pigeonhole Principle 31. (2.3) Substituting this expression for 45 into (2.2), we obtain 91 = 2 · 45 + 1 = 2 · (2 · 22 + 1) + 1 = 22 · 22 + 2 + 1. (b) There exists x such that x ∈ X and x ∈ Y . Thus p = 11 makes the given statement true. To prove that a relation R on a set X is symmetric, show that for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R. For example, if P is a propositional function with domain of discourse Z+ , we obtain the class of propositions P(1), P(2), . Whatever is claimed about x must be true no matter what value x might have in D. For example, f = {(1, a), (2, b), (3, c), (4, d)} is a bijection from X = {1, 2, 3, 4} to Y = {a, b, c, d}. We assume any string 1 1 1 in L of length k < n has an equal number of a's and b's. Since r is true, ¬r is false. Give an example of a graph that has an Euler cycle and a Hamiltonian cycle that are not identical. an = an−1 + 4; a1 = 3 2. † The floor of x, x, is the greatest integer less than or equal to x. Which algorithm is preferred in a particular situation depends on factors such as the size of the data and how the data are represented. You choose second. The Euclidean algorithm is based on the fact that if r = a mod b, then gcd(a, b) = gcd(b, r). We can also determine whether the argument in Example 4.4 is valid or not by examining the truth table of Example 4.2. In the third row of the table, the hypotheses are true and the conclusion is false; thus the argument is invalid. Any shortest-path algorithm that receives as input Kn , the complete graph on n vertices, must examine all of the edges of Kn at least once. Next, we show that p(n) = (nk). A(2, 2) = 7, A(2, 3) = 9 43. Implement the minimax procedure as a program. 25. Thus the assertion cn ≤ 2n fails for n = 4. As shown, Google found about 91,000 Web pages containing "Alfred Hitchcock" and (Herrmann or Waxman) and (not tv). 625431 6. To convert the decimal number n to base b, divide by b, divide the resulting quotient by b, divide the resulting quotient by b, and so on, until obtaining a zero quotient. gcd(m, n) · lcm(m, n) = mn 1. In Section 2 we discuss the behavior of the price through time by analyzing an explicit formula for the price pn . = 2 bn+1 = b(n+2)/2 + bn/2 + c(n+2)/2,n/2 ≥ bn/2 + bn/2 + cn/2,n/2 = bn . , (3.3) and the initial condition b0 = 2. Therefore, n has a divisor d (namely d = q) satisfying 2 ≤ d ≤ n. Write tournament sort as a formal algorithm. The proofs of parts (b) and (c) are left as exercises (see Exercises 27 and 28). temp1 := buyer [Name = Danny's] temp2 := temp1 [Part No = Part No] supplier temp2 [Dept] 04, 96 17. Find the least n such that among n persons, the probability that at least two persons have birthdays onApril 1 (but not necessarily in the same year) is greater than 1/2. Prove that for all integers m and n, if m and n are odd, then mn is odd. Since the statement is of the form p if and only if q, we have two tasks: (1) Prove if p then q. B = {1, 2, 3, 4}, C = {2, 4, 6, 8}, A = B ∩ C In Exercises 29–32, determine whether each pair of sets is equal. The entry in row i, column j, i = j, is the number of edges incident on i and j. .). Among the axioms are ■ ■ Given two distinct points, there is exactly one line that contains them. Now ρ(R1 ∪ R2) = (R1 ∪ R2) ∪ D = (R1 ∪ D) ∪ (R2 ∪ D) = ρ(R1) ∪ ρ(R2). X × X 60. Let e be the edge of minimum weight incident on v. State Kuratowski's theorem. Therefore, 2 is not an integer. A directed graph consists of a set V of vertices and a set E of edges such that each edge e ∈ E is associated with an ordered pair of vertices. In words, for every person x, there exists a person y such that x loves y. , vn−1 , en , vn), in which edge ei is incident on vertices vi−1 and vi for i = 1, . We now state a necessary and sufficient condition for a graph to be planar. If 2m−1 < (n − 1)/2, we may use the inductive assumption to conclude that the loop iterates at most m − 1 more times. (Alternative: Ten tails were obtained.) 16. If some integer in this list divides n, then n is composite. To prove a statement about the Fibonacci numbers, use the equation fn = fn−1 + fn−2 for all n ≥ 3. An initial condition for a sequence is an explicitly given value for a particular term in the sequence. Could the size of an optimal vertex cover of a graph with n vertices equal n? 11 7 7 6 4 31. Find the probability that at least one passes. P = p, Q = p ∨ q p p q p imp2 q T T F F T F T F T T F T F T F F (a) Show that (p imp2 q) ∧ (q imp2 p) ≡ p ↔ q. Let f be the function from X = {0, 1, . For this tree, the inequality (5.1) becomes an equality. How many symmetric and antisymmetric relations are there on an n-element set? Proof Let T be the decision tree that represents the algorithm for input of size n and let h denote the height of T. Now 1 ,b1) max(a2 ,b2) k ,bk) pmax(a p2 · · · pmax(a 1 k is divisible by both m and n and if any exponent, max(ai , bi), is decreased, the resulting integer will fail to be divisible by m or n (or both). Notice that each component of the graph G has less than n edges and that in each component of the graph G , every vertex has even degree. Draw a decision tree that can be used to determine who must file a federal tax return. (1, 2, 3, 1) A cycle is path of nonzero length from v to v with no repeated edges. Proof In Algorithm 3.5, the comparison of elements in the sequences occurs in the while loop at line 10. By Theorem 5.6, lg t ≤ h, where t is the number of terminal vertices in T ∗ . If there is a unique edge e associated with the vertices v and w, we write e = (v, w) or e = (w, v). .) eliminate a, . Find the probability of obtaining a 4–4–4–1 distribution, that is, four cards in each of three different suits and one card of a fourth suit. For example, suppose that for any input of size n, algorithm A requires 300n units of time and algorithm B requires 5n2 units of time. Previously, we observed these properties graphically (see Example 1.9). The cycle C = (a, b, c, d, a) is a Hamiltonian cycle for the graph G of Figure 3.8. Replacing any of the edges in C by either of the edges labeled 11 would increase the length of C; thus C is a minimum-length Hamiltonian cycle for G. The data are arranged so that, for each vertex v in T , each data item in the left subtree of v is less than the data item in v, and each data item in the right subtree of v is greater than the data item in v. Thus the basis step is proved. In a weighted graph, the length of a path is the sum of the weights of the edges in the path. Notice that G1 and G2 have the same numbers of edges and vertices. We must show that if f(α) = f(β), then α = β. an = b2n + d4n + 1 15. Such an approach to problem-solving is not new; indeed, the word "algorithm" derives from the name of the ninth-century Persian mathematician al-Khow¯arizm¯ı. Algorithm B is superior if 2 ≤ n ≤ 15. Notice that ∃x∀y L(x, y) is not a correct interpretation of the original statement. Met the Mayor. We assume that there is at least one bad coin and at least one good coin among the n coins. Therefore, x ∈ X ∪ (Y − X). P ≡ Q 63. Since 2 is eliminated, 1 survives. For every integer x, there exists an integer y (e.g., y = x − 1) such that x ≤ y is false. This implies that we must include edges (h, g) and (b, c). Since the simple path from the root to any given vertex is unique, each vertex is on a uniquely determined level. Exercise 23 31. What does a matching represent? Furthermore, the polynomial in Theorem 3.4 is (nk), but the expression in Example 3.8 is (nk+1). 527 Trees 36. The smallest case is n = 1. Write the negation of the assertion of Exercise 21 symbolically and in words. Then s > 0. If John has sophomore, junior, or senior standing, he may or may not take calculus. Tell whether each proposition is true, is false, or has unknown status at this time. A Huffman code is most easily defined by a rooted tree (see Figure 1.10). Certain edges cannot appear in the Hamiltonian cycle: If a graph has a vertex v of degree greater than 2, only two edges incident on v can appear in the Hamiltonian cycle. 2! 7! 3! 4! 2! 3! 4! We can write the binomial coefficients in a triangular form known as Pascal's triangle (see Figure 7.1). (a) ∀x∀y P(x, y) (b) ∃x∀y P(x, y) (c) ∃x∃y P(x, y) Attacking the Problem Let's begin with part (a). In Example 5.4, we found that the probability of obtaining no defective microprocessors is 0.903735781. The three equivalence classes of the relation R of Example 4.10 appear in the digraph of R (shown in Figure 4.2) as the three subgraphs whose vertices are {1, 3, 5}, {2, 6}, and {4}. By the Principle of Mathematical Induction, the theorem is established. Thus, by the Multiplication Principle, there are 1 · 4 · 3 · 2 = 24 strings that start with the letter B. 1 8 28 56 70 56 28 8 1 16. If an integer n > 1 is composite, then it has a positive divisor d other than 1 and itself. If f ◦ g is one-to-one, then f is one-to-one. This makes the last digit 7. We then ,set the entry in row x and column y to 1 if x R y and to 0 otherwise. 2 [ ) 1 [ ) 16. An initial condition is c1 = 2, since there are two strings (0 and 2) that contain an even number (namely, zero) of 1's. For part (b), let's look at the 3 × 5 grid. Exercise 7, Section 1 10. Therefore, b1 = 0. 11 divides 77. Write a program that prints a solution to the n-node communication program (see Exercise 51, Section 1). This statement is false because, for every positive integer x, there is at least one positive integer y, namely y = x + 1, such that x ≥ y is false. You run 10 laps daily, but you are not healthy. The relational database model, invented by E. What is a recurrence relation? Find an explicit formula for An . Actors who did not appear in a film with Bacon but who appeared in a film with an actor whose Bacon number is one are said to have Bacon number two. Since x10 = 3, which is the value of the seed, the sequence now repeats: 3, 4, 0, 5, 7, . , xn }. Describe how to solve the Instant Insanity puzzle. A coin is loaded so that a head is five times as likely to occur as a tail. See if you can find a solution before looking at a solution given in Figure 3.3. a b h c d a e k l f g i j n m f b g o e n p t o p q r a b g q s h i o j r k m l t d s Figure 3.1 Hamilton's puzzle. Suppose that a professional wrestler is selected at random among 90 wrestlers, where 35 are over 350 pounds, 20 are bad guys, and 15 are over 350 pounds and bad guys. The composition of R1 and R2 is the relation from X to Z 16. The idea is that as the price increases, the manufacturer is willing to supply greater quantities. When multiplying modulo z, compute the remainders as soon as possible to minimize the sizes of the numbers involved. If n = 0, put one unlabeled vertex at (−1, 0) and stop. i=1 Fix kn disks at the bottom of peg 1. What is compound interest and how can it be described by a recurrence relation? y m2 /n2 n1 m2 Since x/y is the quotient of integers, it is rational. number." The domain of discourse is R. Isomorphic as rooted trees and as free trees. k=1 86. Property (8.2) assures us that each cube is used exactly once. (Although eligible to take calculus, he may have decided not to.) The "if p then q" formulation emphasizes the hypothesis, whereas the "p only if q" formulation emphasizes the conclusion; the difference is only stylistic. gcd(47, 11) = gcd(11, 3) = gcd(3, 2) = gcd(2, 1) = gcd(1, 0) = 1, s = 30 39. Prefix 0 to the label of each vertex above the x-axis, and prefix 1 to the label of each vertex below the x-axis. Solve the recurrence relation √ √ √ an = an−1 + 2 an−2 a1 = 10 19. ("Characteristic function" is defined before Exercise 82, Section 1.) Define a relation R on X by x R y if f(x) = f(y). For an input size of n = 5, algorithm A requires 1500 units of time and algorithm B requires 125 units of time, and thus algorithm B is faster. j=1 Let a be the sequence defined by the rule ai = 2(−1)i , i ≥ 0. Suppose that the input to Algorithm 3.2 is not in nondecreasing order. 26. 6⌋ 8 The entry from row d, column e is 6, which means that there are six paths of length 4 from d to e. Proof of Theorem 5.6 Let X = {n − dk | n − dk ≥ 0, k ∈ Z}. k=1 Since the sequence P1 , P2 , . n which follows from the fact that each pair of classes is mutually exclusive and each item to be classified belongs to one of the classes. Continuing in this way, we ultimately obtain gcd(r0 , r1) = rn−1 = t0 r1 + s0 r0 = t0 b + s0 a. Thus this sum cannot equal 20. Thus the first listed string will be 12 · · · r and the last listed string will be (n − r + 1) · · · n. For each statement that is a propositional function, give a domain of discourse. Chapter Self-Test Section 1 1. You run 10 laps daily or you take multi-vitamins, and you are healthy. Suppose that the entry in row i, column j, i = j, of A1 is equal to k. (Assume that all the values are real numbers.) The algorithm returns true if si + sj = x, for some i = j, and false otherwise. Example 1.19 Particular regions in Venn diagrams are depicted by shading. What is the probability of doubles? Draw all nonisomorphic rooted trees having five vertices. 7711 45. When d is set to 2, . Suppose that X1 , . {{1, 2, 3}, {4}} 19. The statement ∀x∃y P(x, y) is false if there is at least one x ∈ X such that P(x, y) is false for every y ∈ Y . Writing these equations in the form r = n − dq, where r is the remainder and q is the quotient, yields 33 = 425 − 196 · 2 31 = 196 − 33 · 5 10. How many (base-10) digits does this number have? We then repeat this process. The capacity of each edge is a nonnegative integer. Thus the vertices are processed in the order ABC. C A B postorder DBECA postorder EDCBA C * B Section 6 inorder BDAEC inorder EDCBA / * 12. Headline over a Martha Stewart column: All lampshades can't be cleaned. j=n while (j ≥ 1) { for i = 1 to j x=x+1 j = j/3 } i=n while (i ≥ 1) { for j = 1 to n x=x+1 i = i/2 } 26. Pseudocode is so named because it resembles the actual code of computer languages such as C++ and Java. Functions, Sequences, and Relations Problem-Solving Corner Problem Let U be a universal set and let X ⊆ U. p → r p→q 7. b a C2 C3 C2 e C3 C3 d c C1 C1 C1 C2 8. Express each decimal number in Exercises 14–19 in octal. .} and {2, 4, . If R and S are reflexive, then R ◦ S is reflexive. To see that R is transitive, suppose that s1 and s2 have the same number of zeros and that s2 and s3 have the same number of zeros. What is the probability of choosing the winning numbers, assuming that the contestant chooses three distinct numbers? What is the worst-case time of an algorithm? Section 6 Exercises 21–23 refer to the following binary tree. An Euler cycle visits each edge once, whereas a Hamiltonian cycle visits each vertex once; however, the problems are actually quite distinct. The proof is illustrated in Figure 4.3. X X1 (r elements) X2 (r elements) Xk (r elements) X rk Figure 4.3 The proof of Theorem 4.16. Now m1 /n1 m1 n2 x = = . If the disk got full, the system could deadlock or even crash. [Part(b)] {San Francisco, San Diego, Los Angeles}, {Pittsburgh, Philadelphia}, {Chicago} 204 37. In the current addressing scheme, known as IPv4 (Internet Protocol, Version 4), the addresses are divided into five classes—Class A through Class E. 220, 1400 4. There are Cn−k such routes. Therefore f is not one-to-one. Therefore, d divides mn (with quotient qn). [Hinz] is a comprehensive discussion of the Tower of Hanoi with 50 references. Example 6.12 Two fair dice are rolled. Thus there are 9 · 4 hands that are consecutive and of the same suit. Since v5 and v6 are adjacent and v5 is in V2 , v6 is in V1 . Six vertices; four edges v1 14. The idea is to use short bit strings to represent the most frequently used characters and to use longer bit strings to represent less frequently used characters. Alpha-beta pruning deletes (prunes) parts of the game tree and thus omits evaluating parts of it when the minimax procedure is applied. In words, this last proposition can be stated as: Every bird can fly. p → (r ∨ q) r → ¬q 8. Therefore, there exists an integer k such that p = 2k. In Figure 5.6, we show the full binary tree that results from modifying the binary search tree of Figure 5.4. The added vertices are drawn as boxes. Given the matrix of a relation, how can we determine whether the relation is transitive? The literal meaning is: No environmental problem is a tragedy. The second half is devoted to data structures. The n-cube is an important model of computation because several such machines have been built and are running. Explain how Algorithm 3.11 computes an . C for all n ≥ 1. We must show that the algorithm is correct for n > 2. Then m + n = 2k1 + 2k2 . We first swap ai and aj , where i = 1 and j = rand(1, 5) = 5. If we divide both sides of this last inequality by x2 + 1, we obtain 1 ≤ 1. a a c 7. d d Thus, we may compute the quotient q as n/d. There are C(7, 4) committees with no men. Thus j becomes m+1 and the algorithm returns i. In 2000 the population was 10,000. Therefore, the graph G of Figure 7.6 is not planar, since it contains a subgraph homeomorphic to K3,3 . (n + 1)n1 · · · nk n+1 n+1 n1 · · · nk C1 [The term 1/(n + 1) results from the subset {n + 1}.] By the inductive assumption, 1 1 1 1 1 + + · n = 1. (d) Let us consider the activity of assigning Dolph, Francisco, and one other person to offices to be made up of three successive steps: Assign Dolph; assign Francisco; fill the remaining office. a b c 1 2 3 f 1 Y X Figure 1.12 The inverse of the function in Figure 1.11. [The vertices f(vi), i = 1, . Cryptology is the study of systems for secure communications. In words: Someone is not taller than someone. Suppose that n = 2m+1 . The Cartesian product of X and Y is the set of all ordered pairs (x, y) where x ∈ X and y ∈ Y . In Figure 1.3 we see two sets A and B within the universal set U. Let G be a weighted graph in which the weight of each edge is a positive integer. Suppose that we prove (2.3). Sets and Logic 15. A unary operator on X is a function from X to X. , Xt } is a pairwise disjoint family (i.e., if i = j, Xi ∩ Xj = ∅), the number of possible elements that can be selected from X1 or X2 or . Exercises In Exercises 1–6, tell whether the statement is a propositional function. The Basis Step is i = 1. How many outcomes have at least one die showing 2? State the involution law for sets. If New Atlantic gets ∀x∃y(x = y3) is true or false. Therefore a2 is not included in the product, and Result is unchanged. Define X intersect Y . In this example, n0 in inequality (5.1) is equal to 4. This notation is called the theta notation. ∃x∃y(x2 + y2 ≥ 0) x) ∧ (z < y)) is true or false. Let bn = n (i + 1)2 − i2 . Theorems 2.17 and 2.18 allow us to easily determine whether a graph has an Euler cycle: A graph G has an Euler cycle if and only if G is connected and every vertex has even degree. j=1 ↓ 001 010001 ↑ i=1 (1) j=1 ↓ 001 010001 ↑ i=3 (4) j=1 ↓ 001 010001 ↑ i=4 (7) j=2 ↓ (×) 001 010001 ↑ i=1 j=1 ↓ (×) 001 010001 ↑ i=2 (2) (3) j=2 ↓ 001 010001 ↑ i=3 j=3 ↓ (×) 001 010001 ↑ i=3 (5) j=2 ↓ 001 010001 ↑ i=4 (6) j=3 ↓ 001 010001 ↑ i=4 (8) (9) Figure 2.1 Searching for "001" in "010001" using Algorithm 2.1. The cross (×) in steps (2), (3), and (6) marks a mismatch. 67 Sets and Logic 2. − |B ∩ M| + |F ∩ B ∩ M| = 65 + 76 + 63 − 36 − 20 − 18 + 10 = 140. This deletion will not affect the value of v. 27. while (i ≤ m − 1) { 24. Show that a tree is a planar graph. Let us modify the example slightly. Since f(2, 1) = 22 + 12 = 12 + 22 = f(1, 2), f is not one-to-one. Hence t = t1 + t2 . Let E denote the event "getting a sum of 10," and let F denote the event "at least one die shows 5." The event E ∩ F is "getting a sum of 10 and at least one die shows 5." Since only one outcome belongs to E ∩ F , P(E ∩ F) = 1 . Since the n-cube is a simple graph, m = 2. Therefore, a1 = 1. There are C(11, 2) ways to choose positions for the two P's. In Exercises 22–27, find the number of integer solutions of x1 + x2 + x3 = 15 subject to the conditions given. An expression such as (6.1) can be represented as a binary tree. x = 1 y=2 z=x+y z=z+1 51. (6.1) This standard way of representing expressions is called the infix form of an expression. Therefore, T is connected. In the following section, we will relax the assumption that all outcomes are equally likely. One fair die is rolled. Let A be the adjacency matrix of the graph Km,n . We next show that Prim's Algorithm is correct. Let f be the function from X = {0, 1, 2, 3, 4} to X defined by f(x) = 4x mod 5. Proof The proof is sketched before the statement of the theorem. Thus the ikth entry in An+1 gives the number of paths of length n + 1 from i to k, and the Inductive Step is verified. The winners, Seles and Graf, then played, and Graf defeated Seles. (If it were acyclic, it would be a tree and, therefore, have n − 1 edges.) 518 Trees Section Review Exercises 1. Proof Since E1 and E2 are mutually exclusive events, E1 ∩ E2 = ∅. Unlike actual computer languages, which must be concerned about semicolons, uppercase and lowercase letters, special words, and so on, any version of pseudocode is acceptable as long as its instructions are unambiguous. Suppose that GK4 has a Hamiltonian cycle C = (v1 , v2 , . Example 1.23 Since each element of X = {1, 2, 3, 4, 5, 6, 7, 8} is in exactly one member of S = {{1, 4, 5}, {2, 6}, {3}, {7, 8}} , S is a partition of X. Cube 1 is to have red and white opposing faces. If f and g are onto, then f ◦ g is onto. Also, logb n ≥ C loga n for all n ≥ 1; so logb n = (loga n). (For Exercise 11) 10 5 12 12 10 9 8 4 10 5 9 1 5 3 10 2 11 3 15. Trees 26. 2091, 4807 8. Answer the following where R serves as a universal set: 1. , 10}. Find 6 i=1 23. Such a tree is called a decision tree. Since S is the sum of even numbers, S is even. i=1 70. Inductive Step We assume that for all k, 1 ≤ k < n, it takes k − 1 multiplications to compute the product of k numbers if parentheses are inserted in any manner whatsoever. The Petersen graph is much used as an example; in fact, D. Therefore 2 x − 1 ≤ 2x . Write an algorithm that receives as input the n × n matrix A and outputs the transpose AT . Theorem 7.1 Binomial Theorem If a and b are real numbers and n is a positive integer, then (a + b)n = n C(n, k)an−k bk . 5 ➜ Representations of Graphs In the preceding sections we represented a graph by drawing it. Use Mathematical Induction to prove the general InclusionExclusion Principle for finite sets X1 , X2 , . The graph 2 3 1 6 4 5 has an Euler cycle because it is connected and every vertex has even degree. 195 Functions, Sequences, and Relations 16. 5. The pair z, n is then made public. A proof by resolution proceeds by repeatedly applying the rule in Exercise 1 to pairs of statements to derive new statements until the conclusion is derived. C(2n, n)/(n + 1) 5. 209D 31. Thus at the next iteration of the loop, a sequence of length 2m is processed. This contradiction shows that it is impossible to mark k squares in the upper-left k × k subgrid and k squares in the lower-right k × k subgrid so that no two marked squares are in the same row, column, or diagonal of the 2k × 2k grid. E v4 v5 v2 v6 v8 w1 v3 v7 w2 w5 T1 w3 w4 w6 w7 w8 T2 Trees 32. We can show that there are three nonisomorphic trees with five vertices. Then, all we had to do was insert the next element si+1 in the proper place. Therefore, an ≤ 3n + 2, and the Inductive Step is complete. Now this change of units does not affect how the worst-case time grows as the size of the input increases but only TABLE 3.2 ■ Comparing growth of t(n) with 60n2 . We let E1 denote the event "get doubles" and E2 denote the event "get the sum of 5." Notice that E1 and E2 are mutually exclusive: You cannot get doubles and the sum of 5 simultaneously. If we have n > 1 disks on peg 1 as in Figure 1.2, we begin by recursively invoking our algorithm to move the top n − 1 disks to peg 2 (see Figure 1.3). Venn diagram 22. , plus three times the twelfth digit. In Exercises 25–33, C0 , C1 , C2 , . 53. 73. Now the value of a2 might be equal to any of the remaining values in the sequence. Write a program that determines whether a graph is connected. Except for constants and a finite number of exceptions, f is bounded above and below by g. In doing so, we must be careful to follow 0111 with every possible four-bit string: 01110000, 01110100, 01111000, 01111100, 01110001, 01110010, 01110101, 01110110, 01111001, 01111010, 01111101, 01111110, 01110011, 01110111, 01111011, 01111111. A − B 4. 1 · 3 · 5 · · · (2n − 1)(2n + 1) 1 2n + 1 ≥ · 2 · 4 · 6 · · · (2n)(2n + 2) 2n 2n + 2 2n + 1 1 1 = · ≥ 2n 2n + 2 2n + 2 15. (3.11) Taking i = n − 4 in (3.10), we obtain rn−4 = rn−3 qn−2 + rn−2 or rn−2 = −qn−2 rn−3 + rn−4 . Graph Theory b c 3 2 2 a b z 1 0 2 d 2 0 c 3 a 1 d 2 z d, 2 a 0 1 2 3 a, 2 c b, 5 2 d, 2 2 z a, 1 1 e c b, 5 a, 2 0 2 1 1 Figure 4.10 The third iteration of Dijkstra's shortest-path algorithm. 10! 71. Since 365 mod 7 = 1, 365 days from Wednesday, it will be one day later, namely Thursday. Write an algorithm that returns the index of the last occurrence of the largest element in the sequence s1 , . Suppose that we can make n-cents postage: n = np p + nq q. Now suppose that x ∈ U. Hint: Consider the partition {1, 10}, {2, 9}, {3, 8}, {4, 7}, {5, 6}. We will show later that if L(v) is the permanent label of vertex v, then L(v) is the length of a shortest path from a to v. i=1 61. The case n is odd is similar. A(n, m) = 1+A(n−1, m−1)+A(n−1, m), n−1 ≥ m ≥ 1, n ≥ 2; A(n, 0) = A(n, n) = 1, n ≥ 0 Recurrence Relations Show that there exist constants b and d such that 39. f(m, n) = mn 19. e1 e2 e6 e5 e3 e4 e7 e8 27. However, assuming all previous cases potentially gives you more to work with in proving case n. Implement a computation of an nonrecursively that uses repeated multiplication and Algorithms 3.11 and 3.12 as computer programs, and compare the times needed to execute each. For n = 2, there is one way for everyone to get the wrong orders: Person 1 gets order 2, and person 2 gets order 1. Suppose that there are n vertices in a balanced binary tree of height h. We must prove that if n is odd then n − 1 is even and if n − 1 is even then n is odd. Use the If S(n) is true, then S(n + 1) is true form of the Inductive Step to prove the statement in Exercise 2. If the roots of t 2 − c1 t − c 2 = 0 are r1 and r2 , r1 = r2 , the solution is an = br n + dnr n for some constants b and d, where r = r1 = r2 . First we select a triangle to which side ab belongs. Definition 5.2 The probability P(E) of an event E from the finite sample space S is P(E) = |E| † . Decrypt 250. By the definition of set difference, since x ∈ X ∩ Y , but x ∈ / X ∩ Z, x ∈ (X ∩ Y) − (X ∩ Z). In this case, the algorithm assigns large the value si+1 . Prove that for every rational number x, if x = 0, then 1/x is rational. An edge exists between xi and xj if the algorithm compares xi and xj . From the definition of "intersection," we conclude that x ∈ X. Put all vertices adjacent to w in V1 . Let c0 , c1 be the smoothing sequence defined by c0 = c1 = 1/2. How many ways are there to select 10 comics? Trace Algorithm 2.19 for the values a = 50, n = 30, z = 11. (e) At lines 2 and 3, since 0 = 0, we return 1. State a necessary and sufficient condition that a graph have an Euler cycle. ∀x∃y T1 (x, y) 3. 210 67. To count the number of items available at Kay's Quick Lunch, we just sum the number of items each category since dividing the items by category naturally splits them into nonoverlapping subsets. Sets and Logic 5. Let Pn be the number of partitions of an n-element set. If the teams are A, B, C, and D and each pair of teams plays at most one time, the most games that can be played are: A and B; A and C; A and D; B and C; [(c) → (a)] Let x ∈ A. To understand what is to be proved, look at some specific values in the domain of discourse. (Gray codes are discussed in Example 3.5.) The coordinates in the horizontal direction are the first p members of an i-bit Gray code, and the coordinates in the vertical direction are the first q members of a j-bit Gray code (see Figure 6.2). Example 6.3 Consider the statement ∀x∀y((x > 0) ∧ (y > 0) → (x + y > 0)). Since the road inspector lives in Greybull, the most economical way to inspect all of the roads would be to start in Greybull, travel each of the roads exactly once, and return to Greybull. Since the first two quantifiers are ∀, Farley chooses values for both x and y. In a model of a computer network, a vertex is a message or switching center, an edge represents a channel on which data can be transmitted between vertices, a flow is the average number of bits per second being transmitted on a channel, and the capacity of an edge is the capacity of the corresponding channel. 2 ➜ Propositions Which of sentences (a)–(f) are either true or false (but not both)? Clarify the headline by rephrasing it and writing it symbolically. Unfortunately, there are problems (an example of which is the tapefilling problem described previously) for which no method much better than the "list all" approach is known. What is a flow in a network? Formally, ∪ S = {x | x ∈ X for some X ∈ S}. As an application of the number theory presented in Sections 1–3, we discuss the RSA system for secure communications in Section 4. At this point, we know that the value x of A must be at least 2, since the value of A is the maximum of its children; that is, x ≥ 2. Write an algorithm that returns the smallest value in the sequence s1 , . The n-disk, four-peg Tower of Hanoi puzzle has the same rules as the three-peg puzzle; the only difference is that there is an extra peg. To justify this last statement, we note that the following arrangements are possible after we insert a5 in its correct position: 9. How simple! If we use the Strong Form of Mathematical Induction, we can assume the truth of the statement for all k < n. 825 mod 315 = 195; 315 mod 195 = 120; 195 mod 120 = 75; 120 mod 75 = 45; 75 mod 45 = 30; 45 mod 30 = 15; 30 mod 15 = 0; so gcd(825, 315) = 15. Clarify each statement by rephrasing it and writing it symbolically. Continue. , x10 } into eight subsets. (Exercise 1) {(Hammer, 8840), (Pliers, 9921), (Paint, 452), (Carpet, 2207)} 18. Example 2.9 The graph G = (V , E) of Figure 2.3 is a subgraph of the graph G = (V, E) of Figure 2.4 since V ⊆ V and E ⊆ E. 22 Sets and Logic Example 3.4 Let p : 1 > 2, q : 4 < 8. ¬p From 3 and 5 Now 4 and 6 combine to give a contradiction. Example 2.11 Prove that there exists a prime p such that 2p − 1 is composite (i.e., not prime). , f(vn). That is, assume the Strong Form of Mathematical Induction and prove the alternative form; then assume the alternative form and prove the Strong Form of Mathematical Induction. s1 , s2 , . 513 Trees 12. 21. For example, the fundamental cycle matrix of the graph G of Figure 3.1 relative to the spanning tree shown in Figure 3.1 is ⎛ e7 e6 e11 e10 e2 e1 e3 e4 e5 e8 e9 e12 ⎞ (abdca) (efdbace)⎟ ⎜ (ageca) ⎟ ⎜ ⎝ (aga) (abga) 16. (x, y) ∈ R if x > y. Exercises 11–13 refer to the sequence c1 , c2 , . An algorithm is a step-by-step method of solving a problem. , xn }: (unordered) subset of {x1 , . To prove equation (1.4), let x ∈ (X ∩ Y) − (X ∩ Z). What is a supersource? 31. To solve a recurrence relation involving the sequence a0 , a1 , . There are 7! arrangements for the remaining Jovians. i=0 12. We call the proposition q → p the converse of the proposition p → q. For large n, the term 60n2 is approximately equal to t(n) (see Table 3.2). The variable n is incremented and becomes 3. The next theorem relates matching networks and flows. z = pq = 13 · 17 = 221, φ = (p−1)(q−1) = 12 · 16 = 192 14. Thus Sn = Sn−1 + Sn−2 . Figure 1.12 The complete graph K4 . (1 + x)n ≥ 1 + nx, for x ≥ −1 and n ≥ 1 1 1−r for all n ≥ 0 and 0 < r < 1. We will build a binary tree with terminal vertices labeled s1 , . n = 100, φ = 231 41. The decision tree analysis shows that at least lg 5! = 7 comparisons are required to sort five items in the worst case. , (vn−1 , vn), 58. To test whether a relation is transitive, let A be its matrix. Example 2.29 Let X = {a, b}. Then, all we had to do was look at the next element si+1 and, if si+1 was larger than large, we simply updated large. In this way, we store all of the words in the tree and thus create a binary search tree. ; select the eighth bit. Suppose that p → q and p are true. Why is 5 the quotient in Example 5.5? Because q = 5 119 Proofs makes the remainder n − dq nonnegative and as small as possible. Alert the reader where you are headed. is true. If we want to locate a stored value n, we compute m = h(n) and begin looking at location m. Notice that if n ≥ 2 and m ≥ 1, 2m + 5n2 ≥ 2m + 5 · 22 > 20, so the only possible solution is if n = 1. Modify Algorithm 4.1 so that it accepts a weighted graph that is not necessarily connected. [Chu] showed that the converse is true, except when n is 5. n = 15 58. 364 n/2 k=0 C(n, 2k) = 2n−1 = n/2 k=1 C(n, 2k − 1). , 2n + 1} and m is the greatest element in X, there exist distinct i and j in X with m = i + j. What is the probability that at least two of the lockers are consecutive? Only one link exists at a time, and after a link is established and the files are shared, the link is deleted. Find the 6-combination that will be generated by Algorithm 4.9 after 145678 if n = 8. 1 2 n q Section 5 1. (b) We first evaluate ¬r. In order for the given statement to be true, we must find a propositional function P(x, y) satisfying for every x, there exists y such that P(x, y) is true. We decrease the flow by 1 in the improperly oriented edge (c, b) and increase the flow by 1 in the properly oriented edges (a, b), (c, d), and (d, z) (see Figure 2.6). From the San Antonio Express-News:All old things don't covet twenty-somethings. One type of "box play" win requires that three distinct numbers match in any order those randomly drawn by a lottery representative, repetitions allowed. Exercise 1, Section 1 15. In this case we conclude that 1 + 2 + · · · + n = (n), and while the preceding expression is true, we cannot deduce a -estimate for 1 + 2 + · · · + n, since the upper bound n2 and lower bound n are not equal. One common form is [p1 → p2 .] Proof of p1 → p2 goes here. Table 7.1 shows all the possibilities. , Xn } is a partition of X, the Addition Principle applies and |X| = n |Xi |. Since n mod 2 is equal to 1, result is set to (result∗x) mod z = 20 mod 11 = 9. What is an n-level search? A path represents a sequence of moves. How many outcomes give an even sum? Explain how to prove that a universally quantified statement is true. We then compute the quotient when 7 is divided by 2. , n by n in the sum 1 + 2 + · · · + n, the sum does not decrease and we have 1 + 2 + · · · + n ≤ n + n + · · · + n = n · n = n2 for all n ≥ 1. The words have different lengths, and each letter in the shorter word is identical to the corresponding letter in the longer word. Show that R is transitive if and only if τ(R) = R. Since five trominoes are required, the missing square cannot be a 2. Explain why all the diagonal elements of An are equal and all the off-diagonal elements of An are equal. Example 5.4 Five microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. The composition function from X to Z is defined as [ ) 3. Show that a connected graph G is bipartite if and only if every closed path in G has even length. Example 4.3 The matching for the graph of Figure 4.2, shown with black lines, is a maximal matching and a complete matching. Some additional properties of divisors are given in the following theorem and will be useful in our subsequent work in this chapter. We use strong induction on the length n of α to show that if α ∈ L, α has an equal number of a's and b's. , 2n−1 − 1 (see Figure 1.11). Therefore X ⊆ Y . Show that 1 p < . Theorem 1.5 is the same as the flow into the sink z, Fcz + Fez ; both values are 5. How many dinners at Kay's Quick Lunch (Figure 1.1) consist of an optional appetizer, one main course, and an optional beverage? (X ∩ Y) ∪ (Y − X) = Y for all sets X and Y . Such a function may be constructed as follows. Again, we'll need to come up with a counterexample. X+ is the set of all nonnull strings over X. Deep Blue won two games, drew three, and lost one. Why is An symmetric about the main diagonal for every positive integer n? sits on an (infinitely) long table (see Figure 4.1) and that some blocks are marked with an "X." (All of the blocks visible in Figure 4.1 are marked.) Suppose that The first block is marked. To show that q ≤√ n, we use √ proof by contradiction. If we now input n = 91 to Algorithm 1.8, the algorithm returns the prime 7 because 7 divides 91, specifically 91 = 7 · 13. 372 26. Example 6.18 Joe and Alicia take a final examination in discrete mathematics. How many reflexive, symmetric, and antisymmetric relations are there on an n-element set? For example, entering mathematics produces a (huge!) list of pages that contain the word "mathematics." Some search engines allow the user to use and, or, and not operators to combine keywords (see Figure 2.1), thus allowing more complex searches. † 25. ∀x∃y P(x, y) 73. First, suppose that the Mathematics Department gets an additional $60,000. 59. (b) Show that Sn,n = 1 for all n ≥ 1. Write selection sort in pseudocode. Example 4.7 Use mathematical induction to show that n fk = fn+2 − 1 for all n ≥ 1. The odds against this feat are 6 billion to 1. In this graph only, if the left pan is heavier, go right. Thus, whether one of the trees is empty or not, at most 3n + 2 comparisons are required in the worst case. We must show that for every y ∈ Y , there exists x ∈ X such that f(x) = y. The inductive argument is complete and this portion of the proof is complete. In other words, there must be edges of the form (a, v) and (v, c). Solve the recurrence relation of Exercise 66 in case n is a power of 2, assuming that b1 = 0. 195 + 76E 37. After values are chosen for all the variables, if P(x, y) is true, you win; if P(x, y) is false, Farley wins. Let R be an equivalence relation on a set A. Index: in the sequence {sn }, n is the index 17. For example, if a particular hypothesis refers to an even integer n, you know that n is of the form 2k for some integer k. We relate the (n + 1)st case to the nth case by writing 5n+1 − 1 = 5n − 1 + to be determined. The first player can always win in nim by adopting the following strategy: Play nim exactly like nim unless the move would leave an odd number of singleton piles and no other pile. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... (p ∨ q) ∧ ¬r 8. A binary search tree is a binary tree T in which data are associated with the vertices. The complete graph on n vertices has one edge between each distinct pair of vertices. Prove that n n = aij j=i i=1 j n j=1 aij . b 3, 0 a c 2, 0 4, 0 (a, 3) (b, 2) (,) 2, 0 (c, 2) (a, 5) 5, 0 d 2, 0 z 4, 0 e Figure 2.7 After the first labeling. In [Frey], the minimax procedure is applied to a simple game. Basis Step (n = 1) We have already observed that the theorem is true if n = 1. x4 y7 ; (x + y)11 17. The cases n = 1 and n = 2 may be directly verified. Not a proposition; it is a question. At line 17 we examine the edges (a, b) and (a, d) since neither b nor d is labeled. Notice that If the Cubs sign a right-handed relief pitcher, then they win the World Series, is not an equivalent formulation. A string of length 4 can be constructed in four successive steps: Choose the first letter; choose the second letter; choose the third letter; and choose the fourth letter. If we roll two dice, the events "get an odd number on the first die" and "get an even number on the second die" are independent. Determine the truth value of each statement in Exercises 49–54. The domain of discourse is the set Z × Z. Greedy algorithm Section 5 25. Describe the following properties an algorithm typically has: input, output, precision, determinism, finiteness, correctness, and generality. Often the context will make clear which number system is in effect; but when we want to be absolutely clear, we subscript the number to specify the base—the subscript 10 denotes the decimal system and the subscript 2 denotes the binary system. We will use variables, such as p, q, and r, to represent propositions, much as we use letters in algebra to represent numbers. If e is an edge in G whose weight is less than the weight of every other edge, e is in every minimal spanning tree of G. List the elements of the event E1 or E2 . Since the probability of each attempt is 1/3 = 0.3333, after three attempts the probability of completing the marathon is 0.9999, which means that the runner is almost, but not quite, certain to complete the marathon. Then T ∗ is connected and acyclic, and because T ∗ contains n vertices, by the inductive hypothesis T ∗ contains n − 1 edges. This time b = 0, so we skip to line 10, where we return a (36), the greatest common divisor of 396 and 504. ¬p ∨ r From 1 and 2 5. 3! 13. Because of the tremendous increase in the size of the Internet, IPv6 (Internet Protocol, Version 6) uses 128-bit addresses rather than 32-bit addresses. The recurrence relation preceding Exercise 56, Section 1 27. Algorithm 2.3 shows how to convert an integer in base b to decimal. Now suppose that n > 2. v and the edge incident on v are removed so that the inductive hypothesis can be invoked. (e) Show that sn,2 = (n − 1)! 1 + for all n ≥ 2. A set is a collection of objects. The set X ∪ Y = {x | x ∈ X or x ∈ Y } is called the union of X and Y . How do we denote X is a proper subset of Y ? We must show that s = s. Summary of Problem-Solving Techniques ■ When examples get too wordy, develop a notation for concisely describing the examples. In Exercises 55–59, represent the proposition symbolically by letting p : You run 10 laps daily. We define "pseudo equivalence classes" as in Theorem 4.8: [a] = {x ∈ X | x R a}. , sn , again say si , and places it second by swapping s2 and si . In the problem of counting the number of dinners consisting of one main course and one beverage, the first step is "select the main course" and the second step is "select the beverage." Thus n1 = 3 and n2 = 4 and, by the Multiplication Principle, the total 306 Counting Methods and the Pigeonhole Principle Cheeseburger Hamburger Fish Filet Tea Milk Cola Root Beer Tea Milk Cola Root Beer Tea Milk Cola Root Beer HT HM HC HR CT CM CC CR FT FM FC FR Figure 1.2 An illustration of the Multiplication Principle. We have already noted that if every even number greater than 2 is the sum of two prime numbers (which is currently unsettled), the algorithm will not terminate. Inductive Step Assume that statement (4.12) is true for n. It is denoted Kn . Example 1.26 If X = {1, 2}, Y = {a, b}, Z = {α, β}, then X × Y × Z = {(1, a, α), (1, a, β), (1, b, α), (1, b, β), (2, a, α), (2, a, β), (2, b, α), (2, b, β)}. Since the goal involves m + n, we can add the equations m = 2k1 + 1 and n = 2k2 to obtain a fact about m + n, namely, m + n = (2k1 + 1) + 2k2 . ∃x (P(x) ∨ Q(x)) 41. Example: If the sequence is AMY BRUNO ELIE, the reversed sequence is ELIE BRUNO AMY. 29 60. x∈S 2. Two graphs are homeomorphic if they can be reduced to isomorphic graphs by performing a sequence of series reductions. Consider the integer m = p1 p2 · · · pn + 1. Let V be the set of vertices in Ti . Since n mod 2 is equal to 1, result becomes result ∗ x = 1 ∗ a16 = a16 . W. Since X ∪ Y = X ∪ Z, x ∈ X ∪ Z. We now have x ∈ X and x ∈ / X, which is a contradiction. What is the probability that the wrestler selected is over 350 pounds or a bad guy? } 31. 244 Algorithms Problem-Solving Tips A recursive function is a function that invokes itself. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to determine how many students are not taking any of the three courses. We present a solution, using a graph model, that makes it possible to discover a solution, if there is one, in a few minutes! B Cube 1 R Cube 2 W Cube 3 B Cube 4 G R B W G Figure 8.2 A solution to the Instant Insanity puzzle of Figure 8.1. G R W R R WB B W G G 1 2 3 4 Figure 8.1 An Instant Insanity puzzle. Therefore 2m < n + 1 ≤ 2m+1 . Let X be a set with least upper bound a, and let t < 0. Therefore, all nonisomorphic binary trees with three vertices are shown in Figure 8.12. Implement a hashing system for storing integers in an array. In late 2006, the 44th known Mersenne prime was found, 232,582,657 − 1, a number having 9,808,358 decimal digits. = 2n−1 c1 + 2n−2 + 2n−3 + · · · + 2 + 1 = 2n−1 + 2n−2 + 2n−3 + · · · + 2 + 1 = 2n − 1. Otherwise, we could check whether s2 ≥ A and if true, swap them. Assume that the input size is n and that the run time of the algorithm is the number of comparisons made at line 3. A minimal spanning tree is a spanning tree with minimum weight. In the notation of Definition 4.1, p = q = 4, s1 = 1, s2 = 3, s3 = 2, s4 = 4, t1 = 1, t2 = 3, t3 = 4, and t4 = 2. Equivalence relation: [1] = [3] = {1, 3}, [2] = {2}, [4] = {4}, [5] = {5} 40. Recurrence Relations Input: a (a real number), n (a positive integer) Output: an 1. an = 7an−2 − 6an−3 6. y √ Notice that 1/ y is defined because y is a positive real number. Show that for Figure 8.5, Instant Insanity, as given in the text, has a solution, but the modified version does not have a solution. C 1C 2C 3C 4 : C 5C 6C 7C 8 C 1C 2C 5 : C 3C 4C 6 C1 : C2 C 1, L C 6, H C 5C 6C 1 : C 7C 8C 2 C7 : C8 C 2, L C 8, H C3 : C4 C 7, H C 3, L C 5, H C5 : C6 C 4, L C 5, L C 2, H C3 : C4 C 6, L C 4, H C7 : C8 C 3, H C 7, L C 1, H C 8, L C 1 C 2 C 3 : C 9 C 10 C 11 C 9 : C 10 C 10, H C 11, H C 9, H C 1 : C 12 C 12, H C 12, L C 9 : C 10 C 9, L C 11, L C 10, L 7. We count the number of n × n matrices that represent reflexive relations on an n-element set X. For example, Bela Lugosi has Bacon number three. An n × n matrix has n2 entries and the diagonal contains n entries. Since x2 is the quotient of integers, x2 is rational. Find a formula for the sequence d defined by Answer 31–34 for the sequence τ0 = 5, dn = τ2 = 5. Notice that the relation R of Example 5.4 is reflexive and that the main diagonal of the matrix of R consists of 1's. If d divides n, n = dq, we call q the quotient. Show that there is a unique binary tree with six vertices whose preorder vertex listing is ABCEFD and whose inorder vertex listing is ACFEBD. Therefore, the total number of functions from X into X having the property that if i is in the range of f , then so are 1, . an = 2n an−1 ; a0 = 1 a0 = 1, 16. 170 Functions, Sequences, and Relations Eliminating the cases x = y and y = z leaves only the following to be explicitly checked to verify that the relation is transitive: Pairs of Form (x, y) (y, z) (x, z) (1, 2) (1, 2) (1, 3) (2, 3) (1, 3) (1, 4) (1, 4) (2, 4) (2, 3) (2, 4) (3, 4) (3, 4) The digraph of a transitive relation has the property that whenever there are directed edges from x to y and from y to z, there is also a directed edge from x to z. (For Exercise 1) In symbols: ∃x∃y¬T1 (x, y). Using the formula fk fk+2 − fk+1 21, we obtain 1+ n (−1)k+1 k=1 fk fk+1 = 1+ n fk fk+2 − f 2 k+1 k=1 = 1+ n fk+2 k=1 = 1+ fk fk+1 fk+1 f2 f3 − f2 f1 − fk+1 fk + f3 f4 − f3 f2 4. A counterexample is m = 2 and n = 4.) Given a universally quantified propositional function, showing that the propositional function is true for specific values in the domain of discourse is not a proof that the propositional function is true for all values in the domain of discourse. The union of all connected subgraphs containing G is a component. .} 97. If the desired item is greater than the middle item, binary search recursively searches in the right half of the sequence. We are given a sequence s1 , 7. Example 1.2 The Fibonacci sequence is defined by the recurrence relation fn = fn−1 + fn−2 , n ≥ 3, and initial conditions f1 = 1, Example 1.3 384 f2 = 1. [3] = {3, 4}. Let Dn denote the number of ways that n persons can all receive the wrong orders. In Exercises 6 and 7, solve the recurrence relation subject to the initial conditions. Since this increase is 10 percent of the size at time n − 1, we obtain the recurrence relation dn − dn−1 = 0.1dn−1 , which may be rewritten dn = 1.1dn−1 . If the program is readable, then it is well structured. Therefore, a dag has at least one vertex with no out edges. Database management systems are programs that help users access the information in databases. Every time we fill seven of the nine blanks with seven vertical bars, we obtain a partition of {x1 , . 34. Extend Tables 3.1 and 3.2 to the range 0 to 21. gcd subtract(a, b) { while (true) { // make a largest if (a < b) swap(a, b) if (b == 0) return a a=a−b } } 20. Yes. Such problems are called feasible or tractable. Let f(n) denote the number of weighings needed to solve the n-coin problem in the worst case. The algorithm returns result, which is equal to a30 . There is no hurricane. j=n while (j ≥ 1) { for i = 1 to j x=x+1 j = j/2 } Let t(n) denote the number of times we execute the statement x = x + 1. 564 1 0 0 1 0 0 1 0 1 1 1 0 0 0 1 1 0 0 0 Trees is valuable to the first player and we label it with a value greater than 0, say 1 (see Figure 9.2). Using this notation, the given information may be written P(H) = 0.15, P(H) = 0.85, P(Pos | H) = 0.95, P(Pos | H) = 0.02. There are 3 ways to choose an element of X for the first member of the ordered pair (shown at the top of the diagram) and, for each of these choices, there are 2 ways to choose an element of Y for the second member of the ordered pair (shown at the bottom of the diagram). Algorithm 3.14 Insertion Sort (Recursive Version) 57. What is the value of x mod y? True 22. Also add the vertices incident on these edges. By Exercise 75, k vertical steps can occur in C(k, k/2) ways, since, at any point, the number of up steps is greater than or equal to the number of down steps. e5 is a loop. This last equation can be verified by reducing each side to terms involving α and β. There are three routes that first meet the top edge at the second position from the left: meet the top edge at the third position from the left is equal to the number of routes from the lower-left corner to the upper-right corner of a 2 × 2 grid—namely, C(2 + 2, 2) = 6. 8 The probability of an event E is defined as the sum of the probabilities of the outcomes in E. The codomain of each function is Z. The constants in the Strassen and Coppersmith-Winograd algorithms are so large that they are faster than Algorithm 3.19 only for very large matrices. Such algorithms are called serial algorithms. If a vertex has one child, that child is designated as either a left child or a right child (but not both). We next prove that the number of primes is infinite. Define the sequence c1 , c2 , . If x ∈ X, then x ∈ X ∪ Y . Continue in this way until all the elements of X have been assigned to a set. Region 6 represents elements in X and Z, but not in Y . 97 Proofs 3 ➜ Resolution Proofs In this section, we will write a ∧ b as ab. What must a graph look like if some row of its incidence matrix consists only of 0's? Graph Theory In Exercises 24 and 25, draw the graphs represented by the incidence matrices. We have generated the 5-combination 23567, which follows 23467. f(x) = 3 log2 x, X = set of positive real numbers 1 35. The value Fai = Fiz i i is called the value of the flow F. Find an expression, which is the and of clauses, equivalent to (p ∨ q) → r. 34 Since 144 > 34, it is immediately inserted to 34's right 20 34 144 Since 55 < 144, 144 must move one position to the right 20 250 34 144 57 i j 135 101 72 4. .} = n−1 2 n+1 2 . An existentially quantified statement is a statement of the form for some x in the domain of discourse, P(x). Example 4.11 The equivalence classes appear quite clearly in the digraph of an equivalence relation. In a Venn diagram, a rectangle depicts a universal set (see Figure 1.3). Thus the total number of coins is at least 1 + 2 + 3 + · · · + 9 = 45. Now, consider the problem of assigning values to the internal vertices. Continue until swapping sn−1 and a randomly chosen element in sn−1 , sn . At line 4, i is reset to 4 and we return to line 2. In this case, if the remaining (n−2)bit string does not contain 000, neither will the n-bit string. As we have already noted, there are ways other than going to the Eiffel Tower to ensure that Maria visits France. Now n + 1 = 2k + 2 = 2(k + 1), which is even. 507 Trees Seles Seles Navratilova Graf Sabatini WIMBLEDON CHAMPION Graf Graf FINALS SEMIFINALS Figure 1.1 Semifinals and finals at Wimbledon. Prove that k=1 k−1 k 1 i=1 kC(n, k). Then there exists k such that n = 2k. Let x ∈ AB. I'll buy the compact disc player. If n = 3, it is easy to check that the loop iterates one time. 6b. In our algorithm, we will list the r-combination {x1 , . 290 Introduction to Number Theory Algorithm 3.9 Recursive Euclidean Algorithm This algorithm recursively finds the greatest common divisor of the nonnegative integers a and b, where not both a and b are zero. Recurrence relations, recursive algorithms, and mathematical induction are closely related. From p → q and q → r, we may use the hypothetical syllogism to conclude p → r. Trace Algorithm 2.3 for the input 34 10. c ⎝0 1 1 0⎠ d 0 0 0 1 Notice that the matrix of a relation on a set X is always a square matrix. How many subtractions are required by the algorithm of Exercise 17 in the worst case for numbers in the range 0 to m? If this problem were not associated with a particular section or chapter in the book, we might try several approaches—one of which might be to model the problem as a graph. 349 Counting Methods and the Pigeonhole Principle We must have P(2) = 1/8. Show that the maximum number of edges in a simple, disconnected graph with n vertices is (n − 1)(n − 2)/2. Formally, if S is a finite set of integers and a is a sequence, ai i∈S denotes the sum of the elements {ai | i ∈ S}. Example 1.1 For the set A in (1.1), we have |A| = 4, and the cardinality of A is 4. If a string has less than four 1's, it will have two 0's in a row. Attacking the Problem To better understand the definitions, let's construct examples, write out the definitions in words, look at negations of the definitions, and draw pictures. However, for a large number of cities, the brute-force technique would take too long. The relation R = {(1, 2), (2, 3), (3, 4), (4, 1)} on {1, 2, 3, 4} 24. Since the cases are mutually exclusive and cover all nbit strings (n > 3) not containing 000, we have Sn = Sn−1 + Sn−2 + Sn−3 for n > 3. 3. Write a recurrence relation that describes the worst-case time required by binary search. Inductive Step Assume that (1.7) is true for n − 1. The graph produced is still connected. (a) n2 + 2n is an odd integer (domain of discourse = Z+). Define concatenation of strings. , xn be the vertices of a graph. In this section we show how trees can be used to develop gameplaying strategies. Combining (8.1) and (8.2), we obtain 4 + aL + aR ≤ 4 + (3L + 2) + (3R + 2) = 3(2 + L + R) + 2 = 3n + 2. We assume that the fraction p/q√is in lowest terms so that p and q are not both even. This explains the need to make the substitution in part (b). The border consists of 1's, and any interior value is the sum of the two numbers above it. A move consists of (a) Moving to an adjacent point (b) Moving from the putahi to a point (c) Moving from a point to the putahi provided that one or both of the adjacent points contain the opponent's pieces PUTAHI 27. Do any of them depend on the truth of Goldbach's conjecture (which mathematicians have not yet settled)? C(n + 1 − i, i), n = 1, 2, . By the definition of intersection, since x ∈ X and x ∈ Y , x ∈ X ∩ Y . The body of the for loop (line 12) takes time O(n). c b a g c d d f f e e In Exercises 6–8, determine whether each graph is planar. Example 4.3 Which rule of inference is used in the following argument? The nth term is typically denoted Cn , although to be consistent with the more general function notation, it could be (and sometimes is) written C(n). Explain how to prove that an existentially quantified statement is false. Exercises In Exercises 1–3, find the matrix of the relation R from X to Y relative to the orderings given. 2n n!(2n − 1)(2n − 3) · · · 3 n!n! 1 [(2n)(2n − 2) · · · 2][(2n − 1)(2n − 3) · · · 3] n+1 n!n! 1 1 (2n)! = = C(2n, n). We must prove that if X contains an element less than or equal to n + 1, then X has a least element. The statement ∀x P(x) is false if P(x) is false for at least one x in D. The statement is false if there is at least one x ∈ X such that P(x, y) is false for every y ∈ Y . This technique can be iterated to find the prime factorization of n, again for relatively small values of n. How many elements are in the domain of a? If h = 0, the binary tree consists of a single vertex. At this point, we will have discovered a cycle, which is a contradiction. Add to the tree all edges incident on this single vertex that do not produce a cycle when added to the tree. Example 1.18 Using the notation of Theorem 1.17, we have 82320 = 24 · 31 · 51 · 73 · 110 265 Introduction to Number Theory and 950796 = 22 · 32 · 50 · 74 · 111 . If the statement is true, prove it; otherwise, give a counterexample. As examples, the constant terms on the two sides of the equation must be equal, and the coefficient of n on the left side of the equation must equal the coefficient of n on the right side of the equation. For example, the ID Number 01180 in the fifth row (01180, Homer, 1b, 37) of Table 6.1 matches the PID in the fourth row (01180, Mutts) of Table 6.2. These tuples are combined by first writing the tuple from Table 6.1, following it by the tuple from Table 6.2, and eliminating the equal entries in the specified attributes to give (01180, Homer, 1b, 37, Mutts). i=1 55. From x ≤ d and d ≤ d1 , deduce x ≤ d1 . In Exercises 22–26, draw a graph having the given properties or explain why no such graph exists. (4) T is acyclic and has n − 1 edges. Therefore, the inverse of 110 modulo 273 is 206. v1 v2 v3 v5 w1 v4 v6 w2 w3 w5 T2 T1 w4 w6 T2 T2 561 Trees v1 4. Proof It suffices to find one real number x satisfying a < x < b. How many selections are there in which either Ben is chairperson or Alice is secretary or both? Let n = 4. If n = 2, then j = i + 1. Not a proposition; it is a command. If edge e is associated with vertices v and w, e is said to be incident on v and w. For example, the greatest common divisor of 4 and 6 is 2, and the greatest common divisor of 3 and 8 is 1. 101 32. A Tiling Problem A right tromino, hereafter called simply a tromino, is an object made up of three squares, as shown in Figure 4.4. A tromino is a type of polyomino. Counting Methods and the Pigeonhole Principle 13. If a function from X to Y is one-to-one, each element in Y in its arrow diagram will have at most one arrow pointing to it. If each person chose a cola randomly, what is the probability that at least one person did not choose Coke? k! k! k! Section 3 Review 1. Let E1 denote the event "over 350 pounds," and let E2 denote the event "bad guy." Then P(E1 ∪ E2) = P(E1) + P(E2) − P(E1 ∩ E2) = 40 35 20 15 + − = . (The proposition is true because the hypothesis x > 1 is false. How is the height of a decision tree that represents an algorithm related to the worst-case time of the algorithm? Example 2.12 The graph G of Figure 2.1 has one component, namely itself. Since d divides n, by Definition 1.1 there exists√an integer q satisfying√n = d q. Suppose that the graph has a Hamiltonian cycle. This leaves the remaining four terms for the z's. A set is a collection of objects; order is not taken into account. According to Definitions 7.3 and 7.5, any graph is homeomorphic to itself. (7.2) This result is known as the Binomial Theorem. What is the postfix form of an expression? Example 5.7 The matrix of the relation R = {(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)} on {a, b, c, d}, relative to the ordering a, b, c, d, is ⎛ ⎞ 1 0 0 0 ⎜0 1 1 0⎟ ⎜ A=⎜ ⎝0 1 1 0⎠ . Prove that for all sets A, B, and C, A ⊆ C and B ⊆ C if and only if A ∪ B ⊆ C. Then the ijth entry of either A1 or A2 is 1. In words, a relation is antisymmetric if whenever x is related to y and y is related to x, then x = y. It is not the case that you play football or you miss the midterm exam, or you pass the course. Sometimes we generalize the notion of sequence as defined in this section by allowing more general indexing. 1 a b 2 a 3 b a b (1,a) (1,b) (2,a) (2,b) (3,a) (3,b) Figure 1.8 |X × Y | = |X| · |Y |, where X = {1, 2, 3} and Y = {a, b}. It is always the case that two equivalence classes are identical or have no members in common (see Theorem 4.8). Evaluate the root of the tree of Exercise 35 using the minimax procedure with alpha-beta pruning. After selecting the three items, we may arrange them in E3 ways. m vertices In Exercises 45 and 46, tell whether each assertion is true or false. † Informally, a sequence is a list of elements in which order is taken into account, so that s s2 is the second element, and so on. 331 Counting Methods and the Pigeonhole Principle (A route cannot first meet the top edge in two or more distinct positions.) Notice also that every route meets the top edge somewhere, so ■ Every route is in some class. The negation of proposition p is the proposition not p. Recursively construct an optimal Huffman coding tree using the modified list. Therefore, R−1 is transitive. Find a recurrence relation and initial conditions for the sequence {Sn }. Therefore, (x, z) ∈ τ(R) and τ(R) is transitive. Proof When we sum over the degrees of all the vertices, we count each edge (vi , vj) twice—once when we count it as (vi , vj) in the degree of vi and again when we count it as (vj , vi) in the degree of vj . In how many ways can these books be arranged on a shelf if the two art books are not together? If p and q are false and r is true, the hypotheses are true, but the conclusion is false. Suppose by way of contradiction that w is in T . There are n! permutations. 2n−1 (2n − 1)(2n − 3) · · · 3 C1 (n + 1)n(n − 1) · · · 3 n 1 2 (2n − 1)(2n − 3) · · · 3 = n+1 n(n − 1) · · · 3 · 2 3. Find the maximum possible increase in the flow obtainable by altering the flows in the edges in the path. We then replace the original hypothesis ¬(a ∨ b) by the two hypotheses ¬a and ¬b. This discussion motivates the following definition. This is not to say that this is the only approach possible. In the worst√case (when n is prime and the for loop runs to completion), Algorithm √ 1.8 takes time (n). The first weighing either compares two coins against two coins or one coin against one coin. Let S be a unit square including the interior, as shown in the following figure. Then P(x , y) is false. Similarly, if k = 2, then n = 1, and if k = 3, then n = 4, both of which contradict the fact that n ≥ 5. In Exercises 1–4, use a graph to model the tournament. The algorithm returns result, which is equal to an mod z = 510 mod 21 = 16. Again, arguing as in Example 1.4, we find that there are 224 24-bit strings. Show that an ≤ 3n lg n for n = 1, 2, 3, . Since Y ⊂ Z, there exists z ∈ Z such that z ∈ / Y . In a real problem, an appropriate value for S might be selected by trial and error or the value of S might be selected automatically according to some predetermined criteria. 606 Definition 3.1 A cut (P, P) in G consists of a set P of vertices and the complement P of P, with a ∈ P and z ∈ P. , sn−1 . Since block 1 is marked, by (4.2) (taking n = 1), block 2 is also marked. If f(n) = Cn , it can be shown that g(n) = C Cn if C is not a root of (2.22), g(n) = C nCn if C is a root of multiplicity one, and g(n) = C n2 Cn if C is a root of multiplicity two. Having computed the quotient q, we may compute the remainder as r = n − dq. Suppose that (x, y), (y, z) ∈ R ∩ S. We call the level of the root level 0. It is because this correspondence is one-to-one that the government uses social security numbers as identifiers. What is wrong with the following argument, which supposedly shows that the twelve-coins puzzle requires at least four weighings in the worst case if we begin by weighing four coins against four coins? What is the entry in row a, column d of A5 ? Person 1 gets either order 2, or 3, . Measure the time used by the program for various keys and for various values of n. , n; it is denoted X1 × X2 × · · · × Xn . (In this problem, we extended a sum that ends at index i − 1 to a sum that ends at index i.) Algorithms ■ Don't repeat computations. Partition of X: a collection S of nonempty subsets of X such that every element in X belongs to exactly one member of S 25. The problem may be written + 84F 42EA We begin in the rightmost column by adding F and A. ⎜ ⎜⎝1 0 1 0 0⎠ 0 1 1 0 0 We will show that if A is the adjacency matrix of a simple graph G, the powers of A, A, A2 , A3 , . The runner will stop if the marathon is completed or after three attempts. Give the truth table for the negation of p. if (n > m) { 15. Because p1 and p1 → p2 are true, p2 is true; because p2 and p2 → p3 are true, p3 is true; and so on. 124 5. In how many ways can 10 balls be selected if at least one red ball must be selected? for t. Give an optimal algorithm to solve a variant of the Tower of 47. (The case n = 2 is the Basis Step.) Inductive Step Assume that ak ≤ (3k/2) − 2 for 2 ≤ k < n. Next, at lines 4 –7 we set all labels to null. We must prove that the order in which funnyorder visits the nodes of T is revpost(T). Randomized algorithms can be used to search for non-random goals. Multiple quantifiers such as ∀x∀y are said to be nested quantifiers. Include data sorted in nondecreasing order, data sorted in nonincreasing order, data containing many duplicates, and data in random order. Thus, simply to copy the final value into Result takes time at least (n lg a), which is exponential in the size of n (see Example 2.1). 281 Introduction to Number Theory The following shows the computation of 57229 mod 713: 5722 mod 713 = (572 mod 713)(572 mod 713) mod 713 = 5722 mod 713 = 630 5724 mod 713 = (5722 mod 713)(5722 mod 713) mod 713 = 6302 mod 713 = 472 5728 mod 713 = (5724 mod 713)(5724 mod 713) mod 713 = 4722 mod 713 = 328 57216 mod 713 = (5728 mod 713)(5728 mod 713) mod 713 = 3282 mod 713 = 634 5725 mod 713 = (572 mod 713)(5724 mod 713) mod 713 = 572 · 472 mod 713 = 470 57213 mod 713 =

(5725 mod 713)(5728 mod 713) mod 713 = 470 · 328 mod 713 = 152 57229 mod 713 = (57213 mod 713)(57216 mod 713) mod 713 = 152 · 634 mod 713 = 113. Therefore, x ≤ d1 and x ≤ d2 . Exercises 9 and 10 concern the following variant of the coinweighing problem. To draw the digraph of a relation on a set X, we first draw dots or vertices to represent the elements of X. Show that τ(σ(ρ(R))) is an equivalence relation containing R. In fact, if R = {(1, 2)}, R is transitive. Canceling 2 gives q3 = 4k3 . Use Exercises 34 and 35 to give another proof that |xy| = |x||y| for all real numbers x and y. 75. , 5. 2 + 4 + 8 + 16 + · · · + 2n In Exercises 13–15, select a theta notation for f(n) + g(n). Since a = c and b = d, f is one-to-one. Also, 3 · 5 = 15 is also a common divisor of 30 and 105. Assume that a, b, and c are real numbers. A 6. Simple path 10. ∀x ∃y P(x, y) must be true. Trace Algorithm 4.4 when n = 4 and the missing square is the upper-left corner square. There are two possibilities: 1. [It is not reflexive because, for example, (b, b) ∈ / R. How is "invariant" related to isomorphism? Everyone is taller than everyone. A shipment of 100 compact discs contains five defective discs. {{1}} 76. The first version of the Pigeonhole Principle that we will discuss asserts that if n pigeons fly into k pigeonholes and k < n, some pigeonhole contains at least two pigeons (see Figure 8.1). The choices are made by giving conditions on the attributes (see Example 6.3). 61 Sets and Logic Notes General references on discrete mathematics are [Graham, 1994; Liu, 1985; Tucker]. By the note in the preceding paragraph, we have τ(R1 ∪ D) ⊆ τ(τ(R1) ∪ D). If the cryptosystem is secure, unauthorized persons will be unable to discover the decryption technique, so even if they read the encrypted message, they will be unable to decrypt it. In general, there will be many ways to place data into a binary search tree. This r-permutation can also be constructed 373 Counting Methods and the Pigeonhole Principle by first choosing an r-combination [C(n, r) ways] and then ordering it [r! ways]. If A is the adjacency matrix of the graph of Exercise 5, what does the entry in row v2 and column v3 of A3 represent? ai+j i=0 Bi . Let a, b, and c be integers. It is possible (see Exercise 8) to implement Prim's Algorithm so that only (n2) edges are examined in the worst case. (e) T is acyclic and has n − 1 edges. If we set a = 2 and b = −1 in the Binomial Theorem, we obtain Therefore, 3 3 1 · = = P(S ∩ G), 4 2 8 and S and G are independent. The input must be sorted. In Example 5.11, we showed that an existentially quantified statement was false by proving that a related universally quantified statement was true. What is a sample space? preorder ABCDE 6. Find all players' names and ages. Given that Dale made the call, the table shows that 20 percent of the persons hung up; therefore, P(H | D) = 0.2. Similarly, P(H | R) = 0.55, P(H | L) = 0.3. To compute P(D | H), we use Bayes' Theorem: P(H | D)P(D) P(H | D)P(D) + P(H | R)P(R) + P(H | L)P(L) (0.2)(0.4) = = 0.248. (d) If x is an ancestor of y, y is a descendant of x. Suppose that x, y, and z are vertices in T and that (v0 , v1 , . Show that c > 0. Any algorithm, if there is one, that solves an intractable problem is guaranteed to take a long time to execute in the worst case, even for modest sizes of the input. (f) x + 4 = 6. ¬(∃x P(x)) and ∀x ¬P(x) have the same truth values. Thus n= 56. (b) Give a proof using mathematical induction that your algorithm for part (a) is correct. (Again, we chose b = 6 to avoid a clash with the other numbers already chosen.) Now R = {(5, 2), (6, 3)} and S = {(1, 5), (2, 6)}. If n = 2, there are two paths of length 2 from v1 to v1 : (v1 , v1 , v1) and (v1 , v2 , v1); thus, s2 = f3 . If the initial price is p0 , the manufacturer will be willing to supply the quantity q1 , at time n = 1. Let k ∈ X. A student is selected at random. a (a real number), n (a positive integer) an exp1(a, n) { if (n == 1) return a m = n/2 return exp1(a, m) ∗ exp1(a, n − m) } Let bn be the number of multiplications (line 5) required to compute an . The conclusion expresses a necessary condition. , xn in the domain of discourse. What is wrong with the following argument, which supposedly shows that any relation R on X that is symmetric and transitive is reflexive? In a multiple-choice exam, one question has three choices: A, B, C. What is wrong with the following "proof" that the greedy algorithm is optimal for all amounts of postage for the denominations 1, 5, and 6? A probability function P assigns to each outcome x in a sample space S a number P(x) so that 0 ≤ P(x) ≤ 1, and P(F) = P(F | Ci)P(Ci), i=1 for all x ∈ S P(x) = 1. There are n! permutations of X, so the number of orderings of S is n! . Since 13 is less than or equal to 16, we insert (i.e., copy) 16 to the third index: 8 13 16 20 27 This subsequence is now sorted. First, we choose positions for N, O, and S; then we place the I's and L's. How do we denote X is equal to Y ? Backtracking can also be used to search for Hamiltonian cycles in a graph, to generate permutations, to solve Sudoku puzzles, and to determine whether two graphs are isomorphic. Since walk(n) = walk(n − 1) + walk(n − 2), the algorithm returns the correct value of walk(n). Suppose that E and F are independent, that is, P(E)P(F) = P(E ∩ F). 1 Let us interpret this product. Here the relationship is "gets along with," so we'll put an edge between two vertices (people) if they get along. Postorder is CEDBIJHGFA. Other important bases for number systems in computer science are base 8 or octal and base 16 or hexadecimal (sometimes shortened to hex). Example 2.10 14. The assumption that G is connected can be dropped (see Exercise 9). .), the proof, called an existence proof, consists of showing that there exists at least one x in the domain of discourse that makes the statement true. {x | x2 + x = 2}, {1, −1} 32. For all positive integers n, n and n + 2 are prime. (A ∪ B) ∩ (C − A) 47. C 49. Let x ∈ X ∪ Y . These start-up values are called initial conditions. The assumptions imply exponential population growth. Since U also satisfies (2.13) and (2.15), it follows that Un = an , n = 0, 1, . What does it mean to "encrypt a message"? The complexity of the graph isomorphism problem is discussed in [K¨obler]. Thus an equivalent way to represent the second interpretation is ¬(∀x P(x) → Q(x)). 83 Proofs universal set U. Show that if the available denominations are 1, 5, and 25 cents, the algorithm produces the fewest number of stamps to make any given amount of postage. (A + B) ∗ (C − D) 7. Suppose that the proposition ¬(∀x P(x)) is true. Basis Step For n = 1, we have t1 = 1; thus, the Basis Step is verified. 36. You play football and you miss the midterm exam. Example 3.8 If k is a positive integer and, as in Example 3.7, we replace each integer 1, 2, . Input: s, n 20. Next, just as in proof by induction, look for smaller instances within instance n. We turn next to a special class of recurrence relations. Since 2m−1 < n ≤ 2m , m − 1 < lg n ≤ m. If it is not raining (i.e., p is false) and it is not cold (i.e., q is also false), then we would consider the proposition, p ∨ q: It is raining or it is cold, to be false (i.e., p ∨ q is false). At the present time, no sharper lower bound is known. , sr , the previous r-combination (The last r-combination precedes the first r-combination.) prev comb(s, r, n) { s0 = n // dummy value // find rightmost element at least // 2 larger than its left neighbor m=r // loop test always fails if m = 1 while (sm − sm−1 == 1) m=m−1 sm = sm − 1 if (m == 1 ∧ s1 == 0) m=0 // set elements to right of index m to max values for j = m + 1 to r sj = n + j − r } 12. if (j == i + 1) { if (si > sj) { large = si small = sj } else { small = si large = sj } return } Let bn be the number of comparisons (lines 6b, 10, and 14) for an input of size n. n1 ! n2 ! · · · nt ! Example 3.3 In how many ways can eight distinct books be divided among three students if Bill gets four books and Shizuo and Marian each get two books? Next, we label the rows and columns of a matrix with the ordered vertices. Digraph of a binary relation Functions, Sequences, and Relations 34. w6 T2 v2 v3 T1 w5 v1 w 11 w 9 w 10 w 12 v 11 v 12 v6 12. un = 3 + un−1 , n ≥ 2. Let x ∈ A ∪ ∅. We'll have a solution, provided that the Dn−1 term appears in the remaining case: Person 1 gets order 2, but person 2 does not get order 1. The for loop terminates when i = n. (4.2) If I holds, the shorter word precedes the longer. Unlike the situation in Example 1.5, there are efficient algorithms for finding shortest paths. P(2) 14. Problem-Solving Tips To set up a recurrence relation, first define, say An , to be the desired quantity. Example 1.13 If A = {a, b, c}, the members of P(A) are ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}. The sequence is divided into two nearly equal parts (line 4). In Example 4.1 a matching consists of finding jobs for qualified persons.Amaximal matching finds jobs for the maximum number of persons. However, this is surely not what Shakespeare intended. Since 129 mod 10 = 9, the check character for the ISBN 978-1-59448-950-1 is 10 − 9 = 1. In how many ways can 10 balls be selected if twice as many red balls as green balls must be selected? Add an edge incident on this vertex to the tree, and add the additional vertex v incident on this edge. flow into x will still equal the flow out of x. Therefore, if the computer has one gigabyte of memory, then the sonics will be impressive. ∀x∀y P(x, y) 91. (a, d, c, d, e) 4. Inductive Step Assume that Gn−1 is a Gray code. g(x) = −x 8. Give an example of a graph that has neither a Hamiltonian cycle nor an Euler cycle. Proof Let X = {a1 , . The edges (a, b), (a, g), (b, c), and (c, k) must be in H since each vertex in a Hamiltonian cycle has degree 2. Base the terminating condition on the fact that the last r-combination has every element si equal to its maximum value. Graph Theory 8. (The case a2 > a4 is symmetric and for this reason that part of the algorithm is omitted.) At this point we know that a1 < a2 < a4 and a3 < a4 . p ∨ ¬r 2. Other techniques have been combined with alpha-beta pruning to facilitate the search of a game tree. The outer face D is considered to be bounded by the cycle (1, 2, 3, 4, 6, 1). The test is, whenever the ijth entry is 1, i = j, then the jith entry is not 1. Let g(x) = log2 x and h(x) = x2 + 2. We see that (1.2) and (1.3) are equivalent to instructions 1 and 2. w2 w3 v 11 w 7 v 10 w8 w 11 w 10 w9 v5 v7 v6 v8 w8 w1 v4 T1 w4 w5 w6 T2 13. The binary search tree T of Figure 5.4 was constructed in the following way. a d g h i 29. Then w(x, y) ≥ w(j, k). How many binary operators are there on {1, 2, . Yes Yes No Polish ? If E1 and E2 are events, what does the event E1 ∪ E2 represent? What is a complete matching? (Do not include any managers.) 6. The directed edges represent subpipelines of the system and show the direction the oil can flow. For an arrow diagram to be a function, Definition 1.1 requires that there is exactly one arrow from each element in the domain. At most one token can occupy a point of the star or the putahi. During the period 7:00 a.m. to 8:00 a.m., the average trip times are 2. V v1 a v2 v3 We will estimate the number of edges of each type. Since |A| = 3 and |P(A)| = 23 = 8, |P(A)×A| = 8 · 3 = 24. Example 1.13 We show how the proof of Theorem 1.12 produces a prime larger than 11. † 71. In general, ri = ri+1 qi+2 + ri+2 . If f ◦ g is one-to-one, then g is one-to-one. A rooted tree with four vertices in which the root has degree 2 must be isomorphic to the tree in Figure 8.9(b). Since R is symmetric, (b, a) ∈ R. An algorithm is made up of one or more pseudocode functions. 9an = 6an−1 − an−2 ; (n − 1)xn − nxn−1 + 1 . Explain how files and folders in a computer system are organized into a rooted tree structure. tr , find the rightmost element sm that is not at its maximum value. (a) Prove that if v and w are nonadjacent vertices in G, then δ(v) = δ(w). We conclude that X is a subset of Z and we write X ⊆ Z. Example 1.27 If A is a set of appetizers, E is a set of entrees, and D is a set of desserts, the Cartesian product A × E × D lists all possible dinners consisting of one appetizer, one entree, and one dessert. Since d divides m, there exists q such that m = dq. Since A = A and B = B, A ⊆ B. No x satisfies P(x). The absolute value |x| of a real number x is defined to be x if x is positive or 0 and −x otherwise. 38 Sets and Logic Example 5.6 Consider the universally quantified statement ∀x(x2 − 1 > 0). The Hamiltonian cycle is shown in Figure 3.3. The graph does not have an Euler cycle, because all of the vertices have odd degree. Exercise 12 23. To start, we create a vertex and place the first word old in this vertex. Section 8 43. 566 Trees A x C y B 2 D 3 3 2 6 4 1 ? We let w(i, j) denote the weight of edge (i, j). In reading articles and books about graphs, it is necessary to check on the definitions being used. We then define a function f on the vertices of M by f(v) = vx vy . We call q the quotient and d a divisor or factor of n. B ∩ C 3. Therefore, the total number of possibilities is 6 · 5 · 4 = 120. It is denoted p ∧ q. (This exercise is relevant to the last paragraph of the proof of Theorem 2.3.) 33. To prove that a relation R on a set X is transitive, show that for all x, y, z ∈ X, if (x, y) and (y, z) are in R, then (x, z) is in R. If j = rand(2, 5) = 5, after the swap we have 5 21 17 23 ↑ i 9 ↑ j Next, i = 3. , xn) where xi ∈ Xi for i = 1, . Find the diameter of the graph of Figure 2.10. The decision tree analysis shows that any algorithm requires at least five comparisons in the worst case to sort four items. Once Egbert has been assigned and the highest remaining office filled, there are four ways to fill the last office. This string is written baac. (We could use either of the parallel edges incident on a and g.) Repeat this procedure with the vertices on level 1 by examining each in order: b: Include (b, d). It is denoted p ↔ q. Therefore, for all i = 1 to n, pi does not divide m. Would you prefer this method or the preceding algorithm? We may then use addition to conclude ¬p ∨ s. In illustrating the algorithm, we will circle vertices having permanent labels. ■ After developing an algorithm, take a close look at the pseudocode to see where it can be improved. The relation {(1, 2)} is an antisymmetric relation on {1, 2}. In this solution, the induction is over the set X of odd integers n > 5, where 3 divides n2 − 1. An algorithm to solve the puzzle is given in Figure 7.3 as a decision tree. In Exercises 24–31, determine whether each relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order. A collision occurs for a hash function H if H(x) = H(y) but x = y. The choices are made by giving conditions on the attributes. Our last example is of this form. Consider a graph where the vertices are the persons and an edge connects two vertices (people) if the people get along. Thus p and q are both even, which contradicts √ our assumption that p and q are not both even. What is an axiom? Use depth-first search (Algorithm 3.7) with the vertex ordering dhcbefag to find a spanning tree for graph G of Figure 3.1. In Exercises 7–9, find a spanning tree for each graph. We also have 2m ≤ n − 1 < 2m+1 and 2m−1 ≤ (n − 1)/2 < 2m . Definition 2.20 Example 2.21 A string over X, where X is a finite set, is a finite sequence of elements from X. The angle of a point is the angle from the positive x-axis counterclockwise to the ray from the origin to the point. The graph T cannot contain any loops because loops are cycles and T is acyclic. Our algorithms list permutations and combinations in lexicographic order. Now suppose that 2i+1 persons are arranged in a circle. Five vertices each of degree 3 d v5 v3 v6 v9 13. Input: A connected graph G with vertices ordered v1 , v2 , ..., vn Output: A spanning tree T bfs(V, E) { // V = vertices ordered v1 , . By the inductive assumption, within the first n − 1 elements all of the zeros precede all of the ones. If n = 1, A1 is simply A. (a) Derive a recurrence relation for pm . , n − 1, Ti is contained in a minimal spanning tree. 0)⎢ ⎢0⎥ ⎝1⎠d 1 471 Graph Theory The only way a nonzero product appears in this sum is if both entries to be multiplied are 1. How many are greater than 72? Recall that the first of De Morgan's law for logic states that the propositions ¬(P1 ∨ P2 ∨ · · · ∨ Pn) and ¬P1 ∧ ¬P2 ∧ · · · ∧ ¬Pn have the same truth values. (b) Let I(x) be the propositional function "x is an integer" and let P(x) be the propositional function "x is a positive 71. True 28. The algorithm then computes tn−1 + tn−2 + tn−3 . If the join condition is satisfied, the tuples are combined to form a new tuple. If n > 1, an is at most the sum of the numbers of comparisons in the worst case resulting from the recursive calls at lines 7 and 8, and the number of comparisons in the worst case required by merge at line 10. We must find the value of "to be determined." Now 5n+1 − 1 = 5 · 5n − 1 = 4 · 5n + 1 · 5n − 1. No. Let f(x) = x and g(x) = x2 . What does it mean to sort a sequence? Example 1.1 Euclidean geometry furnishes an example of a mathematical system. Dijkstra's shortest-path algorithm Section 5 44. , 2n − 1, and (s2n , s1) are distinct. Since the negation applies to more than one term, we use the first of De Morgan's laws ¬(a ∨ b) ≡ ¬a ¬b, ¬(ab) ≡ ¬a ∨ ¬b (3.3) to obtain an equivalent expression with the negation applying to single variables: ¬(a ∨ b) ≡ ¬a ¬b. X × Y × Y Less Filling 69. Since g(e) is incident on f(v) in G2 , by Definition 6.1, e is incident on v in G1 . We have already noted that 4 is an upper bound for X. If we let a = 2k, then m = a2 . We will show that if you can always win regardless of how Farley chooses values for the variables, the quantified statement is true, but if Farley can choose values for the variables so that you cannot win, the quantified statement is false. Implement the method of playing tic-tac-toe in Example 9.1 as a program. 4 57. Define a function S from P(X) to the set of bit strings of length 3 as follows. True 27. One would expect that if agreement has not been reached on the definition of "graph," most other terms in graph theory would also not have standard definitions. (No route is counted twice since the classes do not overlap, and every route is counted once since each route is in some class.) Evidently, the equation we're supposed to prove results from equating the sum of the number of routes in each class to the total number of routes. No 49. If n = 1, we proceed to line 4 since n = 0. 4 6 6 6 4 4 3 2 5 16 6. When you sing, my ears hurt. For example, {x | x is a positive, real number} defines the set consisting of the positive, real numbers. At line 3, b and large are compared. Describe input for which an = an/2 + a(n+1)/2 + n − 1. 1 + 1 19. |4| < 3 if −3 < 4 < 3. The options available on a particular model of a car are five interior colors, six exterior colors, two types of seats, three types of engines, and three types of radios. Then the number of orderings is n! . If x ∈ A, then x ∈ B, which is not the case. We write 25 in the region representing CALC ∩ PSYCH ∩ COMPSCI. (d) Show that S3,2 = 3. Example 1.18 Let the universal set be Z. In this graph, the programs are grouped into three classes: {1, 3, 5}, {2}, and {4}. A universally quantified statement is a statement of the form for all x in the domain of discourse, P(x). By Definition 6.1, if graphs G1 and G2 are isomorphic, there are one-to-one, onto functions from the edges (respectively, vertices) of G1 to the edges (respectively, vertices) of G2 . The adjacency matrix for this graph is b a b c d e c d e Figure 5.1 The graph for Example 5.1. Example 5.2 a Notice that we can obtain the degree of a vertex v in a graph G by summing row v or column v in G's adjacency matrix. Next, we move the remaining disk on peg 1 to peg 3. Using iteration, we have 2(2n − 1) Cn−1 n+1 2(2n − 1) 2(2n − 3) Cn−2 = n+1 n 22 (2n − 1)(2n − 3) Cn−2 = (n + 1)n Cn = Trees = .. To formally verify that this relation satisfies Definition 3.17, we can list all pairs of pairs of the form (x, y) and (y, z) in R and then verify that in every case, (x, z) ∈ R: Pairs of Form (x, y) (y, z) (x, z) Pairs of Form (x, y) (y, z) (x, z) (1, 1) (1, 1) (1, 1) (1, 1) (1, 1) (1, 1) (1, 2) (1, 2) (1, 2) (1, 3) (1, 3) (1, 4) (1, 1) (1, 2) (1, 3) (1, 4) (1, 2) (1, 3) (1, 4) (1, 3) (1, 4) (1, 3) (1, 4) (1, 4) (2, 2) (2, 2) (2, 2) (2, 2) (2, 3) (2, 3) (2, 4) (3, 3) (3, 3) (3, 4) (4, 4) (2, 2) (2, 3) (2, 4) (2, 3) (2, 4) (2, 4) (3, 3) (3, 4) (3, 4) (4, 4) (1, 1) (1, 2) (1, 3) (1, 4) (2, 2) (2, 3) (2, 4) (3, 3) (3, 4) (4, 4) (2, 2) (2, 3) (2, 4) (3, 3) (3, 4) (4, 4) (3, 3) (3, 4) (4, 4) (4, 4) Actually, some of the entries in the preceding table were unnecessary. What is the value of p0 ? j=3 i=4 4. Thus, the number of dinners is 2 · 3 · 4 = 24. Basis Steps (n = 1, 2, 3) If n = 1, 2, 3, at lines 1 and 2 we return the correct value, 1. Since si = sj , either si = A or sj = A. Thus in the worst case, the time will be equal to at least lg t = lg(n + 1). Recurrence Relations 8. Use proof by cases to prove that max{x, y} + min{x, y} = x + y for all real numbers x and y. Divide the number to be converted to binary by 2. What is the sample space for the experiment of Exercise 8? At lines 15 and 16, vertex 4 is added to the minimal spanning tree and edge (3, 4) is added to E. T is connected and has n − 1 edges. Suppose that we have a 2 × n rectangular board divided into 2n squares. † The real numbers can be constructed by starting with a more primitive notion such as "set" or "integer," or they can be obtained by stating properties (axioms) they are assumed to obey. x2 Solving for x, we find 1 x = ±√ . Algorithm 4.1, due to E. Write a program that checks whether a proposed path is a Hamiltonian path. Expand the expression (s−r)4 using the Binomial Theorem. Therefore, n i=0 Example 2.18 ir n−i = n+1 (j − 1)r n−j+1 . We begin by setting f0 (2) = 2 and g0 (2) = 22 . In Exercises 13–15, G is a bipartite graph, A is the matrix of Exercise 12, and F is a flow in the associated matching network. Example 2.15 Let a be the sequence defined by an = 2n, n ≥ 1. Inorder traversal 35. What is ∩ D? g(n) = (n2) g(n) = (n lg n) g(n) = (n5/2) In Exercises 16–25, select a theta notation from among 6. To put it another way, after a route meets the circled dot, there is only one way to finish the trip. 15 68. In this chapter we develop several tools for counting. We leave the proof of this last statement as an exercise. Use proof by cases to prove that |x| = sgn(x)x for every real number x. We look at subclasses of trees (e.g., rooted trees and binary trees) and many applications of trees (e.g., spanning trees, decision trees, and game trees). Repeat Exercise 21 with "adjacency" replaced by "incidence." 23. , ak−1 = Ck−1 , uniquely defines a sequence a0 , a1 , . If we traverse the tree of Figure 6.7 using postorder, we obtain AB+C∗DE/− . Thus a is chosen the first time we arrive at line 9. A graph has a path with no repeated edges from v to w (v = w) containing all the edges and vertices if and only if it is connected and v and w are the only vertices having odd degree. 71. Therefore, for all n ≥ N, |f(n)| < |g(n)| and f(n) = O(g(n)). If P(di) is true for every di , the condition ¬P(di) in the if statement is always false. Every pair of vertices is connected by an edge. for i = 1 to n for j = 1 to n for k = 1 to i x=x+1 for i = 1 to n for j = 1 to i for k = 1 to j x=x+1 24. Example 4.4 Represent the argument If 2 = 3, then I ate my hat. ∀x (P(x) → Q(x)) 36. For example, Gray codes have been used in converting analog information to digital form (see [Deo]). Example 6.12 Consider the statement ∃x∃y((x > 1) ∧ (y > 1) ∧ (xy = 7)). The border consists of 1's, and any interior value is the sum of the two numbers above it: = C(n, k)an+1−k bk + C(n, n)a0 bn+1 k=0 1 n C(n, k)an−k bk is C(k + 2 + n − k − 1, n − k) = C(n + 1, k + 1). Thus |R(V ∗)| = |W1 | + |W2 | < |V ∗ |, which contradicts (4.1). Model the system of Exercise 4 as a network assuming that well w1 can pump at most 2 units, well w2 at most 4 units, and well w3 at most 7 units. i=1 Because each item to be classified belongs to one of the classes, we have F = (F ∩ C1) ∪ (F ∩ C2) ∪ · · · ∪ (F ∩ Cn). Example 4.1 Suppose that four persons A, B, C, and D apply for five jobs J1 , J2 , J3 , J4 , and J5 . Write a program that, given the adjacency matrix of a tree and a vertex v, draws the tree rooted at v using a computer graphics display. Let x and y be rational numbers. ¬q ∨ ¬r min{min{a, b}, c} = min{a, c} = a = min{a, b} min{min{a, b}, c} = min{a, c} = min{a, min{b, c}}. Let H1 and H2 be two (n − 1)-cubes whose vertices are labeled in binary 0, . gcd(3, 2) = gcd(2, 1) = gcd(1, 0) = 1, s = 2 36. (In Figure 6.2, we may take p = 6, q = 4, i = 3, and j = 2. Valid 4. In the minimax procedure, values are first assigned to the ter- Section 9 6 1. Show that there is an edge y in T but not in T such that (T − {x}) ∪ {y} and (T − {y}) ∪ {x} are spanning trees of G. Find v3 . 512 1 1 0 0 7 1 0 8 12 Trees 1 1 1 1 0 1 0 8 12 ! @ Figure 1.14 The tree that replaces the vertex labeled 20 in Figure 1.13. From pn−1 = 12 pn + 12 pn−2 , we obtain pn = 2pn−1 − pn−2 . [Part (b)] Torus σ(R1) = {(1, 1), (2, 1), (1, 2), (3, 4), (4, 3), (4, 2), (2, 4)} 53. (Since X is nonempty, X contains an integer n. If we square both sides of this last equation, we obtain n2 = (2k + 1)2 = 4k2 + 4k + 1 = 2(2k2 + 2k) + 1. Continuing in this way, we eventually produce an acyclic, connected subgraph T. Thus n= together with the initial condition A0 = d defines the sequence {An }. Consider K5 with the weight of every edge equal to 1. Assume that f(n) ≥ 1 and g(n) ≥ 1 for all n = 1, 2, . Of course, Table 3.1 is really just a set of ordered pairs. Show, by giving a proof by contradiction, that if 40 coins are distributed among nine bags so that each bag contains at least one coin, at least two bags contain the same number of coins. Exercises 18 and 19 refer to the sequence c1 , c2 , . Explain what it means for two events to be "independent." 17. No bad movie is short enough. Since limn→∞ |f(n)|/|g(n)| = d > 0, taking ε = d/2, there exists N such that |f(n)| < d/2, − d |g(n)| for all n ≥ N. 408 Recurrence Relations Attacking the Problem Before attacking the problem, consider what is required for part (a). Inductive Step Suppose that 8 divides 3n + 7n − 2. Next we count the number of arrangements in which P is not alone. , sn), n, and key Output: i (the index of the last occurrence of key in s, or 0 if key is not in s reverse linear search(s, n, key) { i=n while (i ≥ 1) { if (si == key) return i Algorithms i=i−1 } return 0 } 17. Sn = Sn−1 + Sn−3 + Sn−4 + Sn−5 + · · · + S1 + 3. Let P = {p1 , p2 , p3 , p4 , p5 } be a set of five (distinct) points in the ordinary Euclidean plane each of which has integer coordinates. Suppose that n is odd. The fundamental cycle matrix of a graph G has its rows indexed by the fundamental cycles of G relative to a spanning tree T for G and b e1 in G of weight k by k unweighted edges in series: 26. The algorithm is not correct. We show that G has either one or two components. (The process is analogous to changing the variable in an integral in calculus.) Example 2.17 Changing the Index and Limits in a Sum Rewrite the sum n ir n−i , i=0 replacing the index i by j, where i = j − 1. {(x, y) | x and y have the same color hair} In Exercises 15–20, list the members of the equivalence relation on {1, 2, 3, 4} defined (as in Theorem 4.1) by the given partition. 543 Trees END START START END A A C D G C J I Figure 6.5 Preorder traversal. If S = {A1 , A2 , . In this case, p1 , p2 , . It contains 2 1 3 R G 2 G2 2 4 B 1 7. How many do not contain the digit 0? (For Exercise 37) ∃x∃y¬P(x, y) or ∃x∃y(x < y) 42. First Form: If n pigeons fly into k pigeonholes and k < n, some pigeonhole contains at least two pigeons. Since going from an (n − 1)element set to an n-element set doubles the number of subsets, we obtain the recurrence relation Sn = 2Sn−1 . More precisely, a property P is an invariant if whenever G1 and G2 are isomorphic graphs: If G1 has property P, G2 also has property P. n di 10i 2. 450 Graph Theory e e v1 e1 e v1 e v2 1 (a) v2 v3 e2 v1 C P e1 v2 v3 C P e2 v3 e2 v (b) (c) Figure 2.9 The proof of Theorem 2.18. If α is a bit string, let C(α) be the maximum number of consecutive 0's in α. A direct proof by resolution proceeds by repeatedly applying (3.1) to pairs of statements to derive new statements until the conclusion is derived. We write on each edge the time to move the drill b 8 a 4 d 2 c 6 3 5 4 Figure 1.6 A sheet of metal with holes for bolts. The path must begin with the loop (v1 , v1) or the edge (v1 , v2). Imagine trying to find the phone number of a particular individual in the New York City telephone book if the names were not sorted! Many sorting algorithms have been devised (see, e.g., [Knuth, 1998b]). Indeed, a graph is connected if and only if it has exactly one component. If pi appears in a and pj appears in b, include pmax(i,j) in the prime factorization of the least common multiple. In how many ways can we select 10 books? A ⊆ B 3. If you succeed, you are finished—the statement is true and you proved it! If your proof breaks down, look carefully at the point where it fails. Prove that S is onto. Which rule of inference is used in the following argument? 4 Sets and Logic Solving for x, we obtain x = 1 or x = −2. [Photo by the author.] We model the situation as a graph G with seven vertices labeled 0, 1, . Give an algorithm to execute a preorder traversal. , n}. New Atlantic does not build a new stadium. Since n = 1 > 0, the body of the while loop executes. Thus an = (logm n). Solve the recurrence relation (3.12) in case n is a power of 2 to obtain n = 1, 2, 4, . k=1 Algorithm 3.19, which computes the matrix product, is a direct translation of the preceding definition. The n-cube may also be described recursively. with a1 ; but if a5 > a2 , we next compare a5 with a4 . Even if for any input of size n, algorithm A requires exactly C1 n time units and algorithm B requires exactly C2 n2 time units, for certain sizes of inputs algorithm B may be superior. Prove that cn ≤ 4(n − 1)2 for all n ≥ 1. Let Y ⊆ X. [Inductive Step only] Notice that C(n + 1, i)−1 + C(n + 1, i + 1)−1 = n+2 C(n, i)−1 . Since "110" does not occur in t at index 1, the algorithm next tests whether "110" occurs in t at index 2. Prove that X is bounded above and lub X ≤ lub Y . Proof Let d, d1 , d2 , and x be arbitrary real numbers, and suppose that d = min{d1 , d2 } and x ≤ d. 110110101 + 1101101 49. Example 3.15 If a relation has no members of the form (x, y), x = y, we see that the equivalent characterization of "antisymmetric" for all x, y ∈ X, if x = y, then (x, y) ∈ R or (y, x) ∈ R (see Example 3.14) is trivially true (since the hypothesis x = y is always false). x1 ≥ 0, x2 > 0, x3 = 1 26. Number the cubes of an Instant Insanity puzzle 1, 2, 3, and 4. For example, if x = 3, then P(3) and Q(3) are both true (3 > 0 and 32 > 0 are both true), and, by Definition 3.3, (3.4) is true. Now n = 2k + 1. Now (vi , vi+1 , . (v1 , v2 , v3 , v4 , v5 , v7 , v6 , v1) (000, 001, 011, 010, 110, 111, 101, 100, 000) A Hamiltonian cycle would have seven edges. Draw all nonisomorphic rooted trees having three vertices. By definition, when a randomized algorithm executes, at some points it makes random choices. Uranus Aphrodite Eros Kronos Atlas Prometheus Zeus Poseidon Hades Apollo Athena Hermes Ares Heracles Figure 2.1 A portion of the family tree of ancient Greek gods. Preorder traversal 34. 1 1 1 + + ··· + ; 1·2 2·3 n(n + 1) 1 1 · 3 · 5 · · · (2n − 1) ≤ √ , n = 1, 2, , n+1} containing an even number of elements is 2n . Let T be a tree with five vertices and suppose that T has a vertex v of degree 4. If s mod 10 = 0, the check character is 0; otherwise, the check character is 10 − (s mod 10). There are no loops. Show that a directed Euler cycle in G corresponds to a de Bruijn sequence. Counting Methods and the Pigeonhole Principle Six numbers among 52 can be selected in C(52, 6) ways. Problem-Solving Tips It is worth reviewing the Problem-Solving Tips of Section 1. Explain how the divide-and-conquer technique works. (i) Find a formula for Sn,n−2 , n ≥ 3, and prove it. We have proved that f is a bijection. Converse: q → p; if 9 > 12, then 4 < 6; true. Write a backtracking algorithm that outputs all permutations of 1, 2, . The outcomes that satisfy E given that F occurred are precisely the outcomes in E ∩ F . If the graph has an Euler cycle, exhibit one. Thus, exists dj(i) returns true for every i. The next examples are intended to illustrate this thesis. The members of C1 are V, X, and Y ; the members of C2 are X and Z; the members of C3 are V, Y, and Z; and the members of C4 are V, W, X, and Z. Now mn = (2k)(2k) = (2k)2 . The subgraph G of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the component of G containing v. (n2) A and B (n × n matrices) and n true (if A = B); false (if A = B) equal matrices(A, B, n) { for i = 1 to n for j = 1 to n if (Aij ¬ = Bij) return false return true } The worst-case time is (n2). 125 Proofs Using the distributive law and observing that (A ∩ B) ∩ (A ∩ C) = A ∩ (B ∩ C), we find that (A ∩ B) (A ∩ C) = [(A ∩ B) ∪ (A ∩ C)] − [(A ∩ B) ∩ (A ∩ C)] = [A ∩ (B ∪ C)] − [A ∩ (B ∩ C)]. A series reduction consists of deleting the vertex v from the graph G and replacing the edges (v, v1) and (v, v2) by the edge (v1 , v2). Now n + 1 = 2k + 1, which is odd. If T1 and T2 are isomorphic as rooted trees, then T1 and T2 are isomorphic as free trees. Therefore f ◦ g is onto. 1 · 48 (The four aces can be chosen in one way and the fifth card can be chosen in 48 ways.) 46. Invariant Section 7 51. 5 b e 2 Fa,w1 = 1, FcA = 2, Fw2 ,b = 3, FBz = 1, Fdc = 3. If {X1 , . The recurrence relation is dn = 100n + 1.2dn−1 , n > 0. Since n = 2, we proceed immediately to line 6, where we divide the board into four 4 × 4 boards. ■ There is a subset P of real numbers satisfying (a) If x and y are in P, then x + y and xy are in P. Give an upper bound for the number of modulus operations required. Write the negation of each proposition in Exercises 35–42 symbolically and in words. Thus the loop does not execute again. Furthermore, unlike the situation for Euler cycles (see Theorems 2.17 and 2.18), no easily verified necessary and sufficient conditions are known for the existence of a Hamiltonian cycle in a graph. q : It is raining. 13 + 23 + · · · + n3 11. In ordinary language, propositions being combined (e.g., p and q combined to give the proposition p ∨ q) are normally related; but in logic, these propositions are not required to refer to the same subject matter. Therefore, g divides a. Since each edge is incident on a vertex in V and a vertex in W, T is a bipartite graph. Show that ρ(R) is reflexive. Figure 1.3 The arrow diagram of Given a function f from X to Y , according to Definition 1.1, for each element x the set in Example 1.5, which is not in the domain X, there is exactly one y in the codomain Y with (x, y) ∈ f . Connect a and b, b and c, c and d, d and e, and a and e." The graphs produced are shown in Figure 6.1. Surely these figures define the same graph even though they appear dissimilar. Since recursion elegantly handles such a backward computation, we will write a recursive algorithm to compute s and t. The diameter of G is 70. The isomorphism of Example 8.5 is not unique. All accountants own Porsches. To obtain the names, we need only project on the attribute Name. What is a beta value? Since y= 0, m2 = 0. Showing that P(x) is true for a particular value x does not prove that ∀x P(x) is true. We will show that merge sort has worst-case run time (n lg n), so for large input, merge sort is much faster than selection sort (Algorithm 3.1), which has worst-case run time (n2). Since z is not labeled and U is not empty, we move to line 14, where we choose a vertex in U. The following pseudocode determines whether ∀x∃y P(x, y) is true or false: for i = 1 to n if (¬ exists dj(i)) return false return true exists dj(i) { for j = 1 to n if (P(di , dj)) return true return false } If for each di , there exists dj such that P(di , dj) is true, then for each i, P(di , dj) is true for some j. Basis Step (x = 2) a n See Exercise 47. ■ Output The algorithm produces output. Then T ∗ is a full binary tree. Find real numbers x and y satisfying xy = xy − 1. Notice that in Example 1.4, the flow out of the source a, Fab + Fad , 4, 2 5, 3 d 2, 2 e Figure 1.2 Flow in a network. Then √ 3 there exist integers p and q such that 2 = p/q. Write an algorithm that receives as input the matrix of a relation R and tests whether R is antisymmetric. Example 1.4 Since the graph of Figure 1.2 has neither parallel edges nor loops, it is a simple graph. How many strings can be formed by ordering the letters SALESPERSONS if no two S's are consecutive? The rules are the same and, again, if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false. Consider block 2. How is the Fibonacci sequence defined? Find the descendants of Zeus. We show that Algorithm 8.13 is a linear-time algorithm in the worst case. After selection of the chairperson and secretary, the treasurer can be selected in four ways. Prove that 18. Those working on the four-color problem in recent years had one advantage their predecessors did not—the use of fast electronic computers. , xn 4. In general, the entry in row x and column y of the matrix A2 is the number of paths of length 2 from vertex x to vertex y. (b) Explain in words what it means for two sequences in X to be equivalent under the relation R. (2.5) Notice that a linear homogeneous recurrence relation of order k with constant coefficients (2.5), together with the k initial conditions a0 = C0 , a1 = C1 , . T1 and T2 as in Exercise 9 v6 11. 2e = 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5, so e = 14. 108 Proofs A loop invariant is a statement about program variables that is true just before a loop begins executing and is also true after each iteration of the loop. 90 Proofs Example 2.9 Let A, B, and C be sets. Define decreasing sequence. Compare these results with the theoretical estimate for the worst-case time (lg n). It is neither reflexive nor symmetric. f(x) = x2 − x Determine whether each function in Exercises 10–15 is one-to-one, onto, or both. Explain a method of verifying that sets X and Y are equal. We construct a truth table for all the propositions involved: p q p→q ¬q ¬p T T F F T F T F T F T F T T F T F T F F T F F T T We observe that whenever the hypotheses p → q and ¬q are true, the conclusion ¬p is also true; therefore, the argument is valid. How do we denote an equivalence class? If x ∈ A, since A ∩ B = A, again x ∈ B. To prove that the universally quantified statement ∀x P(x) is true, show that for every x in the domain of discourse, the proposition P(x) is true. By the inductive assumption, it takes t − 1 multiplications to compute a1 · · · at and n − t − 1 multiplications to compute at+1 · · · an , regardless of how the parentheses are inserted. At line 4, the function correctly computes the value of 2!: n · (n − 1)! = 2 · 1! = 2 · 1 = 2. Modify Algorithm 4.14 so that line 5 5. Now suppose that the result holds for a connected, acyclic graph with n vertices. 8 9 6 10 an Euler cycle is (10, 9, 6, 5, 9, 8, 5, 4, 8, 7, 4, 2, 5, 3, 2, 1, 3, 6, 10). Now the negation of q is ¬q : X ∩ Y = ∅, and, using De Morgan's law (roughly, negating ∀ results in ∃), the negation of p is ¬p : CX∪Y (x) = CX (x) + CY (x) for at least one x ∈ U. Example 5.5 When we divide n = 74 by d = 13 5 13)74 65 9 we obtain the quotient q = 5 and the remainder r = 9. If x = 2, then x2 + x − 6 = 22 + 2 − 6 = 0. The composition of R1 and R2 , denoted R2 ◦ R1 , is the relation from X to Z defined by R2 ◦ R1 = {(x, z) | (x, y) ∈ R1 and (y, z) ∈ R2 for some y ∈ Y }. x, y; a, b, c. Since lg n < n for all n ≥ 1 (see Figure 3.1), 2n + 3 lg n < 2n + 3n = 5n 222 for all n ≥ 1. Checking that there is no solution to the twoor three-queens problem (see Exercise 10) is straightforward. Let p : I study hard. Algorithm 4.6 5 j 3. ■ If α ∈ L and β ∈ L, then αβ ∈ L. Therefore, C1 C3 |h(n)| ≤ C1 |g(n)| ≤ |f(n)| ≤ C2 |g(n)| ≤ C2 C4 |h(n)| for all but finitely many positive integers n. C(n − 1, 2j − 1)E2j−1 En−2j . Show, by giving a proof by contradiction, that if four teams play seven games, some pair of teams plays at least two times. In this case, n − m + 1 comparisons are made. See Figure 1.9. 14. In determining the postage for the amount n, the greedy algorithm will first choose a 6-cent stamp and then stamps for n − 6 for a total of k + 1 stamps These k + 1 stamps must be optimal or otherwise n − 6 would use less than k stamps. Distinguish between the terms axiom and definition. Example 1.41 Given g = {(1, a), (2, a), (3, c)}, a function from X = {1, 2, 3} to Y = {a, b, c}, and f = {(a, y), (b, x), (c, z)}, a function from Y to Z = {x, y, z}, the composition function from X to Z is the function f ◦ g = {(1, y), (2, y), (3, z)}. Exercises 14–21 refer to a modified version of Instant Insanity where a solution is defined to be a stacking that, when viewed from the front, back, left, or right, shows one color. For example, given the values a = 1, b = 5, c = 3, at line 2, large will be set to 5 regardless of who executes the algorithm. t4 = 3, t5 = 5 15. A counterexample to the statement ∀x P(x) is a value of x for which P(x) is false. Parent Ancestor Child Descendant Sibling Terminal vertex Internal vertex Subtree Acyclic graph Alternative characterizations of trees (Theorem 2.3) Section 3 17. 5! 4. The ordered pair (a, b) can be defined in terms of sets as (a, b) = {{a}, {a, b}}. x is set to x ∗ x mod z = 9 mod 11 = 9, and n is set to 7. We have proved the Basis Step. Since MORE is less than WORD, we go to the left child. 253 Algorithms 10. Verify the ISBN check character for this book. The entry for row v and column e is 1 if e is incident on v and 0 otherwise. 50 6. Since polyominoes were introduced by Solomon W. Example 2.8 The sequence ai = 1 , i i ≥ 1, is decreasing and nonincreasing. The method used in Example 3.4 can be used to derive a general result. Find a Hamiltonian cycle in GK6 (see Example 3.9). Now suppose we can show that for all n, if equation n is marked, then equation n + 1 is also marked. More precisely, if the vertices of G are labeled 1, 2, . (By definition, each loop on v contributes 2 to the degree of v.) In Section 1 we found that if a graph G has an Euler cycle, then every vertex in G has even degree. Answer Exercises 22–25 to give an argument that proves the following result. We assume that initially all values in s are set to 0, except for those values that are specified in the puzzle. The problem is to transfer the disks to another peg by moving one disk at a time beginning with all the disks stacked on one peg in order from largest (bottom) to smallest as in the original puzzle. Since the degree of each vertex is 8 (remember that a loop contributes 2 to the degree), each vertex has even degree. ∀ε > 0 ∃δ > 0 ∀x ((0 < |x − a| < δ) → (|f(x) − L| < ε)) Chapter Self-Test 1. In this case, α is the null string, C2 which has an equal number of a's and b's. Define reflexive relation. For example, the height of the decision tree of Figure 7.3 is 3, so the worst-case time for this algorithm is equal to 3. Our next example shows that in constructing a proof, we may find that we need some auxiliary results, at which point we pause, go off and prove these auxiliary results, and then return to the main proof. If X is a finite set, we let |X| = number of elements in X. The space required by an algorithm is the amount of storage required by the input, local variables, and so on. Functions, Sequences, and Relations 115. Notice that fk2 < n. Since G∗ is connected and has n − 1 edges, by Theorem 2.3, G∗ is acyclic. Initial conditions for the sequence a0 , a1 , . The first time we arrive at the while loop, m = 11 and the condition m > 0 is true; so we execute the body of the while loop. Definition 3.20 A relation R on a set X is a partial order if R is reflexive, antisymmetric, and transitive. Intersection of a family S of sets: ∩ S = {x | x ∈ X for all X ∈ S} 16. If we have disjoint sets of objects and we want to know the total number of objects, we use the Addition Principle. , b, a, . (A is represented as 1, S is represented as 0110, and T is represented as 0111.) Huffman gave an algorithm (Algorithm 1.9) to construct a Huffman code from a table giving the frequency of occurrence of the characters to be represented so that the code constructed represents strings of characters in minimal space, provided that the strings to be represented have character frequencies identical to the character frequencies in the table. Using this result and Theorem 2.17, we may show that decryption produces the correct result. 8 · 4 · 5 13. Recursively invoking this algorithm to move the n − kn disks on peg 2 to peg 4 again takes T(n − kn) moves. To make chili, you do not need red pepper or you do not need onions. The function f = {(1, a), (2, c), (3, b)} from X = {1, 2, 3} to Y = {a, b, c} is one-to-one and onto Y . In Exercises 67–73, state the meaning of each sentence if "or" is interpreted as the inclusive-or; then, state the meaning of each sentence if "or" is interpreted as the exclusive-or (see Exercise 66). Again by Definition 6.13, P(F ∩ Ci) = P(F | Ci)P(Ci). [If (d), then (a).] Suppose that T is acyclic and has n − 1 edges. Example 2.8 The recurrence relation an = 3an−1 an−2 (2.8) is not a linear homogeneous recurrence relation with constant coefficients. Since pq is odd, mk + nk is the highest power of 2 that divides mn, and the result follows. If we have an algorithm to compute the greatest common divisor, we can compute the least common multiple by using Theorem 1.25: mn lcm(m, n) = . But since a Hamiltonian cycle must visit each vertex exactly once, a Hamiltonian cycle in GKn must have length n2 . [Requires calculus] Show that there are n!e − 1 simple paths in Kn . To prove that n is even, we must find an integer k2 such that n = 2k2 . By the Inclusion-Exclusion Principle, the total number of possibilities = number of strings that begin 100 + number of strings that have the fourth bit 1 − number of strings that begin 100 and have the fourth bit 1, so the answer is 25 + 27 − 24 . Discussion We will prove the statement by exhibiting a real number x between a and b. [Requires calculus] Write the negation of the definition of limit (see Exercise 93) in words and symbolically using ∀ and ∃ but not ¬. Write the standard method of adding two positive decimal integers, taught in elementary schools, as an algorithm. If (v0 , v1 , . Then both x and y belong to some set S ∈ S and both y and z belong to some set T ∈ S. Figure 1.4 5. This contradiction shows that r < d. You should indicate clearly what you are about to prove. Use Exercises 38 and 39 to prove that max{x, y} + min{x, y} = x + y for all real numbers x and y. 521 Trees An algorithm for finding a spanning tree based on the proof of Theorem 3.4 would not be very efficient; it would involve the time-consuming process of finding cycles. Every professional athlete plays soccer. Use mathematical induction on n, the number of items in the input sequence, to prove that sort does produce as output a rearranged version of the input sequence in which all of the zeros precede all of the ones. , xn are arbitrary members of the domain of discourse. [Problem 3.] Here we can use the fact that a − ε is not an upper bound (since it's less than the least upper bound) and, as discussed previously, what it means for a value to not be an upper bound. dn = 94. f(n) + g(n) = (h(n)), where h(n) = min{f(n), g(n)} 1 x 53. 2x3 30. Since 0 ≤ x2 , we may add 1 to both sides of this inequality to obtain 1 ≤ x2 + 1. Let us set d1 = 2 and d2 = 4. The subtree whose root is the parent of w may be pruned. We try to prove each of the three statements. Proof We prove (a) → (b), (b) → (c), and (c) → (a). If every element of X is an element of Y , we say that X is a subset of Y and write X ⊆ Y . But adding an edge in parallel introduces a cycle. The statement is not a proposition. We have a contradiction because the number of vertices of odd degree is odd. Notice that the root in Figure 9.2, which represents the original position, has a value of 1. 85. 1 11. Show that dn+1 = 4an ; an+1 = dn + 3an ; an+1 = 3an + 4an−1 . At line 5, j is replaced by n/2 ; hence j ≤ n/2. (2.11) Solving (2.11), we find the solutions t = 2, t = 3. {2} ⊆ P({1, 2}) 82. 31. , n} into X having the property that if i is in the range of f , then 1, 2, . Hint: Use two functions. , sn−1 is recursively sorted. Similarly, if n is 32, the statement x = x + 1 is executed a total of six times. What is an alpha cutoff? Trees A full binary tree is a binary tree in which each vertex has either two children or zero children. Find the names of all employees who work in departments that supply parts for JCN Electronics. Most authors require that V1 and V2 be nonempty in Definition 1.11. (For Exercise 28) Everyone does not love someone. If the pair is not isomorphic, give an invariant that one tree satisfies but the other does not. How does binary search work? Let S denote the event "children of both sexes," and let G denote the event "at most one girl." Then P(S) = 3 6 = 8 4 381 Counting Methods and the Pigeonhole Principle 1 4 = 8 2 3 P(S ∩ G) = . Therefore, R is transitive if and only if whenever entry i, j in A2 is nonzero, entry i, j in A is also nonzero. One one-digit number contains 7. , yj , z1 , . Explain. In every case the loop iterates at most m − 1 more times. It is denoted |α|. One technique is to find a value of x ∈ X (again one value suffices!) that has the property that P(x, y) is true for every y ∈ Y . p q (p ∧ q) ∧ ¬q T T F F T F T F F F F F p q T T F F (p ∨ q) ∧ (¬p ∨ q) ∧ (p ∨ ¬q) ∧ (¬p ∨ ¬q) T F T F F F F F 30. 9 ➜ Game Trees Trees are useful in the analysis of games such as tic-tac-toe, chess, and checkers, in which players alternate moves. (Try Exercises 25 and 27.) Suppose that we have two integers n > 0 and φ > 1 such that gcd(n, φ) = 1. A counterexample is x = 0. There are Sn−1 n-bit strings that begin 1 and do not contain the pattern 00 and there are Sn−2 n-bit strings that begin 0 (since the second bit must be 1) and do not contain the pattern 00. Begin with 01. a b n (= number of modulus operations) 1 2 3 5 8 13 0 1 2 3 5 8 0 1 2 3 4 5 pair a, b, a > b, with a as small as possible, that requires n modulus operations for n = 0, . If E1 and E2 are events, the event E1 ∪ E2 represents the event E1 or E2 (or both), and the event E1 ∩ E2 represents the event E1 and E2 . (a) A designated vertex, the source, has no incoming edges. The words have the same or different lengths, and at some position, the letters in the words differ. Finding web pages containing key words is a searching problem. c 15 2. ∀x∃y P(x, y) Which of Exercises 89–92 ¬(∀x∃y P(x, y))? What is the probability of a correct answer if the student stays with choice A? Prove your answers. i=1 (a) αβ β = caaba, n 116. For example, in Figure 1.6, the price fluctuates between p0 and p1 , whereas in Figure 1.7, the price swings become more and more pronounced. Hk = 29. Derive a recurrence relation for the sequence R1 , R2 , . Each subset in the right column is obtained from the corresponding subset in the left column by deleting the element a from it. By summing we obtain all the routes: C(5 + 3, 5) = C(0 + 2, 0) + C(1 + 2, 1) + C(2 + 2, 2) + C(3 + 2, 3) + C(4 + 2, 4) + C(5 + 2, 5). What is a base case in a recursive function? Give a combinatorial argument to prove that n where B0 , . Similarly, if the prime pi is not a factor of n, we let bi = 0.) Then 1 ,b1) min(a2 ,b2) k ,bk) gcd(m, n) = pmin(a p2 · · · pmin(a . The adjacency matrix is not a very efficient way to represent a graph. Set D= 1 1 + ··· + 1 n . If X is the set of five-bit strings and Xi is the set of five-bit strings that contain i consecutive zeros, the Addition Principle does not apply; the sets Xi are not pairwise disjoint. Equivalence relation: [1] = [3] = [5] = {1, 3, 5}, [2] = {2}, [4] = {4} ρ(R1) = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (3, 4), (4, 2)} 7. (The set D2 is defined before Exercise 1.) 26. (n lg n) Section 3 1. Write a command to search the Web for minor league baseball teams in Illinois that are not in the Midwest League. Similarly, at line 7, we process vertex C. Therefore X ⊆ Z. Thus if n satisfies 2m−1 < n ≤ 2m , the algorithm uses at most m + 1 comparisons. At line 4, i is reset to 2 and we return to line 2. Since there are four ways to select the three suits to have four cards each, there are 4C(13, 4)3 C(13, 1) hands that contain four cards of three suits and one card of the fourth suit. Example 3.4 a b 2 11 2 3 11 d 3 c Figure 3.8 A graph for the traveling salesperson problem. The relation X = {1, 2, 3} 19. Suppose, by way of contradiction, that T is not connected (see Figure 2.5). Write a program to determine whether X ⊆ Y , given arrays representing X and Y . Applicant A is qualified for jobs J1 , J2 , J4 , and J5 ; B is qualified for jobs J1 , J4 , and J5 ; C is qualified for jobs J1 , J4 , and J5 ; D is qualified for jobs J1 and J5 ; E is qualified for jobs J2 , J3 , and J5 ; and F is qualified for jobs J4 and J5 . Let R1 = {(1, 2), (2, 3)}, R2 = {(1, 3), (3, 4)}. Prove that a connected graph with one or two vertices, each of which has even degree, has an Euler cycle. Which are cycles? , sj , i, and j for every positive integer n. Draw the subtree rooted at j. The intended meaning is: Some environmental problem is not a tragedy. Find all buyers who buy parts in the department managed by Jones. We have proved part (a). Once the second letter has been selected, the third letter can be selected in three ways. Since Fbc = Cbc , we do not label vertex c. Tuna 12. By definition, the (k + 1)-sided polygon can be partitioned in Pk−1 ways and the (n−k +2)-sided polygon can be partitioned in Pn−k ways. , n − 1 divides n. Section 3 2. What is the probability of getting an even number? What can you say about a vertex in a rooted tree that has no descendants? Negation: 2 + 5 = 19 4. v1 e1 v4 f 9. By the Multiplication Principle, there are 3 · 2 · 4 = 24 possibilities. From the inequality dn = 100n + 1.2dn−1 = 100n + 1.2[100(n − 1) + 1.2dn−2] = 100n + 1.2 · 100(n − 1) + 1.22 dn−2 = 100n + 1.2 · 100(n − 1) + 1.22 [100(n − 2) + 1.2dn−3] = 100n + 1.2 · 100(n − 1) + 1.22 · 100(n − 2) + 1.23 dn−3 .. We can manipulate this equation and hope to obtain a contradiction. , sn , arranged in nondecreasing order selection sort(s, n) { // base case if (n == 1) return // find largest max index = 1 // assume initially that s1 is largest for i = 2 to n if (si > smax index) // found larger, so update max index = i // move largest to end swap(sn , smax index) selection sort(s, n − 1) } 411 Recurrence Relations As a measure of the time required by this algorithm, we count the number of comparisons bn at line 8 required to sort n items. (b) There is a one-to-one, onto function f from the vertex set of G1 to the vertex set of G2 satisfying the following: Vertices v and w are adjacent in G1 if and only if the vertices f(v) and f(w) are adjacent in G2 . The process continues until n becomes 0. For a feature set F , we have P(F | Cj)P(Cj) P(Cj | F) = n . The following theorem makes this relationship precise. B 1 4 2 W 1 R 1 1 2 2 6. Thus ∅ = { }. Exercise 2, Section 1 16. In our case we add the edge (a, b). Reflexive: Suppose that (x1 , x2) is in X1 × X2 . If x ∈ C, again x ∈ (A ∪ B) ∪ C. Report on how this estimate was obtained. (A nondecreasing sequence is like an increasing sequence except that "" is replaced by "≥.") Example 2.7 The sequence 2, 5, 13, 104, 300 is increasing and nondecreasing. List the first seven terms of t. We call this type of proof exhaustive proof. The degree of a vertex v, δ(v), is the number of edges incident on v. Among the definitions are ■ ■ The elements in P (of the preceding axiom) are called positive real numbers. w1 b 3 c 4 4 3 d 2 5 6 2 15 minutes B to D 15 minutes C to D 15 minutes. Suppose, by way of contradiction, that x is rational. It considers each of the squares covered by the center tromino as missing. Exclusive-or is the intended meaning. Although there are values of x that make the propositional function true, the counterexample provided shows that the universally quantified statement is false. (7.1) In 1752, Euler proved that equation (7.1) holds for any connected, planar graph. We list the primes less than or equal to 11: 2, 3, 5, 7, 11. q n−1 (c) Show that if p/q = 1/n, the proof is complete. If we will not buy a new computer, then 4 megabytes is not better than no memory at all. binary search5(s, i, j, key) { if (i > j) return 0 k = (i + j)/2 if (key == sk) return k if (key < sk) return binary search5(s, i, k, key) else return binary search5(s, k + 1, j, key) } 25. Draw a graph with 10 vertices. In words, a function f from X to Y is not onto Y if there exists y ∈ Y such that for all x ∈ X, f(x) = y. For example, it is perfectly acceptable to write Thus x ∈ X. In the context of parallel computation, what is the meaning of this value? The component of G containing v4 is the subgraph G2 = (V2 , E2), V2 = {v4 }, E2 = ∅. The range is the set of all the reciprocals that can be computed and displayed. Inorder is CBDEAFIHJG. ABC+− 17. If x ∈ / X − Y , then either x ∈ / X or x ∈ Y . (For example, it takes time to manufacture goods and it takes time to grow crops.) We denote the discrete time intervals as n = 0, 1, . We assume that s > 0 and t < 0. ,f(Cn) are distinct. A mathematical system consists of axioms, definitions, and undefined terms. Example 1.25 A restaurant serves four appetizers, r = ribs, n = nachos, s = shrimp, f = fried cheese, and three entrees, c = chicken, b = beef , t = trout. The graph of Exercise 8 is a subgraph of the graph of Exercise 9. A relation R on a set X is a partial order if R is reflexive, antisymmetric, and transitive. for i = 2 to n! { is eliminated. (c) Let Gn−1 denote the sequence obtained by prefixing each member of GRn−1 with 1. Further, Definition 1.1 does not require us to use all the elements in Y . By universal instantiation, we have P(Ralphie) and P(Ralphie) → Q(Ralphie). (The least upper num√ bound of the subset of rational √ bers less than 2 is the irrational number 2.) Exercises 1. Solve the recurrence relation an = 3an/2 + n, in case n is a power of 2. Because the classes partition the set of routes, the Addition Principle applies and the sum of the numbers of routes in each class is equal to the total number of routes. If (x, y) ∈ τ(R) = ∪{Rn }, then there exist x = x0 , . The statement there exists x, P(x) is said to be an existentially quantified statement. Assume that n is even. If G has one component, that is, if G is connected, we may apply the inductive hypothesis to conclude that G has an Euler cycle C . Given that the relation {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)} is an equivalence relation on {1, 2, 3, 4}, find [3], the equivalence class containing 3. The remainder gives the number of 1's. Show that G has a vertex of degree 5. State the Inclusion-Exclusion Principle for two sets and give an example of its use. Exercises 55–57 refer to Algorithm 3.15. Let G be a connected, weighted graph and let v be a vertex in G. In this version, edge (1, 5) is selected. A sequence s is nondecreasing if sn ≤ sn+1 for all n for which n and n + 1 are in the domain of the sequence. 2475, 32670 9. Factors such as the average-case time of the algorithm, the number of items to be sorted, available memory, the data structures to be used, whether the items to be sorted are in memory or reside on peripheral storage devices such as disks or tapes, whether the items to be sorted are already "nearly" sorted, and the hardware to be used must be taken into account. Cryptology Cryptosystem Encrypt a message Decrypt a message RSA public key cryptosystem: To encrypt a and send it to the holder of public key z, n, compute c = an mod z and send c. CX∪Y (x) = 0. An analysis of the other cases also shows that if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false. For example, if x1 < x2 < · · · < x5 , the median is x3 . Proof We must show that every element in X belongs to exactly one member of S. Define a relation R on X × X by (a, b) R (c, d) if a + d = b + c. A search that terminates n levels below the given vertex. In every case, we have |x| = sgn(x)x. Notice that 3 is a common divisor of 30 and 105 since it occurs in the prime factorization of both numbers. 1585 8. Proof by Cases Proof by cases is used when the original hypothesis naturally divides itself into various cases. 154 Functions, Sequences, and Relations Mileage Cost 1 2 3 4 5 6 7 8 9 10 $1.00 1.50 2.00 2.50 3.00 3.50 4.00 4.50 5.00 5.50 The list of fares C1 = 1.00, C2 = 1.50, C6 = 3.50, C7 = 4.00, C3 = 2.00, C8 = 4.50, C4 = 2.50, C9 = 5.00, C5 = 3.00, C10 = 5.50 furnishes an example of a sequence, which is a special type of function in which the domain consists of a set of consecutive integers. Edge e1 is associated with the ordered pair (v2 , v1) of vertices, and edge e7 is associated Graph Theory v1 e1 e3 v2 v4 e2 e4 with the ordered pair (v6 , v6) of vertices. Thus (x, z) ∈ R−1 . What is the probability of getting a 5? 3 ➜ Generalized Permutations and Combinations In Section 2, we dealt with orderings and selections without allowing repetitions. Prove that if X ∩ Y = X ∩ Z and X ∪ Y = X ∪ Z, then Y = Z for all sets X, Y , and Z. Since n = 0 is not greater than 0, the while loop terminates. 00 z=z+1 01 {(1, 2), (2, 3), . , 2k , . In the body of the while loop, at most two multiplications are performed. if (n == 1) { 3. Choose an edge (x, y) in C, different from (j, k), with x ∈ V and y ∈ / V. Describe P − ∞ i=2 Xi . By inspection, we find them to be (d, a, d, c, e), (d, c, d, c, e), (d, a, b, c, e), (d, c, e, c, e), (d, c, e, b, e), (d, c, b, c, e). Thus the Basis Step is proved. As examples, c2 = c2/2 + 2 = c1 + 2 = c1 + 2 = 0 + 2 = 2, c3 = c3/2 + 3 = c1.5 + 3 = c1 + 3 = 0 + 3 = 3, c4 = c4/2 + 4 = c2 + 4 = c2 + 4 = 2 + 4 = 6, c5 = c5/2 + 5 = c2.5 + 5 = c2 + 5 = 2 + 5 = 7. We can also find the greatest common divisor of two integers m and n by looking carefully at their prime factorizations. The equivalence relation that generates this partition is given in Exercise 38. If HT appears, the first player wins; if TT appears, the second player wins. In a similar way, we find that the number of routes that first 332 strings that end UR, since we must choose n − 1 slots from among the first (n − 1) + m − 1 slots for the n − 1 R's. How many have the digits in strictly increasing order? But this last equation tells us that n2 is odd. We say that t(n) is of order n2 and write t(n) = (n2), which is read "t(n) is theta of n2 ." The basic idea is to replace an expression, such as t(n) = 60n2 + 5n + 1, with a simpler expression, such as n2 , that grows at the same rate as t(n). f −1 (y) = 1/(y − 3) 38. p ∨ q, ¬p ∨ r/∴ q ∨ r 2. In an unweighted graph, the length of a path is the number of edges in the path. Ko nigsberg Bridge Problem The first paper in graph theory was Leonhard Euler's in 1736. ∃x∃y(x2 < y + 1) 46. When we are asked to prove a universally quantified statement, showing that the statement is true for specific values does not prove the statement; it may, however, help to understand the statement. By Theorem 2.21, S + T is even. Let each row, column, or diagonal that contains one X and two blanks count 1. What properties must the input have? Theorem 7.6 can also be proved using Theorem 2.17 (Exercise 17 of this section). (¬p ∨ ¬q) ∨ p 52. Given that the IPv4 Internet address of a computer interface is either Class A, Class B, or Class C, how many IPv4 Internet addresses are available? An edge connects two vertices if the binary representation of their labels differs in exactly one bit. i=i+2 29. Show that an ≤ an/2 + a(n+1)/2 + 2 lg n. In most applications, we would also want to identify a shortest path. In a Venn diagram, a rectangle depicts a universal set, and subsets of the universal set are drawn as circles. Suppose that G1 has a vertex v of degree k. i=1 P(F | Ci)P(Ci) P(Cj | F) = n Section 7 n 28. Bayes' Theorem is useful in computing the probability of a class given a set of features. At line 5, we call preorder with PT equal to null since there is no left child. A large number of problems are known to be unsolvable, some of considerable practical importance. For each a ∈ X, let [a] = {x ∈ X | x R a}. How is the intersection of X and Y denoted? The bug is a numerical error. First 20 is inserted in 20 135 101 72 34 We next swap ai and aj , where i = 3 and j = rand(3, 5) = 3. 20. X − (Y × Z) = (X − Y) × (X − Z) for all sets X, Y , and Z. In fact, in April 1994, Arjen Lenstra, Paul Leyland, Michael Graff, and Derek Atkins, with the assistance of 600 volunteers from 25 countries using over 1600 computers, factored z (see [Taubes]). 1/24 25. Thus the set A might just as well be specified as A = {1, 3, 4, 2} . Is a proposition. Because 2k−1 is an integer, we must also have 2k−1 ≤ n/2 < 2k . f(x, y) = x/y, 1 if x ∈ X We call CX the characteristic function of X (in U). Prove that at least three weighings are required in the worst case to identify the bad coin and determine whether it is heavy or light using only a pan balance. Yes 83. Show how Kruskal's Algorithm finds minimal spanning trees for the graphs of Exercises 1–5. Since there is no left child, we conclude that MORE is not in the tree. No 7. How many are taking French or business (or both)? Exercises 46–48 refer to the following graph. [When S is a set, we define f(S) = {f(x) | x ∈ S}.] 72. (e) Similarly, a sufficient condition is a condition that suffices to guarantee a particular outcome. From x ≥ d and d ≥ d2 , we may derive x ≥ d2 from the same previous theorem. Is there a complete matching? We obtain the flow of Figure 2.12. This time at line 2, the condition i ≥ 1 is false. Then ∀x∃y P(x, y) is true since for any x, we may choose y = x−1 to make P(x, y) true. 10 xi . That is, an − an−1 = an−2 . , 10, d does not divide 451 and the condition n mod d == 0 is false. Since e1 , . Repeat this process. {(1089, Suzuki, Zamora), (5620, Kaminski, Jones), (9354, Jones, Yu), (9551, Ryan, Washington), (3600, Beaulieu, Yu), (0285, Schmidt, Jones), (6684, Manacotti, Jones)} 5. , n = n0 +p−1. , nk } of {1, 2, . A connected graph is a graph in which we can get from any vertex to any other vertex on a path. In this context, (v, w) denotes an edge between v and w in an undirected graph and not an ordered pair. The first letter can be selected in five ways. Show that the size of an optimal vertex cover of the complete graph on n vertices is n − 1. Trees Example 4.2 The tree T shown in Figure 4.2 is a spanning tree for graph G of Figure 4.1. The weight of T is 20. How is the floor denoted? If there is a Hamiltonian cycle, exhibit it; otherwise, give an argument that shows there is no Hamiltonian cycle. What proof technique did you use? Encrypt 333 using the public key 713, 29 of Example 4.2. 6. A "straight play" win requires that the numbers be matched in the exact order in which they are randomly drawn by a lottery representative. 3 R 312 B 1 4 1 342 3 W B 1 3 2 2 4 3 2 4 1 G 4 (b) Each cube contains all four colors. Neither the "list all divisors" method of Example 1.15 nor use of prime factorization as in Example 1.18 is an efficient method of finding the greatest common divisor. Use the result of Exercise 28 to show that n (7.5) n 30. Exercises 33–38 refer to Algorithm 3.12. Example 3.9 } // merge merge(s, i, m, j, c) // copy c, the output of merge, into s for k = i to j sk = ck Figure 3.2 shows how Algorithm 3.8 sorts the sequence 12, 30, 21, 8, 6, 9, 1, 7. But the problem of sorting three items has six possible outcomes (when the items are distinct), corresponding to the 3! = 6 ways that three items can be arranged: s1 , s2 , s3 , s1 , s3 , s2 , s2 , s1 , s3 , s2 , s3 , s1 , s3 , s1 , s2 , s3 , s2 , s1 . The solution can be represented by a subgraph. 453 Graph Theory 19. , sj of zeros and ones Output: si , . 1001 + 1111 21. , sn and the length n of the sequence Output: s1 , s2 , . Use breadth-first search (Algorithm 3.6) with the vertex ordering eachgbdfi to find a spanning tree for the following graph. A = {1} × {1, 2}, B = {1} × {1, 2, 3} 36. What is a greedy algorithm? If a connected, planar graph is drawn in the plane, the plane is divided into contiguous regions called faces. The proof is completed by showing that Section 6 Review 1. The partition is [x1], [x2], . In negative statements, the situation changes: Not all x satisfy P(x). (3.3) Combining (3.2) and (3.3), we obtain a = bq + r ≥ b + r ≥ fn+2 + fn+1 = fn+3 . Use mathematical induction to show that for all n ≥ 1, n an algorithm to list all of the ways the robot can walk n meters. There are formulas for the number of nonisomorphic n-vertex free trees, for the number of nonisomorphic n-vertex rooted trees, and for the number of nonisomorphic n-vertex binary trees. Find all buyers of part 20A8. In the oil-pumping example of Figure 1.1, conservation of flow means that oil is neither used nor supplied at pumping stations b, c, d, and e. Again, x ∈ A. Can the permutation following the given permutation have the form 1635−− ? 10 11 Section 2 Review 40. A function f is not one-to-one if ∀x1 ∀x2 ((f(x1) = f(x2)) → (x1 = x2)) is false or, equivalently, its negation is true. Prove that if X ⊆ Y , then X ∪ Z ⊆ Y ∪ Z for all sets X, Y , and Z. For example, the relation R, defined by x R y if x and y are the same color, partitions the set into subsets where each subset contains elements that are all the same color. Suppose that we have the situation shown in Figure 9.4. Circle should move to the position represented by vertex C, since this position is least valuable to box and therefore most valuable to circle. Nim is the game using n piles of tokens as described in this section except that the last player to move wins. Example 2.24 If γ = aab and θ = cabd, then γθ = aabcabd, Example 2.25 θγ = cabdaab, γλ = γ = aab, λγ = γ = aab. Starting at z, we can retrace the labels to find the shortest path (a, d, e, z). Then n + 1 = m + 1. A more advanced treatment is found in [Davis]. We will show that bn ≤ bn+1 , n = 1, 2, . Explain why a forest is a union of trees. An example of such a tree, which is the administrative organizational chart of a hypothetical university, is given in Figure 1.6. President Vice-President for Academic Affairs Chair of Mathematics Vice-President for Administrative Affairs Dean of Arts/ Sciences Dean of Business Chair of Computer Science Chair of Accounting Director of Academic Planning Director of Purchasing Figure 1.6 An administrative organizational chart. For more details on relational databases, the reader is referred to [Codd; Date; and Kroenke]. (a) If d | m and d | n, then d | (m + n). Theorem 4.1 Let S be a partition of a set X. Define binary tree. Was the ad correct? If desired, the integers could be combined into the single integer 20061505011416150626 (note that leading zeros have been added to all single-digit numbers). Since the basis step and the inductive step have been verified, the given equation is true for all n ≥ 1. , n} in lexicographic order. Write a program that generates pseudorandom integers. Therefore (x1 , x2) = (x1 , x2) and R is antisymmetric. You may inspect the aircraft only if you have the proper security clearance. Write a program that reads a logical expression in p and q and prints the truth table of the expression. Pascal's triangle is an arrangement of the binomial coefficients in triangular form. for i = 1 to m println(i) for j = 1 to n println(j) 14. Give an example of a set X and three distinct upper bounds for X, one of which is a least upper bound for X. Does the graph of Exercise 7 have a Hamiltonian path? Section 3 4. Suppose that x ∈ X. Since these cases are disjoint, Rn = Rn−1 + 2Rn−2 . We can guarantee that exactly one comparison is performed for each i if the first character in p does not occur in t. After the swap we have 135 57 10. We can also ask for the maximum time needed to execute the algorithm among all inputs of size n. An example of the use of the Inclusion-Exclusion Principle for two sets is provided by Example 1.13. Third Form: Let f be a function from a finite set X to a finite set Y . What is the probability of winning $40? Therefore, n = (m + n) − m = (2k1 + 2k2) − 2k2 = 2(k1 + k2 − k2). How can we use these facts to reach our goal (1.2)? We state the method of repeated squaring as Algorithm 2.16. Assign each student (pigeon) the first letter of the first name (pigeonhole). The vertices v1 , v2 , v3 , v4 , v5 , v6 , v7 in the rooted tree of Figure 1.4 are on (respectively) levels 0, 1, 1, 2, 2, 2, 2. Under the root and on the same level, we place the vertices, v2 and v3 , that can be reached from the root on a simple path of length 1. ∀xP(x) 12. s: The Olympic Games are canceled. = 2k a20 + k · 2k − 1 − 2 − 22 − · · · − 2k−1 = k · 2k − (2k − 1) = (k − 1)2k + 1. The distance between vertices v and w in G, dist(v, w), is the length of a shortest path from v to w. ⎢ ⎥⎝0 2 0 2 1⎠ 1 1 1 1 2 Consider the entry for row a, column c in A2 , obtained by multiplying pairwise the entries in row a by the entries in column c of the matrix A and summing: a (0 b d 1 0 1 ⎛ c⎞ 0⎟1⎟b⎟ ⎟ ⎟ = 0 · 0 + 1 · 1 + 0 · 0 + 1 · 1 + 0 · 1 = 2. Instant Insanity consists of four cubes each of whose faces is painted one of the four colors, red, white, blue, or green. In the algorithm, all vertices are on the unit circle in the xy-plane. 465 Graph Theory Dijkstra's algorithm involves assigning labels to vertices. Write a program that adds octal numbers. Is f one-to-one or onto? 2 2 To satisfy the initial conditions, we must have √ √ 1+ 5 1− 5 b +d =1 2 2 √ 2 √ 2 1+ 5 1− 5 b +d = 1. Therefore, any algorithm that sorts four items requires at least five comparisons in the worst case. At lines 1–7, there are zero comparisons (of the type we are counting). What is a path? Having chosen a value for x, let y stand for an arbitrary element of Y and show that P(x, y) is always true. Analyze the following comments by film critic Roger Ebert: No good movie is too long. Show that any simple, connected graph with 31 edges and 12 vertices is not planar. A lower bound for X is a real number a having the property that x ≥ a for every x ∈ X. Show that among six pictures, there are either three that are mutually similar or three that are mutually dissimilar. For example, let us perform a join operation on Tables 6.1 and 6.2. As the condition we take TABLE 6.2 ■ ASSIGNMENT PID 39826 26710 58199 01180 Team Blue Sox Mutts Jackalopes Mutts ID Number = PID. 316 87. If for every x, P(x, y) is be 88. ∀n P(n) 20. Algorithm 4.1 finds the length of a shortest path in a connected, weighted graph even if some weights are negative. 5 < 9 or it is not the case that (9 < 7 and 5 < 7). , si is a loop invariant using induction on i. Another tree is shown in the hint for Exercise 24. Because we cannot reuse edges, our choices are limited to three edges (see Figure 8.10). ¬(∀x P(x)) 18. [Halmos; Lipschutz; and Stoll] are recommended to the reader wanting to study functions in more detail. sn ≥ A. Describe, in words, an algorithm that solves the text-searching problem. d divides n: d | n d does not divide n: d /| n d is a divisor or factor of n Prime Composite Fundamental theorem of arithmetic: any integer greater than 1 can be written as the product of

primes Common divisor Greatest common divisor Common multiple Least common multiple Section 2 11. † 42. In addition to finding a theta notation in Exercises 1–28, prove that it is correct. ¬p ∨ q ∨ r ¬q ¬r 3. Let G be the graph with vertex set {1, 2, . If x ∈ / X ∪ Y , then x ∈ / X and x ∈ / Y . 3 4 Section 6 22. For this choice of x, there exists y = y such that P(x , y) is false. f(v1) = w1 , f(v2) = w5 , f(v3) = w3 , f(v4) = w4 , f(v5) = w2 , f(v6) = w6 . Region 7 represents elements in Y and Z, 65 Sets and Logic but not in X. zi . If you can put each vertex into either V1 or V2 , but not both, the graph is bipartite. Then x ∈ A ∪ B. Find the quotient q and remainder r as in Theorem 5.6 when n = 101 is divided by d = 11. The cardinality of the set {R, Z} is 2 since it contains two elements, namely the two sets R and Z. When T has n − 1 edges, we stop. Show how Algorithm 3.2 executes in case key = 'G'. What is wrong with the following argument that purports to show that we cannot simultaneously have f(n) = O(g(n)) and g(n) = O(f(n))? Thus the algorithm uses at most 1 + m = 1 + lg n comparisons. 10 · 5 31. Now suppose that the input is the tree of Figure 6.1. We set PT to the root and call preorder(PT). An arbitrary input size falls between two powers of two. Theorem 4.3 Algorithm 4.2 returns the value of n!, n ≥ 0. Basis Steps (n = 4; n = 5) We can make four-cents postage by using two 2-cent stamps. If f is a one-to-one, onto function from X to Y , the inverse function is Y ... 15. What is a propositional function? By referring to Table 3.1, we can determine the input TABLE 3.1 ■ Number of modulus operations required by the Euclidean algorithm for various values of the input. Now, by the inductive assumption, 5n − 1 is divisible by 4. In weighted graphs, we often want to find a shortest path (i.e., a path having minimum length) between two given vertices. Given the hypotheses 1, 2, and 3, we have proved the conclusion b ∨ d. If x ∈ B or x ∈ C, then x ∈ B ∪ C. We delete edges e1 and e2 , but no vertices, and add an edge e incident on v1 and v3 to obtain the graph G [see Figure 2.9(a)]. In the fallacious proof, only the case n = 3 was proved in the Basis Steps. Define substring. ∃y A(Brit, y) 36. We now use induction on k to prove that we can tile a 2k × 2k deficient board with trominoes for all k ≥ 1. Universal product codes (UPC) are the familiar bar codes that identify products so that they can be automatically priced at the checkout counter. (n + 1)2 = n2 + 2n + 1 ≤ 2n + 2n + 1 ≤ 2n + 2n =2 by Exercise 14 n+1 1 − r n+1 1 < 1−r 1−r 21. Then there is a path P from v to w not including e. A unary operator on a set X associates with each single element of X one element in X. It follows that f(n) = (h(n)). Let a, b be an input pair in the range 0 to m that requires n modulus operations. If R is an equivalence relation, the program outputs the equivalence classes of R. As we showed in part (a), ∀x∃y P(x, y) is true. Let a be the sequence satisfying (2.16) and a0 = C0 , a1 = C1 . 8 8 8 8 Of course, for a fair die (with equally likely probabilities), the probability of an odd number is 1/2. 1 · C(48, 9) (Select the aces, then select the nine remaining cards.) 58. x ∈ X: x is an element of the set X 5. Show that if p is a prime number, a and b are positive integers, and p | ab, then p | a or p | b. Express each decimal number in Exercises 14–19 in hexadecimal. Follow an edge out of v0 to a vertex v1 . 2 A sequence of statements is really being made, namely, Sn = (4.4) 1(2) =1 2 2(3) S2 = =3 2 3(4) S3 = =6 2 .. Example: If the sequence is 1 1 1 1. , Bk−1 is row n of Pascal's triangle (row 0 being the top row). What is an event? In the geometric sum in which a = 0 and r = 0, the ratio of adjacent terms [(ar i+1)/(ar i) = r] is constant. If the root has a left child, the child itself has either a left or a right child. Since r∧s represents the proposition "the Chargers can beat the Jets and the Chargers can beat the Dolphins," we conclude that the conclusion does follow from the hypotheses. G Figure 8.9 A subgraph of Figure 8.6 satisfying (8.1) and (8.2). We next swap ai and aj , where i = 4 and j = rand(4, 5) = 5. The algorithm first sets n to −1. (b) For each x ∈ X, there is exactly one y ∈ Y such that (x, y) ∈ f . The discussion following Example 1.3 shows that An = (1 + i)n A0 . Inorder traversal processes the vertices of a binary tree by beginning at the root and recursively processing the vertex's left subtree, the current vertex, and then the vertex's right subtree. (b) There are nine equivalence classes. Problem-Solving Tips Again, we emphasize that to construct an algorithm, it is often helpful to assume that you are in the middle of the algorithm and that part of the problem has been solved. We would have to eliminate one edge incident at v2 and one edge incident at v4 , since each vertex in a Hamiltonian cycle has degree 2. A theorem is a proposition that has been proved to be true. Similarly, if nk is the highest power of 2 that divides n, then n = 2nk q, where q is odd. Then E1 (f) = f(1) = 1 = g(1) = E1 (g). Give an example of a hierarchical definition tree. What is a truth table? An evaluation function assigns to each possible game position the value of the position to the first player. We now look for a second graph having no edges in common with the graph just chosen. We begin by giving some examples of mathematical systems. It can be shown (Exercise 68) that R is an equivalence relation on V and that if v ∈ V , the set of vertices in the component containing v is the equivalence class [v] = {w ∈ V | w R v}. A binary relation from a set X to a set Y is a subset of the Cartesian product X × Y . Find i=2 4 q5 = 33. Write the negation of ∃x∀yP(x, y) using the generalized De Morgan's laws for logic. Figure 2.4 c a f e e c j 40. The disjunction of propositions p and q is the proposition p or q. Suppose that R is a relation on X that is symmetric and transitive but not reflexive. Let α = λ. For n ≤ N, 6. Since gcd(n, φ) = 1, we use the Euclidean algorithm, as explained previously, to find numbers s and t such that s n + t φ = 1. (a, a), (b, c, g, b), (b, c, d, f, g, b), (b, c, d, e, f, g, b), (c, g, f, d, c), (c, g, f, e, d, c), (d, f, e, d) 22. A ∩ B = A 88. Then (y, x), (x, y) ∈ R. / 82. Sometimes the number of cases to prove is finite and not too large so we can check them all one by one. X × Y = X × Y for all sets X and Y . April, July Section 2 Review 1. Then the while loop iterates at most m − 1 times. Example 3.3 The expression p→q is not a clause since the terms are separated by →. However, not all edges between vertices in V1 and V2 are in the graph. P(1) 13. Functions, Sequences, and Relations 3. If Leah gets an A in discrete mathematics, then Leah studies hard. After we prove that proposition (5.2) is true, we may negate (5.2) and conclude that ¬(∀x ¬P(x)) 44 Sets and Logic is false. 275 Introduction to Number Theory Algorithm 2.7 Converting a Decimal Integer into Base b This algorithm converts the positive integer m into the base b integer cn cn−1 · · · c1 c0 . We say that an alpha cutoff occurs. At line 17, we examine edge (b, c). Graphs with Samantha, Alexandra, Lance, Bret, and Tiffany, and no others. 331, 993 20. Exercises In Exercises 1–11, using induction, verify that each equation is true for every positive integer n. A terminal vertex represents the end of the game. , sn , first swap s1 and a randomly chosen element in s1 , . In the Inductive Step, we assume the truth of the statement for k = n/2. Hash function 4. Again by the Pigeonhole Principle, at least four of these seven colorful pairs are the same color, say red. Thus the number of possible subsets is 2 · 2· · · 2 = 2n . Since these sets of selections are pairwise disjoint, we may use the Addition Principle to conclude that there are 15 + 10 + 6 = 31 ways of selecting two books from different subjects among the computer science, mathematics, and art books. Thus, if the proposition ¬(∀x P(x)) is true, the proposition ∃x ¬P(x) is true. For example, in base 16, B4F = 11 · 162 + 4 · 161 + 15 · 160 (see Figure 2.3). The ijth entry is 1 if there is an edge from i to j, which is a path of length 1, and 0 otherwise. Next, we evaluate vertex G. Reverse Polish notation 13. Show that R ∩ R−1 is an equivalence relation on X. ■ It's very common in a graph model for the edges to indicate a relationship between the vertices. Show that the Strong Form of Mathematical Induction and the form of mathematical induction where the Inductive Step is "if S(n) is true, then S(n + 1) is true" are equivalent. a b e f h g d 2 5 8 c G1 1 6 7 4 3 G2 Figure 6.5 Nonisomorphic graphs. In words, there is no greatest positive integer. Taking the logarithm to the base 3/2 of each side, we obtain log3/2 (n + 1) > h + 2. For example, p is true if n = 103 and false if n = 8. Now solve C0 = b C1 = br + dr for b and d to obtain the solution of the recurrence relation. is either true or false. c1 (n − 1)r n−1 + c2 (n − 2)r n−2 = 2r(n − 1)r n−1 − r 2 (n − 2)r n−2 = r n [2(n − 1) − (n − 2)] = nr n . Since i = 2 and power = 4, ci ∗ power = 1 ∗ 4 = 4. So dedicated to programming was he that when he was married in 1957, he listed his profession as a programmer. Define the Cartesian product of sets X and Y . (e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower. These three sets partition X. 215 Algorithms Time and Space for Algorithms It is important to know or be able to estimate the time (e.g., the number of steps) and space (e.g., the number of variables, length of the sequences involved) required by algorithms. d a j f b i e k l c g h (b) The children of b. Merge sort 423 Recurrence Relations Chapter Self-Test Section 1 1. Collision resolution policy 6. Exercises In Exercises 1–4, write the relation as a set of ordered pairs. Define degree of vertex v. What is the negation of p? A bicycle collector has 100 bikes. Find t2077 . If X is a finite set, what is X+ ? 295 Introduction to Number Theory It follows that we can make postage for n, n + 1, . For example, Q+ denotes the set of positive rational numbers. satisfies the recurrence relation Dn = (n − 1)(Dn−1 + Dn−2). If x ∈ X, then x2 + x − 2 = 0. Under basement water, which is highlighted, are the files 11-18-03, DSC01007 11-18-03, and others, which appear on the right. j=0 77. B − A 5. A contiguous region d 3. To efficiently compute an integer s, 0 < s < φ such that ns mod φ = 1, first compute s and t satisfying gcd(n, φ) = s n + t φ (see the subsection Computing an Inverse Modulo an Integer). 33. The concatenation of strings α and β is the string consisting of α followed by β. Inequality (5.1) then becomes 1 ≤ n/2 < n. Of course, a different pair of starting and ending vertices might produce an even shorter path. Assume that if i < 1, the comparison si > temp is not made. Show that at least two persons have the same first, middle, and last names. How many paths of length k ≥ 1 are there in Kn ? Since m/b = 1/2 = 0, m is set to 0. We just saw that if the tree input to preorder consists of a single vertex, preorder processes that vertex. The smallest i for which si = ti is i = 1. Draw the tree T of Figure 1.5 as a rooted tree with b as root. , 18} or {1, 2, . Assign values to all vertices so that the resulting tree is analogous to Figure 9.2. Will the first or second player, playing an optimal strategy, always win? However, in binary trees a child is designated a left child or a right child. The first remainder gives the number of 1's, the second remainder gives the number of 16's, and so on; thus we obtain 2038510 = 4FA116 . Let G = gcd(a, b), where a and b are the values input to Algorithm 3.3. We can prove that Algorithm 3.3 is correct by verifying that G = gcd(a, b) is a loop invariant, where now a and b denote the variables in the pseudocode. number of dinners is 3 · 4 = 12. Of course, this last statement is meaningful only if np + s ≥ 0 and nq + t ≥ 0. You should list the possibilities to confirm this last statement. What is a Gray code? We prove X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z) by proving if x ∈ X ∩ (Y − Z), then x ∈ (X ∩ Y) − (X ∩ Z) (1.3) if x ∈ (X ∩ Y) − (X ∩ Z), then x ∈ X ∩ (Y − Z). Is w nondecreasing? Another restatement is then: There is no greatest integer. The property is an invariant. Find an initial condition for the sequence A0 , A1 , . Replacing a by b in the sum yields n n bn+1 − an+1 = ai bn−i < bi bn−i b−a i=0 = 60. A definition creates a new concept in terms of existing ones. If the best-case time required by an algorithm is O(g(n)) and (g(n)), we say that the best-case time required by the algorithm is (g(n)). Alternative ways to write ∃x P(x) are there exists x such that, P(x) and for some x, P(x) and for at least one x, P(x). Our next theorem confirms that our conjecture is correct. {a, b, c} 76. [Johnsonbaugh] discusses the minimal time for sorting as well as lower bounds for other problems. We define these traversals recursively. 4 128 Therefore, the events E1 and E2 are not independent. In how many ways can the vertices of an n-cube be labeled 0, . In other words, most hash functions produce collisions. Therefore, p(n) = O(nk). †A starred 52. Find a formula for the kth term of the subsequence. (2.19) 403 Recurrence Relations Thus we obtain the solution Sn = 2n of (2.18). For example, the proof in Example 2.7 could be written as follows: n is odd if and only if n = 2k + 1 for some integer k if and only if n − 1 = 2k for some integer k if and only if n − 1 is even. What is the formal definition of "odd integer"? Specifically, write S(k, n) in terms of S(k − 1, i) for appropriate i. , sj , . Assume that (a, b) = (c, d). Using repeated squaring, compute a1 , a2 , a4 , a8 , . In how many ways can 10 balls be selected if at most one red ball is selected? In either case, the algorithm correctly computes the maximum value in the sequence. swap(si , si+1) 12. 1 + 3 = 4. The data are available to various applications. Example 1.11 Example 1.7 showed that Z is a subset of Q. 5 + (8 + 7 + · · · + 1) + (7 + 6 + · · · + 1) 64. Adding Binary Numbers This algorithm adds the binary numbers bn bn−1 · · · b1 b0 and bn bn−1 · · · b1 b0 and stores the sum in sn+1 sn sn−1 · · · s1 s0 . Published by Pearson Prentice Hall. We obtain the two binary trees in Figure 8.12(d) and (e). Because f(n) = (g(n)), there are constants C1 and C2 such that C1 |g(n)| ≤ |f(n)| ≤ C2 |g(n)| for all but finitely many positive integers n. The problem is to find a way to maximize the flow from the dock to the refinery and to compute the value of this maximum flow. Show that any tree with two or more vertices has a vertex of degree 1. Write an algorithm that evaluates the root of a game tree using an n-level, depth-first search with alpha-beta pruning. In words: Someone is shorter than someone. Algorithm 3.7 Depth-First Search for a Spanning Tree This algorithm finds a spanning tree using the depth-first search method. P(E | F) = P(E ∩ F)/P(F) 14. It would be particularly easy to prove this statement if we could assume that we can make postage of n − 2 cents. Write a backtracking algorithm that outputs all subsets of {1, 2, . ¬p ∨ t ¬q ∨ s ¬r ∨ st p∨q∨r∨u 2. (1.3) Taking n = 2 in (1.3), we obtain a2 = a1 + 3. We will buy a new computer. Then the 2k marked squares are contained in 2k − 1 diagonals. Show, by giving a proof by contradiction, that if 100 balls are placed in nine boxes, some box contains 12 or more balls. Example 3.15 Find a theta notation in terms of n for the number of times the statement x = x + 1 is executed. For the same reason, 5 is also a common divisor of 30 and 105. Specifically, the demand is given by the equation p = a − bq, Price Supply Demand where p is the price, q is the quantity, and a and b are positive parameters. Write a command to search the Web for information on lung disease other than cancer. (n4) 11. Each element of X is assigned a unique value in Y : 1 is assigned the unique value a; 2 is assigned the unique value b; and 3 is assigned the unique value a. Discussion First, let us consider giving a direct proof of this statement. The next string is 12356, followed by 12357. Integers in the range 0 to m, m ≥ 8, not both zero, are input to the Euclidean algorithm. Show that there are paths with no repeated edges from v1 to v2 and from v3 to v4 such that every edge in G is in exactly one of the paths. Therefore, lacrosse is wildly popular in California. print parents(V, parent) { for each v ∈ V println(v, parent(v)) } 37. The conclusion follows. The subtree of T rooted at x is the graph with vertex set V and edge set E, where V is x together with the descendants of x and E = {e | e is an edge on a simple path from x to some vertex in V }. Then A = B since there is at least one element in A (1 for example) that is not in B. If s is a sequence of length 1, s1 = 9, and key = 8, the algorithm does not terminate. 208 Algorithms Our description of what an algorithm is will suffice for our needs. v2 v3 w1 v4 T1 5. Theorem 1.11 Fundamental Theorem of Arithmetic Any integer greater than 1 can be written as a product of primes. If x < 0, |x| = −x and sgn(x) = −1. x ∈ / X: x is not an element of the set X 6. Canceling 2 gives q2 = 2k2 . Write a recursive version of the Euclidean algorithm that executes the check for a < b and the call to the swap function one time. 72 17. 5 32. Since X ∈ P(X), X ∈ P(Y). Under what conditions is an edge in a connected graph G contained in every spanning tree of G? ∃x ¬P(x) Suppose that the domain of discourse of the propositional function P is {1, 2, 3, 4}. The key idea is to compute the remainder after each multiplication thereby keeping the numbers relatively small. Compute S1 , S2 , S3 , and S4 . From p → q and q → s, we may use the hypothetical syllogism to conclude p → s. a b c d e f g h i j k l b f c e d 487 Graph Theory C1 7. BGFAEDC PROSE NOT NECESSARILY 26. 122. In this case, as shown in the decision tree, we next compare C1 with C5 (which is known to be a good coin) and immediately determine whether the bad coin is C1 or C2 and whether it is heavy or light. Proof To show if |x| < d then −d < x < d, we use proof by cases. In Exercises 44–49, represent the given proposition symbolically by letting p : You run 10 laps daily. Use proof by contradiction to prove that for all x ∈ R, if x2 is irrational, then x is irrational. In this case, there are two comparisons with null at line 1, after which the procedure returns. How many eight-bit strings begin and end with 1? The (n − 3)-bit string following the initial 110 cannot contain the pattern 111; therefore, there are Sn−3 strings of type (c). Show that in any simple, connected, planar graph, e ≤ 3v − 6. In fact, P(E2) = 1/3 = P(E3) because, if the runner completes the marathon, it is not run again. Let n= m bi 2i i=0 be the binary expansion of n. The algorithm begins by setting result to 1 and x to a. Write the prime factorizations of m and n as m = pa11 pa22 · · · pakk and n = pb11 pb22 · · · pbkk . The initial position is shown. Example 7.7 Use the Binomial Theorem to derive the equation n C(n, k) = 2n . We use the notation in the solution to Exercise 2. Show that if T1 and T2 are distinct trees in X1 , then XT1 ∩ XT2 = ∅. Represent the statement If (a ≥ c or b < c), then b ≥ c symbolically using the definitions of Exercise 8. Explain why they are isomorphic. Edges (a, b), (e, f), and (f, z) are properly oriented, so the flow in each is increased by 1. Notice that g is one-to-one. The worst case for the Euclidean algorithm occurs when the number of modulus operations is as large as possible. , v17). There are four possibilities for the orientations of the edges e1 and e2 incident on x. Example 4.5 Represent the argument The bug is either in module 17 or in module 81. p ∨ q 17. Write the converse and contrapositive of each proposition symbolically and in words. 4 ⎝0 0 1 0⎠ 1 1 1 0 0 Obviously, the matrix of a relation from X to Y is dependent on the orderings of X and Y . The set X − Y = {x | x ∈ X and x ∈ / Y} is called the difference (or relative complement). 137. If we add an edge from v to w, the resulting graph has an Euler cycle, namely, the path P together with the added edge. If the root has exactly one child, the number of vertices will be minimized. We must show if |x| < d then −d < x < d and if −d < x < d then |x| < d. Notice that in Example 1.24, |X × Y | = |X| · |Y | (both are equal to 6). 1 1 0 1 4 3 0 1 2 0 2 1 1 0 0 1 0 0 0 1 0 1 1 1 0 0 1 3 0 2 1 0 0 1 0 0 1 1 0 0 0 0 4 5 1 0 0 1 1 2 1 1 0 2 1 0 1 3 0 0 1 0 0 1 1 3 1 2 0 0 1 1 0 0 1 1 0 1 0 0 0 0 1 2 1 1 1 0 0 1 0 0 0 1 0 1 0 0 0 1 0 strategy: Play nim exactly like nim unless the move would leave an even number of singleton piles and no other pile. If R and S are antisymmetric, then R ∩ S is antisymmetric. By Theorem 8.12, there are Cn−1 (n − 1)-vertex binary trees. Basis Step (y = 0) We must prove that A(x + 1, 0) = AO(x + 1, 2, 3) − 3. Our version of Prim's Algorithm examines (n3) edges in the worst case (see Exercise 6) to find a minimal spanning tree for a graph having n vertices. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is D1 × D1 . Suppose that we are given text t (e.g., a word processor document) and we want to find the first occurrence of pattern p in t (e.g., we want to find the first occurrence of the string p = "Nova Scotia" in t) or determine that p does not occur in t. As we can see from Figures 2.1 and 2.2, a connected graph consists of one "piece," while a graph that is not connected consists of two or more "pieces." These "pieces" are subgraphs of the original graph and are called components. , sn is x—the value returned by the algorithm. Since A0 is the beginning amount, we have the initial condition A0 = 1000. The n-cube (see Example 1.7) is another model for parallel computation. 103 Proofs That is, if n ≥ 1, n! is equal to the product of all the integers between 1 and n inclusive. Assume the conditions of Exercise 10 and assume further that one incorrect response to an array comparison is allowed. 184 Equivalence Relations Let's list all of the strings related to 01111010. Two ways of finding the greatest common divisor of a and b were presented. If R is reflexive, then R−1 is reflexive. Contrapositive: ¬q → ¬p; if 9 ≤ 12, then 4 ≥ 6; false. Evidently, after making the substitution, the recurrence relation for Cn can be solved by the methods of Section 2. An equivalent characterization of 174 Functions, Sequences, and Relations "antisymmetric" can also be used: Show that for all x, y ∈ X, if x = y, then (x, y) ∈ R or (y, x) ∈ R. The following figures show how the algorithm draws the 2-cube and 3-cube. Suppose that n is in the second position. Now let's take stock of the situation: We have 25 vertices and each vertex has degree 5. If n = 1, 2, or 3, the algorithm uses an optimal sort. Prove (a + b + c)n = n! ai bj cn−i−j . (a) 14 (b) 18 (c) 192 (d) ank = 4k 6. Either you are healthy or you do not run 10 laps daily, and you do not take multi-vitamins. Example 1.4 In a digital picture, we wish to encode the amount of light at each point as an eight-bit string. Since there is no such loop, we do not select this edge. In Exercises 8–10, write the relation R, given by the matrix, as a set of ordered pairs. A subgraph G of a graph G is obtained by selecting certain edges and vertices from G subject to the restriction that if we select an edge e in G that is incident on vertices v and w, we must include v and w in G . The traveling salesperson problem furnishes another example of a problem in graph theory and operations research. If x ≥ y, max{x, y} = x and min{x, y} = y. c = an mod z = 58441 mod 5959 = 3237 Chapter Self-Test 1. We square x to obtain a4 . [If (a), then (b).] The proof of this result was given before the statement of the theorem. Algorithm 4.1 Dijkstra's Shortest-Path Algorithm This algorithm finds the length of a shortest path from vertex a to vertex z in a connected, weighted graph. n = 637 5. To see this, suppose that there is such a path and consider vertex Wor. d 4. f = {(−5, 25), (−4, 16), (−3, 9), (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25)}. Let P(x) denote the statement "x is a man," and Q(x) denote the statement "x cheats on his wife." Symbolically, the clarified statement is ∃x(P(x) ∧ ¬Q(x)). We discuss that problem in Section 3. retreat(i, j) } Trees } } else advance(i, j) // sets j to 10 if advanced off board if (j == 10) { // Solution! show values() return } } Section 4 Review E = ∅ T = (V , E) while (|E | < n − 1) { among all edges that if added to T would not complete a cycle, choose e = (vi , vj) of minimum weight E = E ∪ {e} V = V ∪ {vi , vj } T = (V , E) } return T 1. Describe the algorithm for generating permutations. True 31. Is ϒ nonincreasing? Suppose that ns mod φ = 1 = ns mod φ, 0 < s < φ, 0 < s < φ. In any graph, must the number of vertices of odd degree be even? Furthermore, for every x (and, in particular, for at least one value of x), there exists y such that x > y is false. If b is greater than large, we execute line 4 large = b but if b is not greater than large, we skip to line 5. , pn are called the hypotheses (or premises), and the proposition q is called the conclusion. Since none of these numbers divides 43, the condition n mod d == 0 is always false. Therefore fk1 > fk2 . P(E) + P(E) = 1 Section 6 1. The graph GK6 has a Hamiltonian cycle. If the path begins with the edge (v1 , v2), the next edge in the path must be (v2 , v1). is to find an explicit formula for the general term an . Then x ∈ / B. Draw K3 and K5 . We derive a recurrence relation for the sequence a0 , a1 , . In the notation of Definition 4.1, p = 5, q = 4, s1 = 1, s2 = 3, s3 = 2, s4 = 4, s5 = 6, t1 = 1, t2 = 3, t3 = 4, and t4 = 2. Represent the Instant Insanity puzzle by a graph. What is the cardinality of {∅}? (The set D1 is defined before Exercise 1.) 16. At lines 16 and 17, s4 is set to 6 and s5 is set to 7. For example, if a credit card number is sent over the Internet, it is important for the number to be read only by the intended recipient. Then, at lines 8 and 9 we label vertex a as (−, ∞). Write a recurrence relation for bn . The reason is that entry i, j in A2 is nonzero if and only if there are elements (i, k) and (k, j) in R (see the proof of Theorem 5.6). Show that for any set X, X is not equivalent to P(X), the power set of X. (1, 1) S x (0, 0) S = {f −1 ({y}) | y ∈ Y }. Find the greatest common divisor of each pair of integers in Exercises 12–24. Prove that m3 + 2n2 = 36 has no solution in positive integers. Aproposition is a statement that is either true or false. ■ Look for patterns. According to the preceding exercise, R is an equivalence relation. In the Inductive Step of the Strong Form of Mathematical Induction, when you assume that the statement S(k) is true, you must be sure that k is in the domain of discourse of the propositional function S(n). Let a and b be vertices of G. a, g 4. Basis Step 31 + 71 − 2 = 8 is divisible by 8. Does the graph of Exercise 6 have a Hamiltonian path? Then G also has a minimal number of vertices and by Exercise 26, G requires more than four colors to color. To decide where to add the vertex and edge, we begin at the root. Sabic claims to have discovered an algorithm that uses at most 100n comparisons in the worst case to sort n items, for all n ≥ 1. The relation is an equivalence relation. A cycle (or circuit) is a path of nonzero length from v to v with no repeated edges. The vertices b, c, d, and e represent intermediate pumping stations. We can use decision trees to estimate the worst-case time to sort. A B 1 1 C 0 Figure 9.4 The second player (circle) should move to position C since it is least valuable (to box). References on resolution and automated reasoning are [Gallier; Genesereth; and Wos]. Therefore an equivalent formulation is If John takes calculus, then he has sophomore, junior, or senior standing. Prove that A is the matrix of R1 ∪ R2 . In nim, if the terminal vertex is a circle, the first player removed the last token and lost the game. Eight coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. , n} containing an even number of elements. Since f(n) = O(g(n)), there exist constants such that f(n) ≤ C g(n) 4. Since there is one winning combination, the probability of choosing the winning numbers is 1 6! = 0.000000049. Call the resulting equation E1 . Start with the original equation an = . Floor of x, x: greatest integer less than or equal to x 7. At line 3, c > large (3 > 5) is false, so we do nothing. Since the degree of every vertex is even, by Theorem 2.18, G has an Euler cycle. Substituting the second-to-last formula for 16 into the previous equation yields 3. R = {(x, a), (x, c), (y, a), (y, b), (z, d)}; ordering of X: x, y, z; ordering of Y : a, b, c, d In Exercises 4– 6, find the matrix of the relation R on X relative to the ordering given. (b) A cycle (c) A simple cycle a 20. Let G be a simple, disconnected graph with n vertices having the maximum number of edges. For example, suppose that the worst-case time of an algorithm is t(n) = 60n2 + 5n + 1 for input of size n. Prove that bn = (lg n). Logic is a tool for the analysis of proofs. We can arrange for nq + t ≥ 0 to hold by choosing nq ≥ −t. What is a cycle? We obtain the new labels L(b) = min{∞, 0 + 2} = 2, L(f) = min{∞, 0 + 1} = 1 (see Figure 4.3). Even though merge sort, Algorithm 3.8, is optimal, it may not be the algorithm of choice for a particular sorting problem. The problem can be modeled as a graph. (If d | n, there exists an integer q such that n = dq. Let An denote the amount at the end of n years and let i be the interest rate expressed as a decimal. Given f = {(a, b), (b, a), (c, b)}, 58. 613 Network Models Example 4.8 For the graph in Figure 4.1, if S = {A, B, D}, we have R(S) = {J2 , J5 } and |S| = 3 > 2 = |R(S)|. Here, there is apparently no formula for membership; the definition just tells us which pairs make up the function. Thus n = m. We then successively use the recurrence relation to replace each of an−1 , . If A is the adjacency matrix of a simple graph, the ijth entry of An is equal to the number of paths of length n from vertex i to vertex j. At this point, we encounter the first character R. Using the result of Example 3.13, we see that the truth values of ∃x(P(x) ∧ ¬Q(x)) and ∃x ¬(P(x) → Q(x)) are the same. We have shown that if (i, k) ∈ R2 ◦ R1 , then the ikth entry in A1 A2 is nonzero. Discussion Here a direct proof seems particularly bleak. The solution is found at vertex 8. Show that S(n, k) satisfies the recurrence relation C(n, j)Fj . Notice that if A is a subset of B, it is possible that A = B. The resulting graph G has n edges; hence, by the inductive assumption, (7.3) holds for G . What is the average-case time of an algorithm? n k=0 28. 6 · 5 · 4 · 3 + 6 · 5 · 4 · 3 · 2 5. a b c e d f g h i 44. Use Exercise 13 to show that K5 is not planar. = min{a, min{b, c}}. Since no larger product of primes is common to both 30 and 105, we conclude that 15 is the greatest common divisor of 30 and 105. Similarly, we write 47 in the region representing CALC ∩ PSYCH ∩ COMPSCI and 14 in the region representing CALC ∩ PSYCH ∩ COMPSCI. from (1, 0) to (k, k − 1) give a good route in the (k − 1) × (k − 1) grid with corners at (1, 0), (1, k − 1), (k, k − 1), and (k, 0). If p and r are true and q is false, the hypotheses are true, but the conclusion is false. Choose x ∈ X. This problem is due to John McCarthy. cos x + cos 2x + · · · + cos nx = cos[(x/2)(n + 1)] sin(nx/2) sin(x/2) provided that sin(x/2) = 0. We will see that this problem can be reduced to finding a maximal flow in a network. Then G1 = G2 , but f(G1) = 1 = f(G2). We may choose y = x+1 to make x > y false. Find numberss and t such that s n + t φ = 1. {1, 2, 2, 3}, {1, 2, 3} 30. If we select a from all factors and multiply, we obtain the term aaa. By the Multiplication Principle, there are 5 · 4 · 3 · 2 = 120 strings. Adding these inequalities yields s1 + s2 + · · · + sn < nA. A rise/fall permutation is a permutation p of 1, 2, . Suppose that there are distinct simple paths P1 and P2 from a to b in T (see Figure 2.6). (Examples are 13, 147, 8.) 62. How many selections exclude Connie? Determine the truth value of each statement in Exercises 42–59. Then x = 1 or x = 2. f(n) = O(g(n)) means that for every positive constant C, |f(n)| > C|g(n)| for infinitely many positive integers n. (∗) sin(x/2) We must show that the right-hand side of (∗) is equal to cos[(x/2)(n + 2)] sin[(n + 1)x/2] . k e a h e c b j l m j f o f h i h b a h h d c k j b e l f 4. Suppose that n = 2k . It then follows that 8 divides 4(7n + 1). ■ List the permutations that begin with sn . 0 ≤ x1 ≤ 6, x2 ≥ 0, x3 ≥ 0 27. Input: a and b (nonnegative integers, not both zero) Output: Greatest common divisor of a and b gcdr(a, b) { // make a largest if (a < b) swap(a, b) if (b == 0) return a r = a mod b return gcdr(b, r) } Algorithm 3.9 first makes a largest. 120. As an example, let us decode the string 01010111. ■ To show that a particular relation R is an equivalence relation, go directly to the definitions. Similarly, all the off-diagonal elements of An are equal. If the domain of the sequence is infinite (as in Example 2.1), we say that the sequence is infinite. (1) In order for (a) to be false, we must have at least one value of x and at least one value of y such that P(x, y) is false. Is E1 onto? Use depth-first search (Algorithm 3.7) with the vertex ordering fdehagbci to find a spanning tree for the graph of Exercise 9. 12, 21 Input: Input: Output: 9. Prove that the following are equivalent for the integer n: (a) n is odd. Cycle, simple cycle 4. Exercises Answer the questions in Exercises 1–6 for the tree in Figure 2.1. 1. Yes 67. Then x ∈ A ∪ ∅. The same argument can be used to show that there are 32 eight-bit strings that begin 111. Input: An array row of size 4 Output: true, if there is a solution false, if there is no solution [If there is a solution, the kth queen is in column k, row row(k).] four queens(row) { k = 1 // start in column 1 // start in row 1 // since row(k) is incremented prior to use, set row(1) to 0 row(1) = 0 while (k > 0) { row(k) = row(k) + 1 // look for a legal move in column k while (row(k) ≤ 4 ∧ column k, row(k) conflicts) // try next row row(k) = row(k) + 1 if (row(k) ≤ 4) if (k == 4) return true else { // next column 524 Trees k =k+1 row(k) = 0 } else // backtrack to previous column k =k−1 } return false // no solution } The tree that Algorithm 3.10 generates is shown in Figure 3.3. The numbering indicates the order in which the vertices are generated. Problem-Solving Corner Problem (a) How many routes are there from the lower-left corner to the upper-right corner of an m×n grid in which we are restricted to traveling only to the right or upward? 284 Introduction to Number Theory Let c be a common divisor of a and b. Desktop Fax My Documents My Computer My Data Sources My Pictures archived basement water 11-18-03 DSC01007 11-18 -03 My Network Places My eBooks DSC01008 11-18-03 Figure 1.8 The structure of Figure 1.7 shown as a rooted tree. Prove that the graph has the specified properties. n 85. The conjecture is correct and the formula can be proved by mathematical induction (see Exercise 1). Proof Let C = (v0 , e1 , v1 , . 28. Let an denote the number of pairs of rabbits at the end of the nth month. For example, the dice-rolling program would need pairs of pseudorandom 139 Functions, Sequences, and Relations numbers, each between 1 and 6, to simulate the outcome of rolling dice. Write the negation of each proposition in Exercises 28–31 in words and symbolically. , qn satisfy pi = qi for i = 1, . Farley goes first and chooses a value for x. How does selection sort work? If x is greater than sn , the maximum value in the sequence s1 , . State the Euclidean algorithm. Therefore, P(2) = P(3) = P(4) = P(5) = P(6) = 1 8 and P(1) = 3P(2) = 3 . (a) p q p→q q→p T T F F T F T F T F T T T T F T 74. A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher. Again using Bayes' Theorem or noting that P(D | H) + P(R | H) + P(L | H) = 1, we obtain P(L | H) = 0.326. Establish the recurrence relation Sn+1,k = Sn,k−1 + kSn,k 62. What is a constructive existence proof? Therefore, Algorithm 1.2 is correct. In this section we discuss isomorphic trees, isomorphic rooted trees, and isomorphic binary trees. Exercises 38–40 concern three persons who each randomly choose a locker among 12 consecutive lockers. Example 1.1 How many dinners are available from Kay's Quick Lunch consisting of one main course and an optional beverage? Exercises Tell whether or not each recurrence relation in Exercises 1–10 is a linear homogeneous recurrence relation with constant coefficients. Find a recurrence relation and initial conditions for the sequence {bn }. Two squares on a single domino can have the same number. Show that when b > k in Example 1.9, the differences between successive prices increase. p ∧ r 63. Use mathematical induction to show that fn2 n f2k−1 = f2n , for all n ≥ 3. We write the problem as + 10011011 1011011 As in decimal addition, we begin from the right, adding 1 and 1. Prove that R is an equivalence relation on V . (f) The terminal vertices are Eros, Apollo, Athena, Hermes, Heracles, Poseidon, Hades, Ares, Atlas, and Prometheus. One approach to searching for p is to check whether p occurs at index 1 in t. FROM PAINT YOUR WAGON Trees form one of the most widely used subclasses of graphs. Such theorems are proved by using the equivalence p ↔ q ≡ (p → q) ∧ (q → p); that is, to prove "p if and only if q," prove "if p then q" and "if q then p." Example 2.7 Prove that for every integer n, n is odd if and only if n − 1 is even. One technique is to let x stand for an 57 Sets and Logic arbitrary element in X and then find a value for y ∈ Y (one value suffices!) that makes P(x, y) false. , n + p − 1. n(n + 1) 3. b 286 a 0 1 2 3 4 5 6 7 8 9 10 11 12 13 0 1 2 3 4 5 6 7 8 9 10 11 12 13 — 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 2 1 2 1 2 1 2 1 2 1 2 1 2 0 1 2 1 2 3 1 2 3 1 2 3 1 2 3 1 2 0 1 1 2 1 2 2 3 1 2 2 3 1 2 0 1 2 3 2 1 2 3 4 3 1 2 3 4 0 1 1 1 2 2 1 2 2 2 3 3 1 2 0 1 2 2 3 3 2 1 2 3 3 4 4 3 0 1 1 3 1 4 2 2 1 2 2 4 2 5 0 1 2 1 2 3 2 3 2 1 2 3 2 3 0 1 1 2 2 1 3 3 2 2 1 2 2 3 0 1 2 3 3 2 3 4 4 3 2 1 2 3 0 1 1 1 1 3 1 4 2 2 2 2 1 2 0 1 2 2 2 4 2 3 5 3 3 2 1 Introduction to Number Theory TABLE 3.2 ■ Smallest input pair that requires n modulus operations in the Euclidean algorithm. this is not the shortest path from a to z. For example, b ∈ X, but (b, b) ∈ / R. The operators +, −, ∗, and / operate on pairs of operands or expressions. We begin with coin puzzles. What is the minimax procedure? Clearly, (T − {y}) ∪ {x} is a spanning tree. Let S = {f −1 ({y}) | y ∈ Y }. This unique a function because there are two value y is denoted f(x). Consider a greedy algorithm that selects stamps by choosing as 533 Trees many of the largest denomination as possible, then as many of the second-largest denomination as possible, and so on. Since there are three kinds of cabs, two kinds of cargo beds, and five kinds of engines, the correct number of ways to personalize the big pickups is 3 · 2 · 5 = 30, not 32. The sequence walk(1), walk(2), walk(3), . Therefore, the probability of obtaining no defective microprocessors is C(980, 5) 980 · 979 · 978 · 977 · 976 = = 0.903735781. The order of the words is determined by the order of the letters at position p. 1 e d f G2 a c h 2 G1 2 1 5 b 3 The second graph is G2 of Exercise 2. The function {(a, 1), (b, 3), (c, 0)} is onto {0, 1, 3}. In ordinary speech and writing, we combine propositions using connectives such as and and or. Example 3.2 If we let X = {Bill, Mary, Beth, Dave} and Y = {CompSci, Math, Art, History}, our relation R of Table 3.1 can be written R = {(Bill, CompSci), (Mary, Math), (Bill, Art), (Beth, History), (Beth, CompSci), (Dave, Math)}. Proof Let X, Y , and Z be arbitrary sets. Various methods to speed up the search of the game tree are discussed and compared. Give an example of a reflexive relation. What is the value of the decimal number dn dn−1 . According to Theorem 2.11, Sn = r n is a solution of (2.18), where r is a solution of t 2 − 4t + 4 = 0. In a direct proof, the negated conclusion is not assumed, whereas in a proof by contradiction, the negated conclusion is assumed. Section 6 21. Now s = (s mod φ)(ns mod φ) = s ns mod φ = (s n mod φ)(s mod φ) = s. As suggested earlier, we consider the cases: x ∈ X ∪ Y and x ∈ / X ∪ Y . This algorithm sorts the sequence s1 , s2 , ..., sn in nondecreasing order by recursively sorting the first n − 1 elements and then inserting sn in the correct position. Thus Algorithm 2.7 runs in time (logb m). Write a program that finds an increasing or decreasing subsequence of length n + 1 of a sequence of n2 + 1 distinct numbers. Finally, show that F(n) + an+1 = F(n + 1). Let n = 2i+1 . And so forth. (8.1) The 302 numbers consisting of (8.1) together with c1 + 1, c2 + 1, ..., c151 + 1 (8.2) range in value between 1 and 301. [Case II: x < 0.] Proof of this case goes here. , pk ; otherwise, g would not divide m or n (or both). 12 17. (b) v0 , . Section 2 Review 1. A decision tree is a binary tree in which the internal vertices contain questions with two possible answers, the edges are labeled with answers to the questions, and the terminal vertices represent decisions. ∅ = U, U = ∅ 36. Example 4.1 TABLE 4.1 ■ Decomposing the factorial problem. Choose v ∈ T with minimum label and remove v from T . n i . Pigeonhole Principle (Second Form) If f is a function from a finite set X to a finite set Y and |X| > |Y |, then f(x1) = f(x2) for some x1 , x2 ∈ X, x1 = x2 . Executing power = power ∗ b sets power to 16. p Example 2.2 If p : It is raining, q : It is cold, then the conjunction of p and q is p ∧ q: It is raining and it is cold. In this case, the algorithm does not change the value of large; thus, large is the largest value in the subsequence s1 , . , 2n − 1 in such a way that an edge connects two vertices if and only if the binary representation of their labels differs in exactly one bit. Many ways are known to minimize the height of a binary search tree (see, e.g., [Cormen]). Example 2.5 Most programming languages define "and" exactly as in Definition 2.3. For example, in the Java programming language, (logical) "and" is denoted &&, and the expression x < 10 && y > 4 is true precisely when the value of the variable x is less than 10 (i.e., x < 10 is true) and the value of the variable y is greater than 4 (i.e., y > 4 is also true). 95. Compare the times required by the programs. Prove that d | gcd(a, b). Show that R is an equivalence relation. Example 1.23 The function f = {(1, a), (2, b), (3, a)} is not one-to-one since f(1) = a = f(3). The next bit is 1, so we move left and encounter the next character A. To obtain all of the permutations, when we find a permutation, we print it and continue (whereas in the four-queens algorithm, being content with one solution, we terminated the algorithm). Similarly, ai i∈S denotes the product of the elements {ai | i ∈ S}. Since R and S are symmetric, (y, x) ∈ R and (y, x) ∈ S. b e1 c e5 e6 e8 e7 d e 21. The number of solutions is also the number of solutions C(k +1+n−k −1, n − k) = C(n, k) with xk+2 = 0 plus the number of solutions C(k + 1 + n − k − 1 − 1, n − k − 1) = C(n − 1, k) with xk+2 = 1 plus · · · plus the number of solutions C(k + 1 + 0 − 1, 0) = C(k, k) with xk+2 = n − k. Prove that for every real number ε > 0, there exists a positive integer N such that L − ε < an ≤ L for every n ≥ N. The situation is different for trees. The degree of c is 3 since c is incident on the three edges (c, b), (c, d), and (c, e). Example: If the sequence is AMY BRUNO ELIE DAN ZEKE, the algorithm returns the value 4. vi . This rule of inference is called universal instantiation. If equality holds in (3.1), then the flow is maximal and the cut is minimal. Similar arguments (see Exercises 74–76) justify the other rules of inference listed in Figure 5.1. Rule of Inference Name ∀xP(x) ∴ P(d) if d ∈ D Universal instantiation P(d) for every d ∈ D ∴ ∀x P(x) Universal generalization ∃xP(x) ∴ P(d) for some d ∈ D Existential instantiation P(d) for some d ∈ D ∴ ∃x P(x) Existential generalization Figure 5.1 Rules of inference for quantified statements. Finally, reverse the substring to the right of d's original position. Then f(k, 0) = k − 0 = k. At line 5, since key ('S') is not equal to s3 ('F '), we proceed to line 7. You can participate. 1 0 1 0 1 0 1 0 32. Thus, if algorithms A and B have run times that are (f(n)) and (g(n)), respectively, and (f(n)) is above (g(n)) in Table 3.3, then algorithm A is more time-efficient than algorithm B for sufficiently large inputs. Prove that S is one-to-one. 1 1 The argument given in Example 5.5 holds for any relations. If 1 holds, then s = 1 and u = 1, so su = 1 and su + tv is nonzero. Example 2.28 Let X = {a, b}. We have shown that if (a, b) = (c, d), then a = c and b = d. For example, we can compute n! recursively because n! = n · (n − 1)! for all n ≥ 1. If n is even and m > 1 or if m is even and n > 1, there is a Hamiltonian cycle. By Theorem 5.14, the truth values of ∃x ¬(P(x) → Q(x)) and ¬(∀x P(x) → Q(x)) are the same. By the Principle of Mathematical Induction, it follows that any 2k × 2k deficient board can be tiled with trominoes, k = 1, 2, . Assume that the ages are given as whole numbers. ti . Two sets X and Y are equal and we write X = Y if X and Y have the same elements. Solve the recurrence relation of Exercise 66 in case n is a power of 2, assuming that b1 = 1. Proof We first prove that if n is odd then n − 1 is even. Let a = b = 2. A directed Euler cycle in G is a sequence of edges of the form (v0 , v1), (v1 , v2), . Define a function from X∗ × X∗ to X∗ as f(α, β) = αβ. on X = {a, b, c} is antisymmetric. Let T1 be a binary tree with root r1 and let T2 be a binary tree with root r2 . a b j c i d h e g a 482 f 2 f Graph Theory 7 ➜ Planar Graphs Three cities, C1 , C2 , and C3 , are to be directly connected by expressways to each of three other cities, C4 , C5 , and C6 . g(S) = {a}, g(T) = {a, c}, g−1 (U) = {1}, g−1 (V) = {1, 2, 3} 75. Now X contains an element less than or equal to n; so it follow that X has a least element.) Basis Step (n = 0) If X contains an element less than or equal to 0, then X contains 0 since X consists of nonnegative integers. The resulting matrix is the adjacency matrix of the graph. Our next algorithm (Algorithm 3.2) is binary search. = fn+2 , fn+1 for all n ≥ 1. Suppose that g(x1) = g(x2). Show that a graph G with n vertices and fewer than n − 1 edges is not connected. Therefore ta is the least upper bound for tX. Next, we circle vertex e and update the label of vertex z (see Figure 4.11). The Quotient-Remainder Theorem states that if d and n are integers, d > 0, there exist integers q (quotient) and r (remainder) satisfying n = dq + r 0 ≤ r < d. State the Strong Form of Mathematical Induction. (Assume that all the values are real numbers.) The algorithm inserts x into the sequence so that the resulting sequence is sorted in nondecreasing order. Exercises 56 and 57 refer to the sequence Sn defined by S1 = 0, S2 = 1, Sn = Sn−1 + Sn−2 , n = 3, 4, . In addition to circling a vertex, we will also label it with the name of the vertex from which it was labeled. In the following example, we use backtracking to solve a puzzle. What is a maximal matching? 58 29. If n = fk1 , then n is trivially the sum of distinct Fibonacci numbers, no two of which are consecutive. At termination, L(z) will be equal to ∞ if there is no path from a to z. Theorem 2.23 A graph has a path with no repeated edges from v to w (v = w) containing all the edges and vertices if and only if it is connected and v and w are the only vertices having odd degree. (The answer to the question is "yes" for n ≥ 5; see ["Problem 1186"]. Therefore there are 2 truth tables for an n-variable function. In positive statements, "any," "all," "each," and "every" have the same meaning. Then (z, y), (y, x) ∈ R. Then there are at least k values a1 , . In view of the preceding result, the smallest possible board that might have a knight's tour is the 2 × 2 board, but it does not have a knight's tour because the board is so small the knight has no legal moves. To prove that ∃x∃y P(x, y) is true, find one value of x ∈ X and one value of y ∈ Y that make P(x, y) true, where the domain of discourse is X × Y . [p2 → p3 .] Proof of p2 → p3 goes here. The quaternions provided inspiration for the development of modern abstract algebra. In general, the cost Cn of traveling n miles is 1.00 (the cost of traveling the first mile) plus 0.50 times the number (n − 1) of additional miles. Therefore, if 4 megabytes is better than no memory at all, then we will buy a new computer and we will buy more memory. For example, the hypothesis "x is a real number" can be divided into cases: (a) x is a nonnegative real number and (b) x is a negative real number. 1 2 3 4 {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} is a partial order on {1, 2, 3}. An unsuccessful search in T corresponds to arriving at an added (box) vertex in T ∗ . A collection of sets S is pairwise disjoint if, whenever X and Y are distinct sets in S, X and Y are disjoint. {(x, y) | x and y have, at some time, lived in the same country} 11. In Section 2, we will show that cn = 2n − 1. f(x) = x − x 8. In Exercises 87–90, what relation must hold between sets A and B in order for the given condition to be true? By the inductive assumption, n − fk1 can be expressed as the sum of distinct Fibonacci numbers fk2 > fk3 > · · · > fkm , no two of which are consecutive: n − fk1 = m 35. If five student records are chosen randomly, what is the probability that they are chosen so that the first record selected has the lowest grade point average (GPA), the second selected has the second-lowest GPA, and so on? (c) For every positive integer n, there is a prime number§ larger than n. Under each of these vertices and on the same level, we place the vertices, v4 , v5 , v6 , and v7 , that can be reached from the root on a simple path of length 2. An edge between processors p and q indicates that p and q can communicate directly with one another. Assume that a person invests a sum of money at r percent compounded annually. At line 17 we examine edges (b, c) and (e, b). Formally, if r is the statement "n2 is even," we have deduced r ∧ ¬r. Answer 95–102 for the sequence r defined by i=1 n 91. In the latter case, the search can be terminated when the index in t is so large that there are not enough characters remaining in t to accommodate p. Are the events "there are children of both sexes" and "there is at most one boy" independent? Following lines 3–7 (left/root/right) of Algorithm 6.3, we obtain the inorder listing CBDEAFIHJG. Theorem 3.6 If integers in the range 0 to m, m ≥ 8, not both zero, are input to the Euclidean algorithm, then at most log3/2 2m 3 modulus operations are required. The worst-case time for input of size n of an algorithm is the maximum time needed to execute the algorithm among all inputs of size n. Give an example of an event when the experiment of Exercise 8 is performed. For example, in Figure 1.7 the length of the path that starts at a, visits c, and terminates at b is 8. Moreover, equality holds in (3.1) if and only if (a) Fij = Cij for i ∈ P, j ∈ P and (b) Fij = 0 for i ∈ P, j ∈ P. An acyclic graph is a graph with no cycles. One idea is to order the children of the vertices to be evaluated so that the most promising moves are examined first (see Exercises 23–26). Hints/Solutions to Selected Exercises Section 1 Review 15. Finally, show values prints the array s. If ai + 3 = aj + 6, two are three apart. The Fibonacci sequence {fn } is defined by the equations f1 = 1 f2 = 1 fn = fn−1 + fn−2 for all n ≥ 3. If there is a child to move to, we repeat this process. Let p be a prime factor of m (m may or may not itself be prime; see Exercise 33). Given statement: q → p; true. m 2 ak = m ⎪ ⎣m − k if k > . At lines 10–13, we find that the largest index k satisfying sk > sm is 5. This brute-force solution could then be compared with the output of the program. A sequence si , si+1 , . Therefore, 4 megabytes is better than no memory at all. One simple collision resolution policy is to find the next highest (with 0 assumed to follow 10) unoccupied cell. The subsets of class 1 are just k-element subsets of X and there are C(n, k) of these. In the present problem, we can encode each route as a string of n R's (right) and m U's (up). b 6 w1 4 2 3 2 c w2 4 3 w3 A B d 3 Figure 1.3 A pumping network. Remove lens cap. where n1 , n2 , . This statement is true because, for every real number x and for every real number y, the conditional proposition (x > 0) ∧ (y > 0) → (x + y > 0) is true. Now x is part of a boundary for two faces. Find a formula for the entries in Aj . Sometimes to specify the domain of discourse D, we write a universally quantified statement as for every x in D, P(x). Yes Aurelio's Pizza Yes Jimmy's Place Cheap ? If n ≥ 0, then n − d ·0 = n ≥ 0 so n is in X. 8840 9921 452 2207 Hammer Pliers Paint Carpet a b b c 3 1 4 1 175 Functions, Sequences, and Relations 3. Choose x = x and y = y in the domain of discourse. Formal Solution Suppose that T has n vertices. To prove that ∃x∃y P(x, y) Section 6 66. The function f is both one-to-one and onto. a d b 5 1 c 4 1 f 6 3 3 8 h 2 g 6 g 3. b1 + b2 + · · · + bn = (a2 − a1) + (a3 − a2) + · · · + (an+1 − an) = −a1 + an+1 = an+1 − a1 6. Give a proof using mathematical induction that your algorithm for Exercise 15 is correct. Modify max sum3 so that it computes not only the maximum sum of consecutive values but also the indexes of the first and last terms of a maximumsum subsequence. Since 100 = 1, the symbol in position 0 represents the number of 100 's or 1's; since 101 = 10, the symbol in position 1 represents the number of 101 's or 10's; since 102 = 100, the symbol in position 2 represents the number of 102 's or 100's; and so on. In particular, we can make postage for k − p. Suppose that the available denominations are 1 = a1 < a2 < · · · < am . The domain of discourse is the set of rock fans. Find z3 . An = (1.17)An−1 , A0 = 4000 3. If p(x1 , x2 , . ai . Since b = 0, we proceed to line 6, where we set r to a mod b = 504 mod 396 = 108. 52. If all outcomes in a finite sample space are equally likely, how is the probability of an event defined? The equivalence class [1] consists of all x with x R 1. Table 3.1 shows the execution times, under this assumption, for various input sizes. The less than or equal to relation on the positive integers is a total order since, if x and y are integers, either x ≤ y or y ≤ x. Therefore, x ∈ A. Chapter Review Section 1 1. 289 Introduction to Number Theory We then substitute this formula for 53 into equation (3.9) to obtain 1 = 27 · 53 − 13 · 110 = 27(273 − 110 · 2) − 13 · 110 = 27 · 273 − 67 · 110. Draw all nonisomorphic, simple graphs having exactly five vertices, two components, and no cycles. (c) A flow whose value is |V | corresponds to a complete matching. 251 Algorithms 42. In particular, we must have 2m2 ≤ 40 and 3n2 ≤ 40. Bush: I understand everybody in this country doesn't agree with the decisions I've made. (b) a, c f e g h i a b c f g e d b a (c) d, a, c, h, j, k, l (d) e d i hj k l c 12. We can consider the numbers as contestants and the internal vertices as winners where the larger value wins. Example 2.6 The graph G of Figure 2.2 is not connected since, for example, there is no path from vertex v2 to vertex v5 . Since ∀x ∀y P(x, y) is true, regardless of which value of x is selected, P(x, y) is true for all y. (2.1) (a) Find s0 . Thus 1 ≤ n/2 < n. To prove p if and only if q, you must prove two statements: (1) if p then q and (2) if q then p. Give an example of a relation that is not an equivalence relation. Looking ahead, problem 2 is to show that there is only one (distinct) upper bound for a set X; thus, we usually say the least upper bound rather than a least upper bound. Then, n − k horizontal steps can be inserted among the k vertical steps in C(n, k) ways. In Exercises 4–9, assume that a sequence from {1, . (3.8) An arbitrary value of n falls between two powers of 2, say, 2k−1 < n ≤ 2k . It proceeds up a // column first. 32 14. Processor 5 is not connected to any processors. This algorithm builds a tree by iteratively adding edges until a minimal spanning tree is obtained. Therefore A = B. Definition 4.1 n factorial is defined as n! = 1 if n = 0 n(n − 1)(n − 2) · · · 2 · 1 if n ≥ 1. Explain why if we allow cycles of length 0, a graph consisting of a single vertex and no edges is not acyclic. Since Not all trominoes of the tiling are shown. There is a hurricane, but it is not raining. If D is greater than the data item at the current vertex v, we move to v's right child and repeat this process. This argument shows that X ⊆ Y . In either case, d ≤ d1 and d ≤ d2 . Prove the absorption laws for sets. Furthermore, lg n < k. Notice that in the digraph of this relation (see Figure 3.2) there are directed edges from b to c and from c to b, but there is no directed edge from b to b. The last permutation precedes the first permutation. s1 , . 12 + 22 + 32 + · · · + n2 = n(n + 1)(2n + 1) 6 5. Since T contains no cycles, T also contains no cycles. If x = 3, then x2 − 4x + 4 = 32 − 4 · 3 + 4 = 1 and again x ∈ A. Call the resulting equation E2 . (e) Show that S4,2 = 7. (Lucas was the first person to call the sequence 1, 1, 2, 3, 5, . Proof We begin by letting x be an arbitrary real number. Thus, even for a fixed size n, the time required by Algorithm 2.3 depends on the input. ¬p ∧ ¬q 7. i=i+2 13. Let T1 , T2 , ..., Tk be the components of T. (d, a, e, b, c, h, g, f, j, i, d) 3. (Later in this section we will define all of these terms carefully.) We have labeled each vertex with the first three letters of the city to which it corresponds. The decision tree analysis shows that any algorithm requires at least ten comparisons in the worst case to sort six items. ¬(∀x P(x)) 24. We then invoke this algorithm with i = 6, j = 5. Prove that if p ones and q zeros are placed around a circle in an arbitrary manner, where p, q, and k are positive integers satisfying p ≥ kq, the arrangement must contain at least k consecutive ones. 63. PROCESSING PRODUCES MANUSCRIPTS C / postfix: EBD/ ∗CA− − parened infix: ((E ∗ (B/D)) − (C − A)) WORD BUT – * B CLEAN 23. a b e d g c f i h k j l G m Figure 3.7 A graph with no Hamiltonian cycle. 458 The cycle (a, b, c, d, e, f, g, a) is a Hamiltonian cycle for the graph of Figure 3.4. The problem of finding a Hamiltonian cycle in a graph sounds similar to the problem of finding an Euler cycle in a graph. If we set k = n − 1 in this last expression, we have an = a1 + (n − 1) · 3. 24 Sets and Logic Example 3.4 shows that the proposition p → q can be true while the proposition q → p is false. Example 4.4 468 Find a shortest path from a to z and its length for the graph of Figure 4.7. We will apply Algorithm 4.1 with a slight modification. Farley can choose values (e.g., x = 1, y = 2) so that no matter which value you choose for z, the expression 4. The difference X − Y consists of all elements in X that are not in Y . Example: If the input sequence is 2 6 12 14 Algorithms and x = 5, the resulting sequence is 2 5 6 12 14. Show that A(x, y) = AO(x, 2, y + 3) − 3 for x, y ≥ 0. The quotient 3 becomes the new value of n. Color the map of Exercise 22, excluding the unbounded region, using four colors. For part (b), we again need the given statement, ∀x∃y P(x, y), to be true and ∃x∀y P(x, y) to be false. By definition, the statement ∀x∀y P(x, y), with domain of discourse X × Y , is true if, for every x ∈ X and for every y ∈ Y , P(x, y) is true. We then return to line 5. The statement for every x, P(x) is said to be a universally quantified statement. in hexadecimal? We would have to eliminate two edges each at b, d, i, and k, leaving 19 − 8 = 11 edges. What is a hypothesis in an argument? Write a program to determine whether one sequence is a subsequence of another sequence. What is a set? A proof by induction shows that n ≤ 2n−1 for all n ≥ 1. State the idempotent laws for sets. Since 3 ∈ A, but 3 ∈ B, A is not a subset of B. ∴p→r ∴q 18. Notice that if ∀x P(x) is true, the for loop necessarily runs to completion so that every member of the domain of discourse is checked to ensure that P(x) is true for every x. Prove that there exists k such that sk > A. Execution resumes in part (d) at line 4 after computing 0! (= 1). , sr (an r-combination of {1, . At lines 15 and 16, vertex 6 is added to the minimal spanning tree and edge (5, 6) is added to E. v2 v2 v3 28. 6 In the solutions to Exercises 55 and 58, a : b means "store item a in cell b." 55. First, we count the number of hands containing cards in spades and hearts. A recurrence relation defines the nth term of a sequence in terms of certain of its predecessors. We turn now to sorting. List the elements of [C], the equivalence class containing C. Write a program that generates a random n-vertex binary tree. c is the first vertex after a on P1 not in P2 . We assume that at least one pile has at least two tokens. Therefore, edges (a, b), (a, j), (j, i), (i, h), (g, f), (f, e), and (e, d) must be included. (The case that each pair is dissimilar is Exercise 14.) If any pair (Pi , Pj), (Pi , Pk), (Pj , Pk) (8.5) is similar, then these two pictures together with P1 are mutually similar and we have found three mutually similar pictures. Consider a 2k+1 × 2k+1 deficient board. If a table has n columns, the corresponding relation is called an n-ary relation. (The last equality follows from the formula for the geometric sum.) Since 2k ≤ n, taking logs, we obtain k ≤ lg n. For example, a graph with four edges and six vertices cannot be a tree because it violates parts (d) and (e). Find the height of the adjacent tree rooted at c. Then P(E) = k and P(xi) P(E) = i=1 n P(xi). Try your hand at the method by finding another solution to the puzzle represented by Figure 8.5. 2 R B 1 2 1 3 4 4 3 W 1 2 G 3 4 Figure 8.5 A graph representation of the Instant Insanity puzzle of Figure 8.1. 490 Graph Theory Example 8.1 2 Find a solution to the Instant Insanity puzzle of Figure 8.6. We begin by trying to construct one subgraph having properties (8.1) and (8.2). Notice that this proves that X has a least element. 20. If R and S are symmetric, then R ∪ S is symmetric. Use the Euclidean algorithm to find the greatest common divisor of the integers 396 and 480. Exercises 36–41 refer to a bag containing 20 balls—six red, six green, and eight purple. If si < A, the proof is complete; so assume that si > A. Dividing by 3, we obtain 1 10,000 ≤k≤ = 3333.333 . The literal meaning is: No man cheats on his wife. The solution counts ordered hands. e (a) (b) 37. 7c. What is a common multiple? Let's compute each side of the equation CX∪Y (x) = CX (x) + CY (x) for each x ∈ U. Thus D3 = 2. Let ri = ai mod 16 and let si = ri if ri ≤ 8 16 − ri if ri > 8. Adding the third edge gives the graph of Figure 8.6. For the same reason, the fourth edge cannot be incident on either of the vertices w1 or w2 of Figure 8.6. Adding the last edge gives a tree isomorphic to the tree T1 of Figure 8.3. Since a tree with five vertices must have a vertex of degree 2, we have found all nonisomorphic trees with five vertices. We use strong induction to prove that cn < 2n Discussion for all n ≥ 1. If x, y ∈ X and either x y or y x, we say that x and y are comparable. Subsets of Y containing a. The last n − 4 numbers can be arranged in En−4 ways. (h) Show that Sn,n−1 = C(n, 2) for all n ≥ 2. Prove that for all real numbers x and integers n, x = n if and only if there exists ε, 0 ≤ ε < 1, such that x + ε = n. Thus, lg n ≤ n−1 for all n ≥ 1. Now suppose that the root of T has children v1 and v2 . This is usually straightforward. 2 n−1 En = 73. Thus there are C(n, j)Fn−j functions of the desired type having exactly j 1's. Then the argument symbolically is p→q q→r ∴r→p The argument is invalid. Show that Sn = fn+2 , n = 1, 2, . (e) (f) (g) (h) If x and y are children of z, x and y are siblings. Define f(n) = (g(n)). C(15 + 6 − 1, 15) 45. by the equations c0 = 0, cn = cn/2 + 3 for all n > 0. (3.3) Using the generalized De Morgan's laws for logic and the fact that ¬(p → q) ≡ p ∧ ¬q, we find that (3.3) is equivalent to ∃x∃y∃z[(x, y) ∈ R ∧ (y, z) ∈ R] ∧ ¬[(x, z) ∈ R] or, equivalently, ∃x∃y∃z[(x, y) ∈ R ∧ (y, z) ∈ R ∧ (x, z) ∈ R]. If d2 < d1 , then d = min{d1 , d2 } = d2 and d = d2 < d1 . The greatest common divisor, written gcd(m, n), is the largest common divisor of m and n. p → q p∨q Use resolution to derive each conclusion in Exercises 2–6. Trace Algorithm 1.1 for the values a = 12, b = 3, c = 0. Write a command to search the Web for national parks in North or South Dakota. From x ≤ d and d ≤ d1 , it follows that x ≤ d1 from a previous theorem (the second theorem of Example 1.5). This accounts for three vertices. Suppose that there exist i and j such that si = sj . (b) List one member of each equivalence class of X × X. Definition 4.3 A relation that is reflexive, symmetric, and transitive on a set X is called an equivalence relation on X. 110. The proof proceeds in this way. Write a program that checks whether a proposed cycle is a Hamiltonian cycle. Exactly how many modulus operations are required by the Euclidean algorithm in the worst case for numbers in the range 0 to 1,000,000? Then a string s is a postfix string if and only if r(s) = 1 and r(s) ≥ 1, for all initial segments s of s. Let c be the first vertex after a on P1 that is not in P2 ; let d be the T1 n1 vertices n 1 − 1 edges T2 n2 vertices n2 − 1 edges Tk nk vertices nk − 1 edges n 1⩽n 2⩽...⩽nk total vertices (n 1 − 1)⩽(n 2 − 1)⩽...⩽(n k − 1) total edges Figure 2.5 The proof of Theorem 2.3 [if (d), then (a)]. Trace Algorithm 4.4 when n = 8 and the missing square is four from the left and six from the top. Input—the algorithm receives input. The hostid is given by the remaining 8 bits, which identifies the computer interface. Replace the terminal vertex w containing m with v. Theorem 1.25 For any positive integers m and n, gcd(m, n) · lcm(m, n) = mn. Indicate clearly where your proof begins (e.g., by beginning a new paragraph or by 81 Proofs writing "Proof."). For an arrow diagram to be a function, there must be exactly one arrow from each element in X to some element in Y . 178 Functions, Sequences, and Relations Example 4.6 The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is not an equivalence relation because R is not symmetric. An example of an expression involving these operators is (A + B) ∗ C − D/E. 714 : 0, 631 : 6, 26 : 5, 373 : 1, 775 : 8, 906 : 13, 509 : 2, 2032 : 7, 42 : 4, 4 : 3, 136 : 9, 1028 : 10 61. In Exercises 56–58, trace Algorithm 2.16 for the given value of n. (((A + B) ∗ C + D) ∗ E) − ((A + B) ∗ C − D) 10. denotes the sequence of Catalan numbers. Direct Proofs Theorems are often of the form For all x1 , x2 , . Suppose, for example, that x = y and (x, y) and (y, z) are in R. We prove the statement by induction on n. 64. [(a) → (b)] We prove the contrapositive: If A ∩ B = A is not a subset of B. Thus there are nsn,k arrangements in which P is not alone. Prove that if ε > 0, then there exists x ∈ X satisfying a − ε < x ≤ a. 422 n > 1, If j > i, the subproblems si , . If so, stop. a a f g b Eliminate edges (a, b) (f, e) f g h b h e e c c d d G Figure 7.6 Eliminating edges to obtain a subgraph. We next formally state the Principle of Mathematical Induction. If G has a path from vertex v to v that traverses every edge exactly once, an even number of edges must touch each vertex. {x} ⊆ {x} 78. 89. Therefore, n − 1 is even. .} and S = {A1 , A2 , . 1 ➜ Introduction Algorithms typically have the following characteristics: ■ Input The algorithm receives input. Equivalence relation: relation that is reflexive, symmetric, and transitive 42. Also, determine whether the trees are isomorphic as free trees or as rooted trees. on 11. Otherwise, each of the pairs (8.5) is dissimilar and we have found three mutually dissimilar pictures. , (n − 2)! v2 74. In the previous figure, the route shown first meets the top edge at the third position from the left. Theorem 3.7 In the worst case, Algorithm 3.5 requires j −i comparisons. Then, as in the example involving the blocks, all of the equations are marked; that is, all the equations are true and the formula (4.4) is verified. n c2 = c1 + 3 = 6 c3 = c1 + 3 = 6 Then an = an+1 − an = 1 −1 1 − = . This leaves 9 students taking none of the three subjects. Model the system of Exercise 5 as a network assuming, in addition to the limitations on the wells, that city A requires 4 units, city B requires 3 units, and city C requires 4 units. Precision—the steps are precisely stated. For example, suppose that the sequence c1 , c2 , . {(1, a, α), (1, a, β), (2, a, α), (2, a, β)} 16. Algorithm 5.10 Constructing a Binary Search Tree This algorithm constructs a binary search tree. The condition given in Definition 1.28 for a function f from X to Y to be onto Y is equivalent to: For all y ∈ Y , there exists x ∈ X such that f(x) = y. The codomain of each function is also the set of all integers. There are four comparisons at lines 1 and 3. X × X × X 40. Merge sort first divides the input into two nearly equal parts. Thus [c] = [d]. Basis Steps (n = 6; 7) We can make six cents postage by using three 2-cent stamps. We can do better using repeated squaring. If x is in the set X, we write x ∈ X, and if x is not in X, we write x ∈ / X. Start/finish Section 4 Review 1. We use the following fact, which can be verified by considering the cases x even and x odd: 3x −2 + 2 3(x + 1) − 2 = 3x − 2 for x = 1, 2, . The values of vertices that are evaluated are written inside the vertices. Then xy ≤ 0 and |xy| = −xy = (−x)(y) = |x||y|. As before, np +2s ≥ 0 holds because np ≥ 0 and s > 0. We present one final example of a recursive algorithm. See [Date] for the use of trees in hierarchical databases. Find an invariant not given in this section or in Exercises 17–21. b j g i 35. ∀x∃y P(x, y) 69. We can make five-cents postage by using one 5-cent stamp. If we define sequence (2.6) by sn = 2n, the subsequence (2.5) is defined by snk = s2k−1 = 2 · 2k−1 = 2k . The set R − Q, called the set of irrational numbers, consists of all real numbers that are not rational. By the inductive assumption, 1 survives. Example 3.6 Restate each proposition in the form (3.2) of a conditional proposition. The composition of f with g, denoted f ◦ g, is the function (f ◦ g)(x) = f(g(x)) from X to Z. Let X = {sa + tb | sa + tb > 0 and s and t are integers}. At least one head was obtained. Let's summarize the argument we've developed through our examples. ∀x∀y T3 (x, y) 20. Other NP-complete problems include the traveling-salesperson problem and the Hamiltonian-cycle problem. 2.666... Contrast the difference between two equivalent sequences in X and two equal sequences in X. This is the same as selecting 12 items (the names of the students) from the set {Anna, Beth, Candy, Dan}, repetitions allowed. Simple Cycle? 191 Functions, Sequences, and Relations Example 6.2 Table 6.1 can be expressed as the set {(22012, Johnsonbaugh, c, 22), (93831, Glover, of, 24), (58199, Battey, p, 18), (84341, Cage, c, 30), (01180, Homer, 1b, 37), (26710, Score, p, 22), (61049, Johnsonbaugh, of, 30), (39826, Singleton, 2b, 31)} of 4-tuples. f(x) = (3 + sin x)4 −4 x 28. The Catalan number C(2n, n)/(n+1) is equal to the number of nonisomorphic binary trees with n vertices. Assume that the only array comparisons allowed are si < key and si == key. , n with n in the second position is C(n − 1, 1)E1 En−2 . To prove the third form of the Pigeonhole Principle, we argue by contradiction. To prove that a relation R on a set X is antisymmetric, show that for all x, y ∈ X, if (x, y) ∈ R and (y, x) ∈ R, then x = y. Quotient-Remainder Theorem: If d and n are integers, d > 0, there exist integers q (quotient) and r (remainder) satisfying n = dq + r, 0 ≤ r < d. The price changes through time may be viewed graphically. Each iteration of the algorithm changes the status of one label from temporary to permanent; thus we may terminate the algorithm when z receives a permanent label. Define a function f from X to Y by the rule (d) a formula for the subsequence of a obtained by selecting every other term of a starting with the first. Example 2.10 The sequence 100 is increasing, decreasing, nonincreasing, and nondecreasing since there is no value of i for which both i and i + 1 are indexes. Therefore T ∗ ∈ X1 . , n, n1 , . Specifically, when nodes A and B are linked, A transmits all its files to B and B transmits all its files to A. (2) We can arrange for (1) and (2) to hold simultaneously if we choose P(x, y) so that for every x, P(x, y) is true for some y, but for at least one x, P(x, y) is also false for some other value of y. For example, in finding the largest element in a sequence s1 , . (d) The subtree rooted at e. Exercise 9 shows that there are 8!/(3! 2!) strings formed by ordering the letters ILLINOIS. (c) In how many ways can this be done if Egbert must hold one of the offices? Example 2.7 The recurrence relations Sn = 2Sn−1 (2.6) fn = fn−1 + fn−2 , (2.7) of Example 2.2 and which defines the Fibonacci sequence, are both linear homogeneous recurrence relations with constant coefficients. The prerequisite to data structures is a course in Java or C++. 61. It is also conceivable that a message could be decrypted by some means other than factoring integers or taking nth roots mod z. ∃x¬(∀y P(x, y)) 90. Show that an = (nlogm c). In what order are the edges added by Prim's Algorithm for the graph of Exercise 13 if the initial vertex is 6? Gray Codes and Hamiltonian Cycles in the n-Cube Consider a ring model for parallel computation that, when represented as a graph, is a simple cycle (see Figure 3.9). If n > 3, the algorithm sorts s1 , . There are two routes from city A to city D. Find x such that x is in X, but x is not in Y . How many values are possible at one point? If m = n − m + 1, either situation is the best case. a2n of 2n bits having the property that if s is a bit string of length n, for some m, s = am am+1 · · · am+n−1 . [An "incorrect response to an array comparison" occurs in one of four ways: (1) si < key is true, but in the algorithm, the relation si < key evaluates to false. Show that any n×n deficient board can be tiled with trominoes if n is even, n > 8, and 3 divides n2 − 1. log 1.14 18. Assume that children are evaluated left to right. Choose m ∈ Z as large as possible satisfying m/n ≤ a. By induction, ti ≤ 2hi , i = 1, 2. If the domain of discourse is Z+ × Z+ , ∀x ∃y P(x, y) is true; however, ∀x ∀y P(x, y) is false. Begin at the root. Assume that G does not have a vertex of degree 5. an = −an−1 + 5an−2 − 3an−3 31. Each corner bore the name of a city and the problem was to start at any city, travel along the edges, visit each city exactly one time, and return to the initial city. The only difference between the assumptions in a direct proof and a proof by contradiction is the negated conclusion. For part (a), we need the given statement, ∀x∃y P(x, y), to be true and ∀x∀y P(x, y) to be false. By the definition of set difference, x ∈ X ∩ Y and x ∈ / X ∩ Z. Now AO(x + 1, 2, y + 4) − 3 = AO(x, 2, AO(x + 1, 2, y + 3)) − 3 by definition = AO(x, 2, A(x + 1, y) + 3) − 3 by the inductive assumption on y = A(x, A(x + 1, y)) by the inductive assumption on x = A(x + 1, y + 1) by (1.12). a g p n i g l m 7. Knowing the time and space required by algorithms allows us to compare algorithms that solve the same problem. The number of lines in the program 2. ; pick or do not pick xn . ∃x ¬P(x) 32. We will imitate this inductive proof to prove that, if gcd(p, q) = 1 for arbitrary p

and q, there exists n such that for all k ≥ n, postage of k cents can be achieved by using only p-cent and q-cent stamps. Notice that the product of the gcd and lcm is equal to the product of the pair of numbers; that is, gcd(30, 105) · lcm(30, 105) = 15 · 210 = 3150 = 30 · 105. These last equations become aS0 + bT0 = a + 0b = 1, aS1 + bT1 = 2a + 2b = 1. Symbol Z Q R Set Integers Rational numbers Real numbers Example of Members −3, 0, 2, 145 −1/3, 0, 24/15 √ −3, −1.766, 0, 4/15, 2, 2.666 . See the hints for Exercises 31 and 32. It then follows that large is the largest value in the subsequence s1 , . Prove that R is an equivalence relation. , an and b1 , . The statement ∀x∀y P(x, y) is false if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is false. If ai = aj + 3, two are three apart. Throughout this section, G denotes a connected, weighted graph. Perform the computation n i i=0 hi 16 in decimal. We compute the number of bits necessary to represent a positive integer n. Signing a right-handed relief pitcher does not guarantee a World Series win. It is denoted X ∪ Y . The time needed to execute an algorithm is a function of the input. Since each edge in P is properly oriented, at line 41 we increase the flow in each edge in P by = 2 to obtain Figure 2.9. You should check that the next iteration of the algorithm produces the labeling shown in Figure 2.9. Increasing the flow by = 2 produces Figure 2.10. n Therefore, (4.11) is true. Let z ∈ tX. Find aj . Section Review Exercises 8. If T is a full binary tree with i internal vertices, how many terminal vertices does T have? The set {(1, a), (2, a), (3, b)} (1.2) is not a function from X = {1, 2, 3, 4} to Y = {a, b, c} because the element 4 in X is not assigned to an element in Y . Let F be the set of students taking French, let B be the set of students taking business, and let M be the set of students taking music. Universal set, universe 20. 2091, 4807 22. A Class A address is a bit string of length 32. d 3 4 f 3 a z 1 2 13. , n + 1} containing an even number of elements that do not contain n + 1, and If we let α = (x/2)(n + 1) and β = x/2, we must show that O1 ∪ {n + 1}, . If the greatest common divisor of m and n is 1, m/n is in lowest terms; otherwise, we can reduce m/n. In this case, if we take n = 4, for all k ≥ 4, postage of k cents can be achieved by using only 2-cent and 5-cent stamps. Now pi → q is true for every i; so (2.2) is also true. How many strings can be formed if we do not allow repetitions? Thus, if three teams play a single-elimination tournament, after the teams are assigned, the tournament can unfold in four ways. Theorem 6.9 says that P(J ∪ A) = P(J) + P(A) − P(J ∩ A). For the Basis Step (i = 1), we note that just before the for loop begins executing, large is set to s1 ; so large is surely the largest value in the subsequence s1 . Therefore a = c. Thus the range Y is a subset of either {0, 1, . R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3)} 12. Summary of Problem-Solving Techniques ■ The set of rational numbers is not√complete. Given a universe U, describe A A, A A, U A, and ∅ A. Let x ∈ X. Show that if V is any vertex cover of a graph G and E is any edge disjoint set for G, then |E | ≤ |V |. , m − 1. We use induction on n to show that the greedy algorithm is optimal for all n. At this point, we may circle z, so the algorithm terminates. Notice that the digraph of this relation (see Figure 3.2) has the property that for every directed edge from v to w, there is also a directed edge from w to v. Find all part numbers. Since the sequence a is nondecreasing, ·2 + ··· 2 amk−1 ≤ an ≤ amk . Recurrence relations such as (2.8) are said to be nonlinear. How can we show that three statements, say (a), (b), and (c), are equivalent? Because large is the largest value in the subsequence s1 , . , pn where p1 p2 · · · pn + 1 12. In Exercises 27–29, find a path of minimum length from v to w in the graph of Figure 1.7 that passes through each vertex exactly one time. The data are arranged so that, for each vertex v in T, each data item in the left subtree of v is less than the data item in v, and each data item in the right subtree of v is greater than the data item in v. Exercises 7–9 refer to selections among Action Comics, Superman, Captain Marvel, Archie, X-Man, and Nancy comics. The sample spaces for the previous experiments are ■ The numbers 1, 2, 3, 4, 5, 6—all possible outcomes when a die is rolled. 3DBF9 40. Implement binary search (Algorithm 3.2) as a program. In the prefix form of an expression, an operator precedes its operands. Definition 3.4 The capacity of the cut (P, P) is the number C(P, P) = Cij . Each subproblem, in turn, is decomposed further until the process yields subproblems that can be solved in a straightforward way. The solutions are √ 1± 5 t= . Taking z = y in the given condition, we have y R x. The binary trees T1 and T2 are isomorphic if there is a one-to-one, onto function f from the vertex set of T1 to the vertex set of T2 satisfying the following: (a) Vertices vi and vj are adjacent in T1 if and only if the vertices f(vi) and f(vj) are adjacent in T2 . The conclusion is what you will work toward— something like the answer in the back of the book to an exercise, except here it is essential to know the goal before proceeding. To save time and money, the drill press should be moved as quickly as possible. Suppose that T is a tree. Figure 1.1 5. Show that there are distinct positive integers i and j such that f i (x) = f j (x) for all x ∈ X. 11 12 23 and key is 12, the algorithm returns the value 1. Since ¬ exists dj(i) is always false, the first for loop eventually terminates and true is returned to indicate that ∀x∃y P(x, y) is true. and/or permuting the order of the objects of type t (but not changing their location); for example, (I1 S1 S2 I2 S3 S4 I3 P1 P2 I4 M) R (I2 S3 S2 I1 S4 S1 I3 P1 P2 I4 M). (T,1), (T,2), (T,3), (T,4), (T,5), (T,6), (H,3) = 1 − P(E) − P(F) + P(E ∩ F). Show that existential instantiation (see Figure 5.1) is valid. At lines 19 and 20 we label vertex c as (b, 2) since predecessor(c) = b and val(c) = min{, 2 − 0} = min{3, 2 − 0} = 2. Define f from X = {1, 2} to {3} by f(1) = f(2) = 3. Assign values to all vertices so that the resulting tree is analogous to Figure 9.2. Assume that the last player to take a token loses. The graph T initially consists of the vertices of G and no edges. n, an integer greater than or equal to 1 tn tribonacci(n) { if (n == 1 ∨ n == 2 ∨ n == 3) return 1 return tribonacci(n − 1) + tribonacci(n − 2) + tribonacci(n − 3) } 16. Solve the recurrence relation of Exercise 29 in case n is a power of 2. Finally a color I can relate to! FROM THE INCREDIBLE HULK All of mathematics, as well as subjects that rely on mathematics, such as computer science and engineering, make use of functions, sequences, and relations. In developing a proof, it is usually helpful to review definitions and other results related to the statement to be proved. The next time we execute the for loop in lines 7–14, the edges with one vertex in the tree and one vertex not in the tree are Edge Weight (1, 2) (1, 5) (3, 4) (3, 5) (3, 6) 4 3 1 6 3 The edge (3, 4) with minimum weight is selected. Two nondecreasing sequences: si , . a 5 b 6 f K5 K3 17. , ck . If x ∈ A ∪ B, then x ∈ A or x ∈ B. (3!)(5!)(2!)(3!) 74. An = (1.14)n 2000 8. 2 4 3 1 2 3 1 4 2 4 1 3 G1 G2 2 2 n=1 n=2 n=3 We argue by contradiction to show that the 4-cube is not planar. , si gets sorted.) Insertion sort then inserts si+1 in s1 , . The symbol ∴ is read “therefore.” The propositions p1 , p2 , . For a set X to not be equal to a set Y (written X = Y), X and Y must not have the same elements: There must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both). Write this definition symbolically using ∀ and ∃. A set may be defined by listing the elements in it. The method generalizes. Give an example of a bipartite graph different from those in the examples of this section. Input: s, n Output: sum seq sum(s, n) { sum = 0 for i = 1 to n sum = sum + si return sum } 14. We assume√that the fraction p/q is in lowest terms so that p and q are not both even. 00, 01, 10, 11 201 Functions, Sequences, and Relations 120. 5 Sets and Logic In Example 1.13, |A| = 3 and |P(A)| = 23 = 8. Now |S| = k ≤ j = |R(S)|. A directed acyclic graph (dag) is a simple directed graph with no cycles. g Figure 5.1 A binary tree. a 4. [(b) → (c)] If x ∈ B, then x ∈ A ∪ B. employee [Name] Suzuki, Kaminski, Jones, Ryan, Beaulieu, Schmidt, Manacotti 8. Then p is not equal to any of pi , i = 1 to n. The solutions to these subproblems can then be combined, by multiplying, to solve the original problem. After values are chosen for all the variables, if the propositional function is true, you win; if it is false, Farley wins. Write the negation of each proposition in Exercises 49–54 symbolically and in words. Hanoi puzzle in which a stack of disks, except for the beginning and ending stacks, is allowed as long as the largest disk in the stack is on the bottom (the disks above the largest disk in the stack can be ordered in any way whatsoever). Show that if S and T are solutions of (2.23), then bS + dT is also a solution of (2.23). Finally, substituting the first formula for 258 into the previous equation yields 1 = 337 · 49 − (490256 − 337 · 1454)64 = 337 · 93105 − 490256 · 64. Let us say that two pictures are similar if their average brightness differs by no more than some fixed value. , N − 1 are required.) In representing an integer, reading from the right, the first symbol represents the number of 1’s, the next symbol the number of 16’s, the next symbol the number of 162 ’s, and so on. Figure 1.2 The real number line. See Example 4.2. 3. Let R1 be a relation from X to Y and let R2 be a relation from Y to Z. After one month, there is still just one pair because a pair does not become productive until after one month. Also state on which vertices edge e3 is incident and on which edges vertex v2 is incident. e g 2 j d 8 6 f 9 FbA = 0, Fw2 ,b = 0, Fa,w3 = 2, 18. is acyclic and has n − 1 edges. The two-queens problem clearly has no solution. Here s = s mod φ = −67 mod 273 = 206. f(x, y) = x ∪ y, X = P({1, 2, 3, 4}) X = {0, 1, 2, . Give an algorithm that sorts four items using five comparisons in the worst case. Example 1.5 Use the Multiplication Principle to show that a set {x1 , . 2 2 Solving these equations for b and d, we obtain 1 b= √ , 5 1 d = −√ . Sorting a sequence s means to rearrange the data so that s is in order (nonincreasing order or nondecreasing order). The term involving a5 b4 arises in the Binomial Theorem by taking n = 9 and k = 4: Example 7.4 C(n, k)an−k bk = C(9, 4)a5 b4 = 126a5 b4 . Find a triangulation of the following graph. x1 ≥ 0, x2 ≥ 0, x3 ≥ 0 9. p − q (∗) Thus D= p − q ≤ = 1 1 + ··· + 1 n 1 1 1 + ··· + + 1 1 n n+1 − 1 1 + ··· + 1 n 1 . By convention, we add the case n = 5 to the Basis Step; thus the cases n = 4 and n = 5 become the Basis Steps. A rooted tree with four vertices in which the root has degree 3 must be isomorphic to the tree in Figure 8.9(a). Given an i-vertex binary tree, at every terminal vertex we add two children. In your model, what is an Erd˝ os number? Let (x, y), (x, z) ∈ τ(R). 010 s(v1 , v3) = 24, s(v2 , v4) = 76, s(v3 , v5) = 20, 101 001 Figure 1.10 The 3-cube. Thus Dijkstra’s algorithm runs in time O(n2). a h 2. There is a lowest line [passing through (0, 0)] but the lines continue indefinitely above the lowest line. If “to be determined” is also divisible by 4, then the preceding sum, which is equal to 5n+1 − 1, will also be divisible by 4, and the Inductive Step will be complete. Transitivity is proved similarly. Table 3.1 shows which students are taking which courses. 2010 cannot represent a number in binary because 2 is an illegal symbol in binary. are considered to have the same meaning as For some x, ¬P(x); whereas Not any x satisfies P(x). Is there a binary tree with vertices A, B, C, D, E, and F whose preorder listing is P1 and whose inorder listing is P2 ? Will any problem arise if we delete data? f j i g h a b c d j d a g i 33. By Theorem 3.5, the number of ways of selecting eight balls is C(8 + 3 − 1, 3 − 1) = C(10, 2) = 45. Two islands lying in the Pregel River in Ko¨ nigsberg (now Kaliningrad in Russia) were connected to each other and the river banks by bridges, as shown in Figure 2.7. The problem is to start at any location—A, B, C, or D; walk over each bridge exactly once; then return to the starting location. The converse is also true, as we now show. Since lg is an increasing function, lg n + lg(n − 1) + · · · + lg 2 + lg 1 ≤ lg n + lg n + · · · + lg n + lg n = n lg n for all n ≥ 1. Thus binary insertion sort is not optimal if n = 5. Selecting a newborn child at St. Rocco’s Hospital. 7. B B 11. e2 e2 v7 e8 e2 15. 3 + 5 < 2 if and only if 1 + 3 = 4. We can quickly determine whether a relation R on a set X is reflexive by examining the matrix A of R (relative to some ordering). Let x and y be points in X ∩ Y . Basis Step 1/2 ≤ 1/2 Inductive Step Assume true for n. Now the left side of the equation is equal to Cn and the right side is equal to −Cn−1 . A ∪ B = B ∪ A, A ∩ B = B ∩ A A U B 30. We prove by induction on the number of nodes in a tree T that the order in which funnyorder visits the nodes of T is revpost(T). + + + ··· + 1·3 3·5 5·7 (2n − 1)(2n + 1) n = 2n + 1 6. We would have to negate definition (a): ¬∀x(x ≤ a) or, equivalently, ∃x¬(x ≤ a) or ∃x(x > a). The following algorithm implements this design. Definition 5.1 Let P(x) be a statement involving the variable x and let D be a set. For example, Example 1.10 ends with: Thus, there exists an integer k (namely k = k1 + k2) such that m + n = 2k + 1. Therefore, the greedy solution uses at least one am coin. † Exercise 7. = |X ∪ Y ∪ Z| = |X| + |Y | + |Z| − |X ∩ Y | − |X ∩ Z| − |Y ∩ Z| + |X ∩ Y ∩ Z| = 3333 + 2000 + 909 − 666 − 303 − 181 + 60 = 5152. 12 40. Make a random decision about whether to stay with the door initially chosen or switch to one of the unchosen, unopened doors. We may construct a dinner consisting of one main course and an optional beverage by a two-step process. Since 2 is the only solution of (2.19), by Theorem 2.14, Tn = n2n is also a solution of (2.18). (1.8) The last equality follows from the recurrence relation for the sequence c1 , c2 , . Then at line 9 there are at most aL + aR additional comparisons. In the absence of parallel edges, in denoting a path we may suppress the edges. We use a truth table to formally define the truth value of p ∧ q. Generality—the algorithm applies to a set of inputs. By the definition of intersection, x ∈ X and x ∈ Y − Z. We prove that m + n is odd. If the desired item is less than the middle item, binary search recursively searches in the left half of the sequence. There are C(6, 1)C(7, 3) committees with exactly one man. Binary search looks for a value in a sorted sequence and returns the index of the value if it is found or 0 if it is not found. Since any (n − 1)-bit string not containing 111 can follow the initial 0, there are Sn−1 strings of type (a). The equality comes from Exercise 24. TABLE 6.7 ■ BUYER TABLE 6.4 ■ EMPLOYEE ID Name Manager Name Part No 1089 5620 9354 9551 3600 0285 6684 Suzuki Kaminski Jones Ryan Beaulieu Schmidt Manacotti Zamora Jones Yu Washington Yu Jones Jones United Supplies ABC Unlimited United Supplies JCN Electronics United Supplies ABC Unlimited Danny’s United Supplies Underhanded Sales Danny’s DePaul University ABC Unlimited 2A 8C200 1199C 2A 335B2 772 900 772 20A8 20A8 42C 20A8 2. Show that 17. Write a program that plays a perfect game of tic-tac-toe. k=0 Attacking the Problem Example 2.22 asked how many paths there were from the lower-left corner to the upper-right corner of an n × n grid in which we are restricted to traveling only to the right or upward. The remaining digits 235 must be in increasing order to achieve the minimum value. If a and b are positive integers, show that gcd(a, b) = gcd(a, a + b). The given statement may be false and your failed proof may give insight into how to construct a counterexample (see Example 1.15.) On the other hand, if you have trouble constructing a counterexample, check where your proposed examples fail. (1.1) We begin at the root. Finally, i = 4. 404 Recurrence Relations Continue until you obtain ak (e.g., a0), whose value is given explicitly as the initial condition an = . 205 Functions, Sequences, and Relations Section 6 1. Let a1 , . Now suppose that x > 1. According to Theorem 3.5, the number of selections is C(29 + 4 − 1, 4 − 1) = C(32, 3) = 4960. We obtain the initial condition a1 = 2. Therefore, x ≥ d1 and x ≥ d2 . Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats. 4 0 0 0 1 186 Functions, Sequences, and Relations When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns. {(x, y) | x and y are the same height} 10. Suppose that R is transitive. "national park" "north dakota" OR "south dakota" Section 3 Review 1. [Requires calculus] Write the definition of "limx→a f(x) does not exist" (see Exercise 93) in words and symbolically using ∀ and ∃ but not ¬. Similarly, vertex d is labeled “a, 1.” Next, we circle vertex d and update the label of the vertex e adjacent to d (see Figure 4.9). (For Exercise 10) In symbols: ∃x∃y¬T2 (x, y). Suppose that we have stamps of various denominations and that we want to choose the minimum number of stamps to make a given amount of postage. Find the exact number of comparisons (lines 10, 15, 17, 24, and 26) required by the following algorithm when n is even and when n is odd. Exclusive-or: To enter Utopia, you must show a driver’s license or a passport but not both. This statement is true because there is at least one integer x > 1 (namely x = 2) and at least one integer y > 1 (namely y = 3) such that xy = 6. To help guide the construction of the proof, it may be helpful to translate the goal using the definition of set difference: x ∈ (X ∩ Y) − (X ∩ Z) means x ∈ X ∩ Y and x ∈ / X ∩ Z. The moves Recurrence Relations y y x x Figure 1.1 Decomposition of a good route. We let T denote the set of vertices having temporary labels. Find i=1 43. Given the pseudorandom number xn−1 , the next pseudorandom number xn is given by the formula. b a f e 4. Since x ∈ / A∩B, x ∈ / A or x ∈ / B. Algorithm 3.12 Computing an Exponential This algorithm computes an recursively, where a is a real number and n is a positive integer. We then invoke this algorithm with i = j = 5 to search for key in s5 = ‘X’. X ∪ Y , X union Y : set of elements in X or Y or both 13. Applicant A is qualified for jobs J1 , J2 , and J4 ; B is qualified for jobs J4 , J5 , and J6 ; C is qualified for jobs J1 and J5 ; D is qualified for jobs J1 , J3 , J4 , and J8 ; E is qualified for jobs J1 , J2 , J4 , J6 , and J8 ; F is qualified for jobs J4 and J6 ; and G is qualified for jobs J3 , J5 , and J7 . Thus the 11 × 11 board can be tiled. Counting Methods and the Pigeonhole Principle For starters, there are 32 combinations of cabs (standard, Club Cab, Quad Cab), cargo beds (6.5 or 8 feet) and engines (3.9-liter V6, 5.2-liter V8, 5.9-liter V8, 5.9-liter turbo-diesel inline 6, 8-liter V10). The rest of the exercises deal with coloring planar graphs. True 34. Give an example of a binary operator. For example, Znonneg denotes the set of nonnegative integers, namely 0, 1, 2, 3, . Here, your proof consists of presenting the counterexample together with justification that the propositional function is indeed false for your counterexample. , g(ek) are incident on f(v). Such a property is called an invariant. Find an Euler cycle for G. Since the graph is (v0 , v1 , . Example 1.5 The integer 23 is prime because its only divisors are itself and 1. If edge e is associated with vertices v and w, v and w are said to be adjacent. (c) Find a formula for si . Next add an edge incident on v that does not produce a cycle when added to the tree, and add the additional vertex incident on this edge. Use your program of Exercise 3 to generate an optimal Huffman code. Implement the RSA public-key cryptosystem. If all outcomes in a finite sample space are equally likely, the probability of an event is defined as the number of outcomes in the event divided by the number of outcomes in the sample space. If D is less than the data item at the current vertex v, we move to v’s left child and repeat this process. FROM THE PRODUCERS An algorithm is a step-by-step method of solving some problem. Then R and S are transitive. ej v1 e1 ei vi vj e j1 e i1 v j1 v 0 v n v en Figure 2.12 A cycle that either is a simple cycle or can be reduced to a simple cycle. No No No Afghan ? 335 Counting Methods and the Pigeonhole Principle Example 3.6 Suppose that there are piles of red, blue, and green balls and that each pile contains at least eight balls. Given an arrow diagram, the relation is reflexive if there is a loop at every vertex. The ikth entry in A1 A2 is computed as a b i s t k u = su + tv. Using the definition of f , we have αR = βR . Furthermore, several other parallel computation models can be simulated by the hypercube. How do algorithms relate to pseudocode functions? We show that the sequence nr n , n = 0, 1, . Inductive Step: assume true for instance n; then prove true for instance n + 1. Then x R c. , vn), v0 = vn . Example 2.18 We show how to compute 57229 mod 713 using Algorithm 2.16 and Theorem 2.17. If you enter a number and hit the 1/x key, the reciprocal of the number entered (or an approximation to it) is displayed. The next 7 bits, called the netid, identify the network. Theorem 4.7 Hall’s Marriage Theorem Let G be a directed, bipartite graph with disjoint vertex sets V and W in which the edges are directed from vertices in V to vertices in W. The relation of Exercise 7 29. There are similar proofs of the two statements. Actors who have appeared in a film with Bacon are said to have Bacon number one. Problem-Solving Tips To derive a big oh notation for an expression f(n) directly, you must find a constant C1 and a simple expression g(n) (e.g., n, n lg n, n2) such that |f(n)| ≤ C1 |g(n)| for all but finitely many n. Since n is the largest integer satisfying (∗), 1 1 p 1 1 + + ··· + + ≥ . f i h 4. The while loop terminates when i = n + 1. a ≥ fn+2 and b ≥ fn+1 1 = (79 − 21 · 3)4 − 21 · 3 = 79 · 4 − 21 · 15. In our earlier discussion, we found that each distinct four-bit string determines an equivalence class. Parallel planes stacked one above another one unit apart. 53. Fishing is a popular sport or lacrosse is wildly popular in California. In this case, di is a counterexample. an = 2(−3)n 41. The while loop terminates when i = n. Let = min X, 599 Network Models where X consists of the numbers Cij − Fij , for properly oriented edges (i, j) in P, and Fij , for improperly oriented edges (i, j) in P. Now n − 1 = (2k + 1) − 1 = 2k. The full binary tree T ∗ has n internal vertices, so by Theorem 5.4, t = n + 1. We then proceed to lines 9–12, where we call the algorithm to tile the subboards. , xn) as well as other axioms, definitions, previously derived theorems, and rules of inference, shows directly that q(x1 , x2 , . We can use the definition of “odd” again if we can show that m + n is equal to 2 × some integer + 1. Such a theorem asserts that (a), (b), (c) and so on are either all true or all false. In Exercises 20–27, two dice are rolled, one blue and one red. {1, 2, 3, 4, 5, 7, 10} 7. This makes n = nq q = −(p − 1)tq. The statement is true if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is true. If events E1 and E2 are mutually exclusive, P(E1 ∪ E2) = P(E1) + P(E2). The host then gives the contestant the option of abandoning the chosen door in favor of one of the two still-closed, unchosen doors. According to Definition 5.9, the existentially quantified statement ∃x P(x) is false if for every x in the domain of discourse, the proposition P(x) is false. False 18. , k, 6 (n − 7) n n < n + 1 ≤ ni for i = 1, . Now the entire sequence s1 , . Example 3.7 In how many ways can 12 identical mathematics books be distributed among the students Anna, Beth, Candy, and Dan? 73. Upper bounds for X are 4, 5, and 6 since x ≤ 4, x ≤ 5, and x ≤ 6 for every x ∈ X. Thus we must add n = 3 to the Basis Step (n = 2). (n) 59. ∀x P(x) Let P(x) denote the statement “x is a professional athlete,” and let Q(x) denote the statement “x plays soccer.” The domain of discourse is the set of all people. The symbols A–F are interpreted as decimal 10–15. Basis Step for the Strong Form of Mathematical Induction: prove true for the first instance. Use depth-first search (Algorithm 3.7) with the vertex ordering hfdbgeca to find a spanning tree for graph G of Figure 3.1. k f 7. X ∩ Y , X intersect Y : set of elements in X and Y 15. Functions related to Ackermann’s function appear in the time complexity of certain algorithms such as the time to execute union/find algorithms (see [Tarjan, pp. Each step of an extended argument involves drawing intermediate conclusions. Example 6.6 Describe operations that provide the answer to the query “Find the names of all persons who play for some team.” If we first join the relations given by Tables 6.1 and 6.2 subject to the condition ID Number = PID, we will obtain Table 6.3, which lists all persons who play for some team as well as other information. Suppose that we have n dollars and that each day we buy either orange juice ($1), milk ($2), or beer ($2). If Katrina passes discrete mathematics, then she will take the algorithms course. The virus was spread by an e-mail message containing an attached word processor document with a malicious macro. For example, a loop invariant for a while loop while (condition) // loop body is true just before condition is evaluated the first time, and it is also true each time the loop body is executed. How many strings begin with A if repetitions are not allowed? Assume that the deer population of Rustic County is 0 at time n = 0. v2 v1 v2 e2 Graph Theory v1 26. (b) The sum of the areas of the rectangles whose bases are on the x-axis and whose tops are above the curve is equal to ≤C for some constant C and for all sufficiently large n. What is a matching? (Entry ij is 1 if and only if entry ji is 1.) Therefore there 2 are 2(n +n)/2 symmetric relations on an n-element set. Prove that for all real numbers d, d1 , d2 , x, if d = max{d1 , d2 } and x ≥ d, then x ≥ d1 and x ≥ d2 . This implies that f0 (n) = O(g0 (n)). Given all the disks stacked on one peg, the problem is to transfer the disks to another peg by moving one disk at a time. Let R1 and R2 be two n-ary relations. Find a recurrence relation for the sequence {bn }. Now R is transitive if and only if whenever (i, k) and (k, j) are in R, then (i, j) is in R. If n is not at this position, we look in the next-highest position (again, 0 is assumed to follow 10); if n is not in this position, we proceed to the next-highest position, and so on. To prove that R∩S is transitive, we must show that if (x, y), (y, z) ∈ R ∩ S, then (x, z) ∈ R ∩ S. For i ≥ 2, define Xi = {ik | k ∈ P}. (c) Is there a complete matching? Cancelling |f(n)| gives 1 > 4, which is a contradiction. f −1 = {(y, x) | (x, y) ∈ f }. Section 7 25. A homomorphism from a graph G1 to a graph G2 is a function f from the vertex set of G1 to the vertex set of G2 with the property that if v and w are adjacent in G1 , then f(v) and f(w) are adjacent in G2 . X 6. By the inductive assumption, aL ≤ 3L + 2 and aR ≤ 3R + 2. If x ∈ (A ∩ B) ∪ C, then x∈A∩B or x ∈ C. Computer systems are capable of storing large amounts of information in databases. Notice that any office can communicate with any other either directly through a communication link or by having others relay the message. Describe a linear-time algorithm to test whether two binary trees are isomorphic. 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 1 1 0 0 1 1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 1 1 1 1 1 0 0 ⎪ ⎪. If we substitute “Le Franc¸ais” for “restaurant,” statement (d) is Le Franc¸ais rated over two stars in Chicago magazine, which is true. What is a left child in a binary tree? It can be shown using elementary methods that GKn has a Hamiltonian cycle for all even n ≥ 6 (see [Schwenk]). Now m > 0, by definition. If√n is composite, it must have a divisor d satisfying 2 ≤ d ≤ n (see Theorem 1.7). (b) Describe the equivalence classes of R1 ∩ R2 in terms of the equivalence classes of R1 and the equivalence classes of R2 . For the inductive step, we assume case n n fk = fn+2 − 1 k=1 and prove case n + 1 n+1 fk = fn+3 − 1. Probability of an event when all outcomes are equally likely Section 6 4. Sets and Logic Richard Johnsonbaugh 1 2. Thus an−k bk appears C(n, k) times. Prove that if X ⊆ Y , then X ∩ Z ⊆ Y ∩ Z for all sets X, Y , and Z. On the other hand, Definition 5.4 assigns a truth value to the quantified statement (5.1). ⎛ w x y z⎞ a 1 b⎛0 c⎝0 d 1 9. R = {(Roger, Music), (Pat, History), (Ben, Math), (Pat, PolySci)} 7. p ∧ q 35. Peel me a grape. Backtracking or depth-first search is especially attractive in a problem such as that in Example 3.9, where all that is desired is one solution. ∀x∃yL(x, y). What is a flow in an edge? Prove that at least two people receive checks on the same day. Write the adjacency matrices of the components of the graphs given by the adjacency matrices of Exercises 13–17. The graph is not planar; the following subgraph is homeomorphic to K5 : 64 = 2 · 32 ≥ 4(32 − 16 + 2) = 72, d a which is a contradiction. Figure 4.7 shows the result of executing lines 2–4 of Algorithm 4.1. First, we circle a (see Figure 4.8). Then p is false and q is true. X × (Y ∪ Z) = (X × Y) ∪ (X × Z) for all sets X, Y , and Z. Only a disk of smaller diameter can be placed on a disk of larger diameter. Suppose that S(1) is true; (4.7) for all n ≥ 1, if S(n) is true, then S(n + 1) is true. The terms point and line are undefined terms that are implicitly defined by the axioms that describe their properties. (Since q and r each consist of terms separated by or’s, where each term is a variable or the negation of a variable, q ∨ r also consists of terms separated by or’s, where each term is a variable or the negation of a variable.) Example 3.4 We prove the following using resolution: 1. 9 outcomes Figure 7.4 A five-coins puzzle algorithm that uses at most two weighings. Rules of inference for quantified statements: universal instantiation, universal generalization, existential instantiation, existential generalization 67. Divide the number to be converted to hexadecimal by 16. 485 Graph Theory We will conclude this section by proving Euler’s formula. + 2. Find the matrix product A1 A2 . Justify each step of the following direct proof, which shows that if x is a real number, then x · 0 = 0. , pn is a list of all of the primes less than or equal to p, we must have p > p. (c) Replacing n by i in definition 2.1, we obtain si = 2i + 4 · 3i . Contrasting statement (b) with the given statement, we see that the quantifiers ∀ and ∃ have been swapped. Algorithms Richard Johnsonbaugh 207 5. Label each entry in A that represents an edge with flow 1. Now suppose that n is 16. The converse was discussed just before the statement of the theorem. A Venn diagram provides a pictorial view of sets. (3.5) Since the sequence a is nondecreasing (a fact that can be proved using induction—see Exercise 5), a2k−1 ≤ an ≤ a2k . We must show that for every x, if x ∈ X, then x ∈ Z. We then move to lines 7 and 8, where we set a to 72 and b to 36. n n bi 2i i=0 4. Edsger W. (2.12) We can verify (see Theorem 2.11) that if S and T are solutions of (2.9), then bS + dT , where b and d are any numbers whatever, is also a solution of (2.9). Prim’s Algorithm to find a minimal spanning tree 24. 189 Functions, Sequences, and Relations Let R1 be a relation from X to Y and let R2 be a relation from Y to Z. Since R1 and R2 are symmetric, (y, x) ∈ R1 and (y, x) ∈ R2 . How many selections are there in which Ben is either chairperson or treasurer? Section 4 Review We use Exercise 16 with a = fn+1 + fn and b = fn+1 to justify the second equality. ∃x∃y P(x, y) 83. In the conditional proposition p → q, q is the conclusion. Define X union Y . The ambiguity results from attempting to distribute ∀ across the or. Here X and Y are not disjoint; for example, 10111011 ∈ X ∩ Y . Example 5.20 Statements in words often have more than one possible interpretation. Find 3 1 1 − , n n+1 wi . If n = 11, enclose the missing square in a corner 7 × 7 subboard (see the following figure). , w1 , w0) = val(z) for i = 1 to k + 1 { e = (wi , wi−1) if (e is properly oriented in P) Fe = Fe + else Fe = Fe − } } // end while (true) loop A proof that Algorithm 2.4 terminates is left as an exercise (Exercise 19). A graph G = (V, E) is bipartite if there exist subsets V1 and V2 (either possibly empty) of V such that V1 ∩ V2 = ∅, V1 ∪ V2 = V , and each edge in E is incident on one vertex in V1 and one vertex in V2 . , 6 (see Figure 2.11). Next, order the children of the root with the most promising moves to the left. 6f. Exercises 14–16 refer to the tribonacci sequence {tn } defined by the equations Section 4 14. What is the complete bipartite graph on m and n vertices? ∀x∀y P(x, y) 75. Write a depth-first search algorithm to test whether a graph is connected. , w1 , v, w2 , . Z × R 67. Cartesian product of X1 , X2 , . n-ary relation: Set of n-tuples Database management system Relational database Key 53. 23 (2n − 1)(2n − 3)(2n − 5) Cn−3 (n + 1)n(n − 1) minal vertices in a game tree. Figure 1.15 The complete bipartite graph K2,4 . A set is determined by its elements and not by any particular order in which the elements might be listed. Find the probability of obtaining a bridge hand with 6–5– 2–0 distribution, that is, six cards in one suit, five cards in another suit, two cards in another suit, and no cards in the fourth suit. , a30 ; a1 + 3, . Since the worst-case time is O(n) and (n), the worst-case time of Algorithm 8.13 is (n). Graph Theory Example 2.5 The graph G of Figure 2.1 is connected since, given any vertices v and w in G, there is a path from v to w. Answer 103–110 for the sequence z defined by wi . We can show that graphs G1 and G2 are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal. The relation R is reflexive if and only if A has 1’s on the main diagonal. 15. Postorder Traversal This recursive algorithm processes the vertices of a binary tree using postorder traversal. Show that Nh = fh+3 − 1, for h ≥ 0. The traditional computer, often called a serial computer, executes one instruction at a time. (d) Use Algorithm 2.4 to find a maximal matching. In Exercises 71–73, give an argument using rules of inference to show that the conclusion follows from the hypotheses. By Example 3.9, this sum is equal to C(k + n − 1, k). It follows that (a + b)n = C(n, 0)an b0 + C(n, 1)an−1 b1 + C(n, 2)an−2 b2 + · · · + C(n, n − 1)a1 bn−1 + C(n, n)a0 bn . Example 5.4 The matrix of the relation R = {(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)} on {a, b, c, d}, relative to the ordering a, b, c, d, is ⎛a b c d ⎞ a 1 0 0 0 ⎜b⎜ ⎜0 1 1 0⎟. For example, suppose that we square the matrix A of Example 5.2 to obtain ⎛ 0 ⎜1 ⎜ A2 = ⎜ ⎜0 ⎝1 0 1 0 1 0 1 0 1 0 1 1 1 0 1 0 0 ⎞⎛ 0 0 ⎟1 1 ⎟⎜ ⎟⎜ 1 ⎟⎜ ⎟⎜0 0 ⎠⎝1 0 0 1 0 1 0 1 0 1 0 1 1 1 0 1 0 0 ⎞ a 0 1 ⎟ ⎟ b 1 ⎟ ⎟= c 0 ⎠ d e 0 ⎛a b c d e ⎞ 2 0 2 0 1 ⎜0 3 1 2 1⎟ ⎜ ⎟ ⎜2 1 3 0 1⎟. (3) We can arrange for (1) and (3) to hold simultaneously if we choose P(x, y) so that for every x, P(x, y) is true for some y and false for some other value of y. It is equal to a(r n+1 − 1) . Definition 4.1 An argument is a sequence of propositions written p1 p2 .. ¬(p ∨ q) ∨ r 43. Let m, n, and d be integers. Now, use an n-level, depth-first search with alpha-beta pruning. Prove that if α has equal numbers of a’s and b’s, then α ∈ L. In Exercises 40–43, represent the given proposition symbolically by letting p : 4 < 2, q : 7 < 10, r : 6 < 6. 525 Trees Algorithm 3.10 generates a solution to the four-queens problem. Informally, if there was a hole in the line, the set of numbers to the left of the hole, although bounded above, would not have a least upper bound: (a) z ≤ ta for every z ∈ tX (i.e., ta is an upper bound for tX), (b) if b is an upper bound for tX, then b ≥ ta (i.e., ta is the least upper bound for tX). Multiplication is implicitly defined by the first axiom and others that describe the properties multiplication is assumed to have. Write a program that randomly generates an n×n adjacency matrix. an−2 . We’ll try this first. Show that C(n, k)2 = C(2n, n). 2 Sets and Logic ... We must show that (x, x) ∈ R ◦ S for all x ∈ X. Example 5.8 The universally quantified statement for every real number x, if x > 1, then x + 1 > 1 is true. Let A = {1, 2, 3}, B = {2, 3, 4}, and C = {1, 2, 4}. Illustrate how your operator works by answering the following query, 23. A partition of a set gives rise to an equivalence relation. Because the events J and A are independent, Definition 6.16 says that P(J ∩ A) = P(J)P(A) = (0.70)(0.95) = 0.665. We draw five balls without replacing them. 31000 In Exercises 8–13, express each binary number in decimal. One tree in X2 is obtained by inserting two new children of v—one is a new left child, which is marked, and the other is the root of the original subtree in T rooted at v. State the commutative laws for sets. The orderings v1 , v2 , v3 , v4 , v5 and w3 , w1 , w4 , w2 , w5 produce equal adjacency matrices. (The numbers Sn,k are called Stirling numbers of the second kind.) (a) Show that Sn,k = 0 if k > n. Give an example to show how Dijkstra’s shortest-path algorithm finds a shortest path. C(52, 13) 30. 4 ➜ Algorithms for Generating Permutations and Combinations The rock group “Unhinged Universe” has recorded n videos whose running times are t1 , t2 , . We must show that x ∈ A ∪ B. } 14. Thus (y, x) ∈ R1 ∩R2 and, therefore, R1 ∩ R2 is symmetric. a 8 z Theorem 4.5 Prim’s Algorithm (Algorithm 4.3) is correct; that is, at the termination of Algorithm 4.3, T is a minimal spanning tree. Use equation (7.4) to write S(k, n) = n S(k − 1, i). For example, in the front/back graph, the edge labeled 1 connects R and W, since the front and back faces of cube 1 are red and white. Hypotheses: If Jill can sing or Dweezle can play, then I’ll buy the compact disc. How is the event E given F denoted? These observations also led to the loop invariant large is the largest value in the subsequence s1 , . By De Morgan’s laws of 85 Proofs logic ¬(x ≥ 1 ∨ y ≥ 1) ≡ ¬(x ≥ 1) ∧ ¬(y ≥ 1) ≡ (x < 1) ∧ (y < 1). (b) Suppose that 0 < p/q < 1 and that all fractions i/q , with 1 ≤ i < p and q arbitrary, can be expressed in Egyptian form. (3.1) Statement (3.1) can be verified by writing the truth table (see Exercise 1). Dale is smart and funny. Such keys are said to be private. ∃x∀y P(x, y) 79. Write a program to generate the complete game tree for nim in which the initial position consists of two piles of four tokens each. Example 5.3 Define the sequence c1 , c2 , . Such a simulation is called a trace. Choose a path P with no repeated edges of maximum length. Then m i an = ai=0 bi 2 = m i a bi 2 . , 6, where i=1 Exercises 83–86 refer to the sequence {sn } defined by the rule n ≥ 1. Give an example (different from those of Example 1.3) of a theorem in Euclidean geometry. ¬(∃xP(x)) and ∀x¬P(x) have the same truth values. Therefore, equation (1.4) holds. (b) x2 − x − 6 = 0 (domain of discourse = R). ∴ If I don’t study hard, then I get rich. Therefore n is even. Give an example of isomorphic, nonidentical graphs. Example 1.33 Prove that the function f(n) = 2n − 1 from the set X of positive integers to the set Y of positive integers is not onto Y . Suppose that x ∈ X ∪ Y . The officers can be selected in three successive steps: Select the chairperson; select the secretary; select the treasurer. 500 Section 3 Review 1. Section 2 1. 28. Since the second quantifier is ∃, you go second. We then return to the top of the while loop (line 3). In general, we can make (n + i)-cents postage n + i = (np p + nq q) + i(sp + tq) = (np + is)p + (nq + it)q using np + is p-cent stamps and nq + it q-cent stamps. Answer 59–66 for the sequence defined by n = 3 for all n. For the Huffman code defined by the tree of Figure 1.10, the character A is represented by a bit string of length 1, whereas S and T are represented by bit strings of length 4. What is a bijection? Example 2.8 We show how Algorithm 2.7 converts the decimal number m = 11 to binary. 3. Substituting this expression for 5 into (2.6), we obtain 91 = 25 · 2 + 24 + 23 + 2 + 1 = 26 + 2 4 + 2 3 + 2 + 1 = 10110112 . (b) For each vertex j, which is neither the source nor the sink,   Fij = Fji . We use this function to compute 2!. One die is rolled. C(20, 5)2 33. The following graph is continued to an arbitrary, finite depth. Write an algorithm that returns true if the values of a, b, and c are distinct, and false otherwise. Clause: consists of terms separated by or’s, where each term is a variable or a negation of a variable. 312 Counting Methods and the Pigeonhole Principle Example 1.13 A committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. Prove or disprove: There exists j such that sj > A. Develop an algorithm for finding a maximal flow in G. Region 1 represents (A ∪ B), the elements in neither A nor B. By the inductive assumption, the loop iterates an additional m − 1 times. We may write 1 + 2 + · · · + n = C(1, 1) + C(2, 1) + · · · + C(n, 1) = C(n + 1, 2) by equation (7.4) (n + 1)n = . , 2n − 1. For example, Algorithm 1.4, derived from recurrence relation (1.4) and initial condition (1.5), computes the sequence of Example 1.3. Algorithm 1.4 Computing Compound Interest This recursive algorithm computes the amount of money at the end of n years assuming an initial amount of $1000 and an interest rate of 12 percent compounded annually. A function f from X to Y is a subset of the Cartesian product X × Y having the property that for each x ∈ X, there is exactly one y ∈ Y with (x, y) ∈ f . 81. How is the length of the string α denoted? (b) The ancestors of Hermes are Zeus, Kronos, and Uranus. ∃x P(x) 30. Example 2.6 Decimal to Binary Write the decimal number 130 in binary. Let G = (V, E) be a simple undirected graph. Give an example of mutually exclusive events. A Huffman code can be defined by a rooted tree. b, j 5. §An integer n > 1 is prime if the only positive integers that divide n are 1 and n itself. We write 34 in the region representing CALC ∩ PSYCH ∩ COMPSCI. If both are nonempty, we first check whether the left subtrees are isomorphic and then whether the right subtrees are isomorphic. α is in L because of either rule 1 or rule 2. If an is the worst-case time for input of size n, an = 1 + an/2 . example. Suppose there is a simple, planar graph that requires more than four colors to color. Let P(x) denote the statement “x is a circumstance,” and Q(x) denote the statement “x is right for a formal investigation.” Symbolically, the clarified statement is ∃x(P(x) ∧ ¬Q(x)). Evaluate the root of each of the trees of Exercises 9–13 using a depth-first search with alpha-beta pruning. The statement to be proved says that if x ≤ d (= 2), then x ≤ d1 (= 2) and x ≤ d2 (= 4). , sn in nondecreasing order by first finding the smallest item, say si , and placing it first by swapping s1 and si . ■ s2n and s1 differ in exactly one bit. (The Basis Step is n = 1.) Let bn denote the number of times sort is called when the input sequence contains n items. Asentence that is either true or false, but not both, is called a proposition. (3) si == key is true, but in the algorithm, the relation si == key evaluates to false. Thus p and q are both √ even, which contradicts our assumption that p and q are not both even. Suppose, by way of contradiction, that 2 is rational. (b) The matrix A2 of the relation R2 (relative to the given orderings). (a) We show symmetry only. Find positive integers a1 and a2 such that a1 > 2a2 > 1, a2 does not divide a1 , and the algorithm, with available denominations 1, a1 , a2 , produces the fewest number of stamps to make any given amount of postage. Selection from First Factor (a + b) Selection from Second Factor (a + b) Selection from Third Factor (a + b) Product of Selections a a a a b b b b a a b b a a b b a b a b a b a b aaa = a3 aab = a2 b aba = a2 b abb = ab2 baa = a2 b bab = ab2 bba = ab2 bbb = b3 The Binomial Theorem gives a formula for the coefficients in the expansion of (a + b)n . n (2.9) n ≥ 1, is minimized when x is equal to a median of x1 , . In particular, |g(n)| > 2|f(n)|. The best chess, backgammon, and checkers programs play at a level comparable to the best human players. x is set to x ∗ x mod z = 16 mod 21 = 16, and n is set to 0. Let α denote the last string in Gn−1 , and let β denote the first string in Gn−1 . x is set to x ∗ x mod z = 36 mod 11 = 3, and n is set to 15. Example 6.2 Suppose that a die is loaded so that the numbers 2 through 6 are equally likely to appear, but that 1 is three times as likely as any other number to appear. Hint: Use induction on n. The binary insertion sort algorithm sorts an array of size n as follows. [Examples: C(10010) = 2, C(00110001) = 3.] Let Sn be the number of n-bit strings α with C(α) ≤ 2. Let X be a set with least upper bound a. This contradiction shows that b is not in any pseudo equivalence class. 25. The algorithm first swaps (i.e., interchanges the values of) a1 and arand(1,n) . > 0 and N for all n ≥ N. Someone loves everybody. (b) For every edge e ∈ E , if e is incident on v and w , then v , w ∈ V . Suppose that after the teams are assigned in the NCAA men’s basketball tournament, someone randomly guesses how the tournament will unfold. Similarly, if f(n) = Cn, it can be shown that g(n) = C1 n + C0 if 1 is not a root of (2.22), g(n) = C1 n2 + C0 n if 1 is a root of multiplicity one, and g(n) = C1 n3 +C0 n2 if 1 is a root of multiplicity two. Find a formula for rn−2 . 168 Functions, Sequences, and Relations Example 3.11 The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is not symmetric. An edge incident on a single vertex is called a loop. For example, in Figure 1.7 the weight of edge (c, e) is 5. When we take k = n − 2, to ensure that n0 ≤ k, that is 4 ≤ n − 2, we must have 6 ≤ n. Is increasing? Proof We first find all nonisomorphic binary trees with three vertices in which the root has degree 2. (c) Part (a) shows that 44. However, in case (2.14) has two equal roots r, we obtain only one basic solution r n. Consider a1 , . Today’s hardware relies on the state of an electronic circuit to represent a bit. Input: A connected graph G with vertices ordered v1 , v2 , . Now P1 followed by P2 is a path from v to x, and thus vRx. Therefore, R is transitive. is defined by the equations† c1 = 0, cn = cn/2 + n for all n > 1. Because p1 is false and pn → p1 is true, pn is false; because pn is false and pn−1 → pn is true, pn−1 is false; and so on. Let R be an equivalence relation on X. 110101110, 1AE 5. This new function, denoted f −1 , is called f inverse. We say that d divides n if there exists an integer q satisfying n = dq. Further, we can show that T does not contain a cycle. Next assume that x < 0 and y ≥ 0. By the inductive assumption, 5n − 1 is divisible by 4 and, since 4 · 5n is divisible by 4, the sum (5n − 1) + 4 · 5n = 5n+1 − 1 is divisible by 4. Assume that n + 2 is even. p ↔ r r ∴ p ∴ s∨t∨u 4 ➜ Mathematical Induction Suppose that a sequence of blocks numbered 1, 2, . In practice, arguments and proofs use rules of inference. Use the logic game (Example 6.15) to determine whether the proposition ∀x∀y∃z((z < x) ∧ (z < y)) is true or false. k b e d c i d e k o s p q m r t d a g i l 8. f(x) = 3x , X = set of real numbers 34. The algorithm is explained in Example 2.15. The algorithm returns result, which is equal to an mod z = 5030 mod 11 = 1. State the Principle of Mathematical Induction. Is w increasing? Before reading on, try to determine whether this is possible. This Problem-Solving Corner gives an extended, detailed exposition of how to do proofs by mathematical induction. , a5 , we first sort a1 , a2 (one comparison) and then a3 , a4 (one comparison). How many are greater than 101 and do not contain the digit 6? The last bits 0111 decode as T. Define a relation on P(X), the power set of X, as (A, B) ∈ R if A ⊆ B. The statement is false if, for every x ∈ X, there is at least one y ∈ Y such that P(x, y) is false. The algorithm does have the determinism property. This technique works well if the numbers a and b are relatively small so that the prime factorizations of each can be found, or if the prime factorizations of each are given. A full m-ary tree is a rooted tree such that every parent has m ordered children. x = 1 y=2 z=y+2 w=x+5 x=z+w 52. Such a tree is called a spanning tree. d k j e h c f 5. At line 5, since key (‘Z’) is not equal to s3 (‘J’), we proceed to line 7. Suppose that each pair is similar. The first time we execute the for loop in lines 7–14, the edges with one vertex in the tree and one vertex not in the tree are Edge Weight (1, 2) (1, 3) (1, 5) 4 2 3 The edge (1, 3) with minimum weight is selected. n+1 Now n2 + n (1 + 2 + · · · + n) + n ≤ n+1 n+1 n(n + 1) = = n. First d is set to 2. Therefore, V ∩ P is nonempty. ¬r From 3 and 4 Thus at most six games can be played. In Exercises 9–12, assume that a person invests $2000 each year in a tax-sheltered annuity at 10 percent interest compounded annually. Implement the minimax procedure with alpha-beta pruning as a program. C(20, 5) 39. An example of a palindrome over X is bbaabb. Thus part (c) must be true. Invalid 17. vertex preceding c on P1 ; and let e be the first vertex after d on P1 that is also on P2 . Give an example of a graph that has an Euler cycle. Explain how to convert from hexadecimal to decimal. Draw all nonisomorphic, cycle-free, connected graphs having six vertices. Show that any coloring of the map of Exercise 18, excluding the unbounded region, requires at least three colors. The shaded area represents the beverage, which has great taste and is less filling. By using Exercise 44 or otherwise, prove that A(m, n) < A(m, n + 1) for all m ≥ 0, n ≥ 0. Finding a Solution We first consider proving the contrapositive If X ∩ Y = ∅, then CX∪Y (x) = CX (x) + CY (x) for at least one x ∈ U. Not isomorphic as binary trees. Use the generalized De Morgan’s laws for logic to write the negation of ∀x∃y∀z P(x, y, z). The Basis Step is m = 2. {1, 4, 5} 94. In an terminology, the classes partition the set of routes. The planes continue indefinitely in both directions [above and below the origin (0, 0, 0)]. Since X is convex, the line segment from x to y is in X. At line 14, s3 is set to 5. In Exercises 36–38, solve the recurrence relation for the initial conditions given. After the student writes the program, it is easy to test it for a small number of cities. A = {1, 2, 3}, B = ∅ 26. This fallacy is known as the fallacy of affirming the conclusion. [If (b), then (c).] Suppose that T is connected and acyclic. 585 Trees 14. 3 List the elements of R1 ◦ R2 and R2 ◦ R1 . To verify equation (1.4), suppose that x ∈ B. Assume that the functions f, g, and h take on only positive values. 6 9 12 e Figure 1.7 A graph model of sheet metal in Figure 1.6. The edge weight is the time to move the drill press. Write 29. Since n = 15 > 0, the body of the while loop executes. and ln(1 + i) is approximately equal to i for small values of i, n is approximately equal to 0.69 . If R is antisymmetric, then R−1 is antisymmetric. Edge e1 is denoted (Gre, She) or (She, Gre), and edge e10 is denoted (Cas, Dou) or (Dou, Cas). Two graphs are isomorphic if and only if for some orderings of their vertices, their adjacency matrices are equal. Thus if n ≥ 23, the probability is greater than 1/2 that at least two persons have birthdays on the same month and date. , sn = 2n, . For example, R = {(a, a), (b, b), (c, c)} a b c Figure 3.3 The digraph of the relation of Example 3.15. At line 2, b > large (1 > 6) is false, so we do nothing. 34 34 34 57 i j 11. Prove that the number of independent sets in Pn is equal to fn+2 , n = 1, 2, . For example, tasks 2 and 3 can be done in either order. Minimal spanning tree algorithms and their implementation can be found in [Tarjan]. The correct interpretation of the statement All that glitters is not gold results from negating the entire statement. h < [log3/2 (n + 1)] − 2 = O(lg n). Prove that if x < x1 or x > xn , then i=1 (i − j)2 = 133. Divide the problem into two subproblems: ■ List the r-combinations containing s1 . (Hypothesis) ··· X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z) (Conclusion) The conclusion asserts that the two sets X ∩ (Y − Z) and (X ∩ Y) − (X ∩ Z) are equal. , xn) is true. Input: A (an m × k matrix of a relation R1), B (a k × n matrix of a relation R2), m, k, n Output: C (the m × n matrix of the relation R2 ◦ R1) comp relation(A, B, m, k, n, C) { // first compute the matrix product AB for i = 1 to m for j = 1 to n { Cij = 0 for t = 1 to k Cij = Cij + Ait Btj } // replace each nonzero entry in C by 1 for i = 1 to m for j = 1 to n if (Cij > 0) Cij = 1 } Section 2 Review 1. What is a premise in an argument? Disproving a Universally Quantified Statement Recall that to disprove ∀xP(x) we simply need to find one member x in the domain of discourse that makes P(x) false. Example 1.16 The complete bipartite graph on two and four vertices, K2,4 , is shown in Figure 1.15. Show how Algorithm 4.6 finds a minimal spanning tree for the graphs of Exercises 1–5. Thus a decision tree of height 2 or less can account for at most four outcomes. Since 15 < 77, 77 must move one position to the right 44 44 7. Thus, there exists an integer k (namely k = k1 + k2) such that m + n = 2k+1. When a pointer moves off the end of one of the sequences, merge concludes by copying the rest of the other sequence to the output. Again, by the Multiplication Principle, there were then 50 · 50 · 50 = 125,000 additional recipients.After one more iteration, there were then 50 · 50 · 50 · 50 = 6,250,000 additional recipients. Write a program that converts among decimal, hexadecimal, and octal. 1 63. Represent the statement a < b or (b < c and a ≥ c) the Olympic Games, then New Atlantic will build a new stadium. (Because m is odd) There exists an integer, say k2 , such that n = 2k2 . In Exercises 13–16, write the relation as a set of ordered pairs. 10. |S| Example 5.3 Two fair dice are rolled. Similarly, a beta value of a circle vertex is an upper bound for v. We begin by eliminating 2, 4, 6, . At the ith iteration of insertion sort, the first part of the sequence s1 , . What is the interpretation of ∃x∃yP(x, y)? Now 0= Fij − Fji i∈V = i∈V Fiz − i∈V + = i∈V i∈V Fzi + i∈V j∈V j =a,z Fiz − Fij − i∈V Fia − i∈V Fai i∈V Fji i∈V Fai i∈V since Fzi = 0 = Fia , for all i ∈ V , and (Definition 1.3b) Fij − Fji = 0 if j ∈ V − {a, z}. an = 2an−1 + 8an−2 n = 0, 1, . If two piles remain, leave piles with equal numbers of tokens. Sections 1 and 2 introduce several proof techniques. (a) (b) (c) (d) T T T T is a tree. 2 e 1 e 1 1 b 2 a, 1 2 Figure 4.8 The first iteration of Dijkstra’s shortest-path algorithm. 439 Graph Theory v1 If we let vi be the vertex corresponding to program i, we obtain v2 s(v1 , v2) = 36, s(v2 , v3) = 38, v3 v4 v5 Example 1.8 110 s(v1 , v5) = 30, s(v3 , v4) = 54, The n-Cube (Hypercube) 111 011 100 000 s(v1 , v4) = 42, s(v2 , v5) = 48, s(v4 , v5) = 46. We use Definition 6.16 to verify that H and T are indeed independent. A counterexample is x = 0, y = 1. Find all part numbers and amounts for Zamora’s department. exercise indicates a problem of above-average difficulty. This completes the proof. To summarize, every recursive function must have base cases. ∃x∃y P(x, y) Problem-Solving Corner Problem Assume that ∀x∃y P(x, y) is true and that the domain of discourse is nonempty. Formal Solution If either p or q equals 1, we may take n = 1; so assume that p > 1 and q > 1. (d) Earth is the only planet in the universe that contains life. D 4 W M F E 1 2 R 3 G 2 G7 (b) Solutions are G1 , G5 ; G1 , G7 ; G2 , G4 ; G2 , G6 ; G3 , G6 ; and G3 , G7 . Solve the recurrence relation (3.12) in case n is a power of 2 to obtain bn = 3n − 2, 2 n = 2, 4, 8, . The least number of edges necessary to connect n vertices is n − 1. G2 has a simple cycle of length 3, but G1 has no simple cycles of length 3. Find a recurrence relation for the sequence {bn } and compute b1 , b2 , and b3 . If X has 10 members, how many members does P(X) have? We may assume that n > 6i. The statement is false since, if x = 1, the proposition 12 − 1 > 0 is false. Then binary search is used to determine the correct position for sn , after which sn is inserted in its correct position. Since there are 3n−1 strings altogether of length n−1 and cn−1 of these contain an even number of 1’s, there are 3n−1 − cn−1 strings of length n − 1 that contain an odd number of 1’s. We compute a2 using the formula a2 mod z = [(a mod z)(a mod z)] mod z. If x ∈ X ∪ Y , then p: CX∪Y (x) = CX (x) + CY (x) for all x ∈ U q: X ∩ Y = ∅, the contrapositive is ¬q → ¬p. 3 ➜ The Euclidean Algorithm In Section 1, we discussed some methods of computing the greatest common divisor of two integers that turned out to be inefficient. 24 2. For example, there are five rise/fall permutations of 1, 2, 3, 4: 76. We argue by contradiction. 307 Counting Methods and the Pigeonhole Principle (b) How many strings of part (a) begin with the letter B? 445 Graph Theory 2 ➜ Paths and Cycles If we think of the vertices in a graph as cities and the edges as roads, a path corresponds to a trip beginning at some city, passing through several cities, and terminating at some city. Suppose that we compute p(t) by a straightforward technique that requires n − k multiplications to compute ck t n−k . a = 143, n = 10, z = 230 61. If f is a function from X to Y and A ⊆ X and B ⊆ Y , we define f(A) = {f(x) | x ∈ A}, f −1 (B) = {x ∈ X | f(x) ∈ B}. In Example 1.8 we obtained the recurrence relation cn = 2cn−1 + 1 (2.3) and initial condition c1 = 1. B ∩ (C ∪ A) 46. Since R is antisymmetric, y = x. In a digraph of a relation on X, vertices represent the elements of X and directed edges from x to y represent the elements (x, y) in the relation. Let pn denote the probability that Toots wins all of Sly’s pennies if Toots starts with n pennies. (c) Show that sn,1 = (n − 1)! for all n ≥ 1. Similarly, there exists q2 and q3 such that a = q2 z + x, b = q3 z + y. R B 3 B K H L I J G 4 W G G1 G2 2 C R 3 W 1 G1 25. Example 6.7 Consider the statement ∀x∃y(x > y). 1 = 1n = [2 + (−1)]n = P(S)P(G) = 23. The number of orderings is then C(n, n1)C(n − n1 , n2) · · · C(n − n1 − · · · − nt−1 , nt), which, after applying the formula for C(n, k) and simplification, gives n!/(n1 ! · · · nt !). Let n, c, and d be integers. x=3 48. Example 5.18 Suppose that the domain of discourse of the propositional function P is {−1, 0, 1}. What is a pairwise disjoint family of sets? It must also contain 3, 5, and 7 [so that 105 divides lcm(30, 105)]. Next 15 is inserted. 65. Each method of walking n meters includes the extra string s in the list. 33. Multiplying by t and noting again that t > 0, we have b ≥ ta. Now let’s compare the values of the expressions CX∪Y (x) and CX (x) + CY (x). Now g(ei) = g(e) = E. Therefore, R is reflexive. On the other hand, if f(n) = (g(n)), we can draw the conclusion that, except for constant factors and a finite number of exceptions, f is bounded above and below by g, so f and g grow at the same rate. 6 3 2 6 4 5 1 6 1 8 3 2 9 1 4 6 2 8 12 3 13. Therefore, at most 4 + aL + aR comparisons are required in the worst case. 6 129. Once the first letter has been selected, the second letter can be selected in four ways. In a binary tree we distinguish the left and right subtrees of a vertex. For example, since log3/2 2(1,000,000) = 33.07 . 23. b1 b0 ? [Gardner, 1992] contains a chapter about the Fibonacci sequence. [Project] Develop a computer program to play a game that has relatively simple rules. a’s and b’s. What is a flow into a vertex? Example 1.31 1 2 3 X f a b c Y Figure 1.11 The function of Example 1.29. Having chosen a value for x, let y stand for an arbitrary element of Y and show that P(x, y) is always false. Since at least 264 − 1 = 18,446,744,073,709,551,615 moves are required to solve the 64-disk Tower of Hanoi puzzle, we can be fairly certain that something would happen to the tower before it was completely moved. Let f and g be functions from the positive integers to the positive integers defined by the equations f(n) = 2n + 1, Find the compositions f ◦ f , g ◦ g, f ◦ g, and g ◦ f . 267 Introduction to Number Theory The proof combines the formulas for the gcd (Theorem 1.17) and lcm (Theorem 1.22) (which require that m > 1 and n > 1) with the fact that min(x, y) + max(x, y) = x + y for all x and y. Now suppose that a > b. and W2 = R(V ∗) ∩ P. Example 2.19 If S denotes the set of prime numbers less than 20, 1 i∈S i = 1 1 1 1 1 1 1 1 + + + + + + + = 1.455. ∀x∀y T1 (x, y) 2. Therefore A ∩ (B C) = (A ∩ B) (A ∩ C). Find the rightmost element r that satisfies d < r. Show that Algorithm 3.3 correctly finds gcd(a, b) even if lines 3 and 4 are deleted. What is an experiment? Use depth-first search (Algorithm 3.7) with the vertex ordering hgfedcba to find a spanning tree for graph G of Figure 3.1. b i d b f g i d g f h 10. We then invoke this algorithm with i = j = 4 to search for key in s4 = ‘S’. Choose one appetizer. 395 Recurrence Relations If we substitute this expression for an−2 into (2.2), we obtain an = an−2 ↓ = an−3 + 3 + 2·3 + 2·3 = an−3 + 3 · 3. The ad said that there were 210 possible dinners. Regardless of what value Farley chose, you can choose y = − x, which makes the statement x + y = 0 true. Let each row, column, or diagonal that contains two X’s and one blank count 5. This notation is called the omega notation. The problem of coloring a planar map G, so that no countries with adjoining boundaries have the same color, can be reduced to the problem of coloring a graph by first constructing the dual graph G of G in the following way. In [Gardner, 1959], Hamiltonian cycles are related to the Tower of Hanoi puzzle. Proof Let us divide the vertices into two groups: those with even degree x1 , . Suppose that Ti has ni vertices. 177 Functions, Sequences, and Relations Example 4.2 Consider the partition S = {{1, 3, 5}, {2, 6}, {4}} of X = {1, 2, 3, 4, 5, 6}. 144 3 12 P(E ∩ F) = P(F) 2 144 23 144 364 365 n . A key challenge in automating a reasoning system is to help guide the search for clauses to combine. R1 = {(x, y) | x + y ≤ 6}; R1 is from X to Y ; R2 = {(y, z) | y = z + 1}; R2 is from Y to Z; ordering of X, Y , and Z: 1, 2, 3, 4, 5 Functions, Sequences, and Relations 19. ≤ c + (1 + logm n)d = O(logm n). Then CX∪Y (x) = 1. This matrix is called the matrix of the relation R (relative to the orderings of X and Y). For a precise statement and a proof of the general result, see [Brualdi]. Does 9450 represent a number in binary? There are two cases to consider. A function f from X to Y is said to be one-to-one (or injective) if for each y ∈ Y , there is at most one x ∈ X with f(x) = y. The edge (1, 4) in G2 has δ(1) = 3 and δ(4) = 3 but there is no such edge in G1 (see also Exercise 21). Proof We will give an argument to show that any tree with five vertices is isomorphic to one of the trees in Figure 8.1 or 8.3. If T is a tree with five vertices, by Theorem 2.3 T has four edges. 2 3 5 7 11 13 17 19 A string is a finite sequence of characters. (Most of these problems belong to the class of NP-complete problems; see [Johnsonbaugh] for details.) An example of an NP-complete problem is: Given a collection C of finite sets and a positive integer k ≤ |C|, does C contain at least k mutually disjoint sets? 1 R 2 4 3 3 4 4 3 2 1 4 G 3 3. We first negate the conclusion and use the first of De Morgan’s laws (3.3) to obtain ¬(b ∨ d) ≡ ¬b ¬d. Thus [a] = [b]. 35 Sets and Logic 6. A = {1, 2, 3}, B = ∅ In Exercises 41–48, draw a Venn diagram and shade the given set. , nm−1 r n are solutions of (2.5). , an) = (b1 , b2 , . The first r-combination follows the last r-combination. (d) v is a right child of w in T1 if and only if f(v) is a right child of f(w) in T2 . We then invoke this algorithm At line 2, since i > j (5 > 5) is false, we proceed to line 4, where we set k to 5. For example, Kronos’s children are Zeus, Poseidon, Hades, and Ares. Probability We can compute the probability that no one receives the correct order if we assume that all permutations are Recurrence Relations equally likely. Deduce that the number of ways to parenthesize the product of n elements is Cn−1 , n ≥ 2. ∃x P(x) 25. By definition, the statement ∃x∀y P(x, y), with domain of discourse X × Y , is true if there is at least one x ∈ X such that P(x, y) is true for every y ∈ Y . 110010000 20. v1 w1 v4 w2 v3 v2 v5 f(v4) = w2 , v6 v7 w4 w3 w5 T1 w6 w7 T2 Figure 8.7 Isomorphic rooted trees. How many distinct dominoes are there? 7 6 7 6 3 8 6 8 7 2 1 3 8 2 4 8 36. Suppose that both roots of t 2 − c1 t − c2 = 0 are equal to r, and suppose that an satisfies an = c1 an−1 + c2 an−2 , kn (2.22) g(n) = C n if 1 is a root of (2.22) of multiplicity one, and g(n) = C n2 if 1 is a root of (2.22) of multiplicity two. [For example, in equation (5.2), t = 2.] There are t terms in the first set of parentheses, 1 ≤ t < n, and n − t terms in the second set of parentheses, 1 ≤ n − t < n. large = c 7. We count the number of n × n matrices that represent reflexive and antisymmetric relations on an n-element set. Several useful rules of inference for propositions, which may be verified using truth tables (see Exercises 24–29), are listed in Figure 4.1. Rule of Inference Name Rule of Inference p→q p ∴q p q Modus ponens Modus tollens Conjunction ∴p→r Hypothetical syllogism p∨q ¬p p ∴p∨q ∴p∧q p→q q→r p→q ¬q ∴ ¬p Name Addition ∴q Disjunctive syllogism p∧q ∴p Simplification Figure 4.1 Rules of inference for propositions. What is the probability that the sum of the numbers on the dice is 8? Using Definition 6.13, the Counting Methods and the Pigeonhole Principle probability of rain given high barometric pressure is P(R | H) = 0.10 P(R ∩ H) = = 0.125, P(H) 0.80 where R denotes the event “rain,” and H denotes the event “high barometric pressure.” Independent Events If the probability of event E does not depend on event F in the sense that P(E | F) = P(E), we say that E and F are independent events. In this connection, Hamilton introduced the term vector. Consider routes from the lower-left corner to the upper-right corner in an (n + 1) × (n + 1) grid in which we are restricted to traveling only to the right or upward. b 5 6 6 a c 3 5 d 10 e 2 f 10 4 z 6 4 g 4 5 6 8 h 10. If we first compare a3 with the second-smallest item among a1 , a2 , a4 , a5 , only one additional comparison will be required, for a total of seven comparisons. 11. From p ∨ q and ¬q, we may use the disjunctive syllogism to conclude p. 0 1 0 1 F D1 0 d 0 E B 1 0 A C 5. 39. Logic is the study of reasoning; it is specifically concerned with whether reasoning is correct. If T1 and T2 are rooted trees that are isomorphic as free trees, then T1 and T2 are isomorphic as rooted trees. Since the value 3 of K is less than the alpha value 6 of I, an alpha cutoff occurs and the subtree with root L may be pruned. i=3 q1 = 8, 42. Finally, solutions to the subproblems are combined to obtain a solution to the original problem. (8.2) ■ The two graphs should not have any edges in common. Example 6.14 Write the negation of ∃x∀y(xy < 1), where the domain of discourse is R × R. We find a vertex of a degree 1 and delete a and the edge x incident on it. What is the difference between a direct proof and a proof by contradiction? 2 4. The reason is that there are 3 ways to choose an element of X for the first member of the ordered pair, there are 2 ways to choose an element of Y for the second member of the ordered pair, and 3 · 2 = 6 (see Figure 1.8). if (j − i is odd ∧ (1 + j − i)/2 is odd) m = (i + j)/2 − 1 else m = (i + j)/2 Show that in the worse case, this modified algorithm requires at most (3n/2)−2 comparisons to find the largest and smallest elements in an array of size n. How many bridge deals are there? In expressions that involve the logical operators ∧, ∨, ¬, and →, the conditional operator → is evaluated last. True (according to Dean Martin’s song, “Everybody Loves Somebody Sometime”). Bayes’Theorem: If the possible classes are C1 , . What is a minimal spanning tree? 2. Show how the algorithm constructs the 2-cube from the 1-cube. Find a formula for the sequence d defined by u1 = 3, zi . Let G = (V, E) be a graph. 93 Proofs Section Review Exercises 1. Can we conclude that α1 = β1 and α2 = β2 ? We suppose that the board has three rows and seven columns. In words: There are two distinct people and the first is shorter than or the same height as the second. Algorithm 4.6 Robot Walking This algorithm computes the function defined by ⎧ n=1 ⎨1, n=2 walk(n) = 2, ⎩ walk(n − 1) + walk(n − 2) n > 2. Give an example of a binary search tree. (The while loop will execute i − 1 times during the ith iteration of the for loop.) We call this time the worst-case time. If e is an edge, w(e) is equal to the weight of e; if e is not an edge, w(e) is equal to ∞ (a value greater than any actual weight). In effect, f counts the elements in Y : f(1) = a is the first element in Y ; f(2) = b is the second element in Y ; and so on. ∃x(x > 1 → x2 > x) 53. How many are odd? By convention, G is said to be obtainable from itself by a series reduction. If T is a tree with six vertices, T must have five edges. i=1 P(F | Ci)P(Ci) The equation P(Cj | F) = P(F | Cj)P(Cj) P(Cj ∩ F) = P(F) P(F) follows from the definition of conditional probability. Use induction to show that bn = 2n − 2 35. One vertex must be the root. If a ∈ Y , set s1 = 1; if a ∈ Y , set s1 = 0. Waterbury is a fine city. Appel and Haken published their solution to the four-color problem in [Appel]. This algorithm computes an recursively, where a is a real number and n is a positive integer. Input: The edges E of an n-vertex, connected, weighted graph. The input is n. G1 : 0 1 GR 1: G1 : G1 : G2 : 1 00 11 00 0 01 10 01 11 10 GR 2: G2 : G2 : G3 : 10 000 110 000 11 001 111 001 01 011 101 011 00 010 100 010 110 111 101 100 We close this section by examining a problem that goes back some 200 years. Proof Let E1 = {x1 , . We first prove part (a). (Figure 8.14 shows this construction for n = 6 and k = 2.) By the Multiplication Principle, this construction can be carried out in ak an−k−1 ways. (5.2) Here we would first multiply a1 by a2 to obtain a1 a2 and a3 by a4 to obtain a3 a4 . Any integer greater than 1 can be written as a product of primes. an = 6an−1 − 8an−2 + 3 i ≤ n. We conclude by discussing a special algorithm, which we will need in Section 4, to compute powers mod z. The Inductive Step is complete; G has an Euler cycle. We then enter the for loop. Since each die can show any one of five values, by the Multiplication Principle there are 5 · 5 outcomes in which neither die shows 2. Anyone who wears sandals is an algebraist. Let f k = f ◦ f ◦ · · · ◦ f denote the k-fold composition of f with itself. To prove that ∀x∃y P(x, y) is false, where the domain of discourse is X × Y, you must show that for at least one x ∈ X, P(x, y) is false for every y ∈ Y . What is the prefix form of an expression? Using the notation of Exercise 31, the solutions are G1 , G5 ; G2 , G5 ; G3 , G6 ; and G4 , G6 . Also, a problem with private keys in general is that the keys have to be securely sent to the sender and recipient before messages can be sent. i=1 P(F | Ci)P(Ci) Proof By Definition 6.13, P(Cj | F) = P(Cj ∩ F) , P(F) P(F | Cj) = P(F ∩ Cj) . Summary of Problem-Solving Techniques ■ Look for a similar problem. (7.4) i=k We use Theorem 7.6 in the form C(i, k) = C(i + 1, k + 1) − C(i, k + 1) to obtain C(k, k) + C(k + 1, k) + C(k + 2, k) + · · · + C(n, k) 363 Counting Methods and the Pigeonhole Principle = 1 + C(k + 2, k + 1) − C(k + 1, k + 1) + C(k + 3, k + 1) − C(k + 2, k + 1) + · · · + C(n + 1, k + 1) − C(n, k + 1) = C(n + 1, k + 1). The first player can always win in nim by adopting the following 9. Example 1.6 Let Sn denote the number of n-bit strings that do not contain the pattern 111. [Knuth, 1997, 1998a, 1998b] is the classic reference for much of this material. 40 58. Therefore, the sequence nr n , n = 0, 1, . a 1 e 2 2 b 5 e d b 6 3 d c 5 13. , we simulate the execution of the algorithm for arbitrary input of size n > 1. Explain how to compute the inverse of n modulo φ. 6i. f = e − v + 2 = 14 − 9 + 2 = 7 12. By Exercise 24, D may be written in Egyptian form: D= n 1 1 + ··· + , n1 nk (n − 6) (n − 6) where the ni are distinct. Cube 4 then gets a green front face and a red back 489 Graph Theory R B B R 2 1 3 4 G (a) Front/back G W G 4 Cube 1 R W 1 2 3 B (b) Left/right Figure 8.4 Rotating a cube to obtain a left/right orientation, without changing the front/back colors. (n), where n is the sum of the lengths of the input sequences 9. If such a proof of p ↔ q is read in one direction, we obtain the proof of p → q, and, if it is read in the other direction, we obtain a proof of q → p. Proof [First proof] We show that for all x, if x ∈ X ∪ (Y − X), then x ∈ X ∪ Y , and if x ∈ X ∪ Y , then x ∈ X ∪ (Y − X). The slopes of the supply Price Price Supply p 0 = p2 = p4 = ... Supply p2 p0 p 1 = p3 = p5 = ... Demand p1 p3 q2 = q4 = ... q1 = q3 = ... Figure 1.6 A cobweb with a fluctuating price. If the characters match, ti+j−1 == pj j is incremented j =j+1 and the next characters are compared. Let Tn denote the highest power of 2 that divides n. 3 1 3 3 4 3 1 1 4 4. Let v be a vertex. It is often readily converted to computer code. Notice that the definition of “path” allows repetitions of vertices or edges or both. 40 42 56 9 23 20 9 22 21 10 6 41 40 20 38 30 3 43 13 37 56 26. Example 1.27 Prove that the function f(n) = 2n − n2 from the set of integers to the set of integers is not one-to-one. You might have wondered if there are formulas for the number of nonisomorphic n-vertex trees of a particular type. We now put this all together to produce a proof. f(n) = (n3/2), 10. The words old programmers never die they just lose their memories (5.5) may be placed in a binary search tree as shown in Figure 5.4. Notice that for any vertex v, each data item in the left subtree of v is less than (i.e., precedes alphabetically) the data item in v and each data item in the right subtree of v is greater than the data item in v. If (j, k) is not in T , T ∪ {(j, k)} contains a cycle C. Give an example of a unary operator. Which of the algorithm properties—input, output, precision, determinism, finiteness, correctness, generality—if any, are lacking in the following? Express each octal number in Exercises 42–47 in hexadecimal. Our first method of representing a graph uses the adjacency matrix. 61 16. , in), which means that f(k) = ik . This result, together with Exercise 25, shows that the worst-case time to search in an n-vertex balanced binary search tree is (lg n). Then G is one of the two graphs shown in Figure 7.8. In either case, the formula holds. Example 8.10 The binary trees T1 and T2 in Figure 8.11 are not isomorphic. Each equivalence class consists of a set of mutually isomorphic trees. ρ(τ(R1)) = τ(ρ(R1)) If X and Y are sets, we define X to be equivalent to Y if there is a one-to-one, onto function from X to Y . Evaluate the game-playing ability of this algorithm. In this case, the result is called the prefix form of the expression (or Polish notation). Since the graph itself is not a cycle, we have a contradiction. Kruskal’s Algorithm finds a minimal

spanning tree in a connected, weighted graph G having n vertices as follows. What is a cryptosystem? If we fail, we may learn how to construct a counterexample. Show that aab is not in L. Example 8.2 366 If 20 processors are interconnected, show that at least two processors are directly connected to the same number of processors. If you are to prove that two sets X and Y are equal from the definition of set equality, you know you must show that for every x, if x ∈ X then x ∈ Y , and if x ∈ Y then x ∈ X. , sn , n Output: large (the largest item in s1 , s2 , . Find a formula for the number of edges in Km,n . (b) R is symmetric. By Theorem 2.18, G has an Euler cycle. In Algorithm 2.19, the numbers multiplied are remainders after division by z and so have magnitude less than z. (n) (n2) 13. p p p p 134 134 134 134 134 q q q q q 2456 2456 2456 2456 r r r r r 123 1234 12344 123445 1234456 si, ..., sm: 134 sm≤1, ..., sj: 2456 ci , ..., cj: Figure 3.1 Merging si , . Although every known algorithm to test whether two graphs are isomorphic requires exponential or factorial time in the worst case, there are algorithms that can determine whether a pair of graphs is isomorphic in linear time in the average case (see [Read] and [Babai]). .. How might one automate the selection of S to group data into classes using a similarity graph? Example 2.11 Most programming languages define "not" exactly as in Definition 2.9. For example, in the Java programming language, "not" is denoted !, and the expression !(x < 10) is true precisely when the value of the variable x is not less than 10 (i.e., x is greater than or equal to 10). If m = 1, the result is immediate, so we assume that m > 1. Theorem 3.2 then tells us that after line 6 executes, gcd(a, b) = gcd(b, r). (1.4) and for every x, To verify equation (1.3), suppose that x ∈ A. We have shown that there exists x ∈ X such that a − ε < x ≤ a. First, we verified that the statement corresponding to n = 1 was true. Let {sn } be a sequence defined for n = m, m + 1, . 541 Trees PT A B C B Figure 6.2 At line 5 of Algorithm 6.1, where the input is the tree of Figure 6.1. Figure 6.1 Input for Algorithm 6.1. Example 6.2 In what order are the vertices of the tree of Figure 6.3 processed if preorder traversal is used? If you run 10 laps daily, then you will be healthy. Define n is prime. {1, 1, 3}, {3, 3, 1} 31. The digit following 16 must exceed 3. Therefore A B ⊆ (A − B) ∪ (B − A). So, if we travel 55 miles per hour for t hours, D = 55t, (1.1) where t is the time and D is the distance traveled. He said, "Morning!" And I said, "You got a fine city here, Mayor." And then he had coffee with me. Exercises In Exercises 1–27, the set D1 consists of three students: Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. 63 5. Show that the representation (5.3) need not be unique by representing 5/6 in two different ways. The maximum capacities of the routes are A to B 3000 vehicles B to C 2000 vehicles A to C 4000 vehicles. What is deductive reasoning? Therefore, 2e ≥ 4f. Find 4 q4 = 28, 26. The next time we execute the for loop in lines 7–14, the edges with one vertex in the tree and one vertex not in the tree are Edge Weight (1, 2) (1, 5) (2, 4) (3, 5) (3, 6) (4, 6) 4 3 5 6 3 6 This time two edges have minimum weight 3. (b, c, d, e, b, b) 7. In this case, a ∈ X. b a c e 29. a c e 12. Thus the edges at Wor occur in pairs. Person 3 can't get order 1 (for then person 2 would get the correct order); thus, person 3 gets order 2. It is denoted P(X). Example 2.9 Decimal to Hexadecimal Convert the decimal number 20385 to hexadecimal. Example 7.1 Figure 7.2 A pan balance for comparing weights of coins. In this problem, the length of a path that starts at vertex v1 and then visits v2 , v3 , . an = (22−n + 3n)/5 47. Derive a recurrence relation and an initial condition for the number of ways to divide a convex (n + 2)-sided polygon, †A starred 392 exercise indicates a problem of above-average difficulty. is connected and acyclic. 4 2 2 Therefore, events H and T are independent. Notice that the sequence c, b is not a subsequence of sequence (2.4) since the order of terms in the sequence (2.4) is not maintained. The converse is not true; however, if we introduce the concept of "homeomorphic graphs," we can obtain a true statement similar to the converse (see Theorem 7.7). Section 1 6. Proof by Contrapositive Suppose that we give a proof by contradiction of p → q in which, as in Examples 2.1 and 2.2, we deduce ¬p. 3 5. The argument is similar to that of the proof of Theorem 2.23. Write a program that lists all permutations of ABCDEF in which C and E are side by side in either order. What is a partition of a set X? an = (lg 2n)an−1 − [lg(n − 1)]an−2 8. Assume that a problem of size 1 is already solved. Suspicious officials at first refused to pay him. X = {1, 2, . e3 v2 v3 v4 v3 24. Prove that for all integers m and n, if m and n are odd, then m + n is even. bn = 2n − 2, 44. (If the first tree is empty, r1 has the special value null. Assign probabilities to the outcomes that accurately model the likelihood of the outcomes to occur. Implement proof by resolution as a program. Show that if a | b and b | c, then a | c. R × Z × Z In Exercises 73–76, list all partitions of the set. Find the descendants of c and of e. For 3 1 3 2 5 4 7 37. Summary of Problem-Solving Techniques Since x ∈ X ∩ Y , x ∈ X and x ∈ Y . Exercises 31–39 refer to a fair coin that is repeatedly flipped. , vn be an ordering of the vertices of G1 . Write each proposition in Exercises 35–42 in words. ¬p ∨ ¬r From 1 and 2 5. As an example of double counting, consider the following faulty argument that purports to show that the graph of Figure 3.6 has no Hamiltonian cycle. Then CX (x) = CY (x), for all x ∈ U. f(n) = n2 − 1 12. Thirteen coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. For example, suppose that one of the hypotheses is ¬(a ∨ b). sn = i=0 159 Functions, Sequences, and Relations We find that sn = 2(−1)0 + 2(−1)1 + 2(−1)2 + · · · + 2(−1)n = 2 − 2 + 2 − ··· ± 2 = 2 if n is even 0 if n is odd. We would assume the hypothesis, that is, that n2 is even. Are E1 and E2 mutually exclusive? If a processor needs to communicate with a nonadjacent processor, the first processor sends a message that includes the route to, and ultimate destination of, the recipient. p ∧ ¬q 58. Trace Algorithm 2.1 for the input t = "balalaika" and p = "lai". 315, 825 18. Definition 8.8 Let T1 be a binary tree with root r1 and let T2 be a binary tree with root r2 . , sj . Determine whether each statement in Exercises 42–54 is true or false. At the time the article was written, it was estimated that it would take 40 quadrillion years to factor z. Such trees are used to show logical relationships among records in a database. Write a program that, given a frequency table for characters, constructs an optimal Huffman code. , Pn are replaced by an arbitrary family P(x), where x is in the domain of discourse, and (5.5) is replaced by ∃x P(x). Taking the preceding equation as the definition of ordered pair, prove that (a, b) = (c, d) if and only if a = c and b = d. We make more precise statements about worst-case searching in a binary search tree. Inductive Step Suppose that G has n edges, n > 0, and that any connected graph with k edges, k < n, in which every vertex has even degree, has an Euler cycle. In the worst case, three comparisons are required to sort three items using an optimal sort (see Example 7.2). After the swap we have 5 44 15 10. Since any of the remaining numbers is less than n, any of them may be placed in the first position. If the domain of the sequence is finite (as in Example 2.2), we say that the sequence is finite. Find the next row of Pascal's triangle given the row 1 n 7 21 35 35 21 7 2k C(n, k) = 3n . Modify Algorithm 3.13 by inserting the lines preceding Exercise 45 after line 6 and replacing line 7 with the following. Exercise 2; a0 = 1 22. Because logb n = (loga n), when using asymptotic notation we need not worry about which number is used as the base for the logarithm function (as long as the base is 223 Algorithms greater than 1). , sn) 1. Write a recursive algorithm that reverses a finite sequence. Fibonacci's original question was: Under the conditions of Exercise 18, how many pairs of rabbits are there after one year? Find a connected subgraph of the graph of Exercise 5 containing all of the vertices of the original graph and having as few edges as possible. This is exactly what occurs in the fourth line of the truth table of Definition 3.3. Even more motivation for defining p → q to be true when p is false is given in Exercises 74 and 75. What is a subgraph? First, perform a two-level search. This happens if there is a vertex v whose entry in row a is 1 and whose entry in column c is 1. The terminology adapted from a family tree is used routinely for any rooted tree. The associated algorithms are known as parallel algorithms. Thus α = β. k=0 Equation (7.3) can also be proved by giving a combinatorial argument. (4.9) Basis Step (n = 1) [Condition (4.7)] We must show that (4.9) is true if n = 1. Apollo, Athena, Hermes, Heracles 4. Thus we assume s1 < A s2 < A .. Problem-Solving Tips A sequence is a special type of function; the domain is a set of consecutive integers. 315 Counting Methods and the Pigeonhole Principle 69. In general, the key to devising a proof by induction is to find case n "within" case n + 1. ■ Determinism The intermediate results of each step of execution are unique and are determined only by the inputs and the results of the preceding steps. From the definition of "intersection," we conclude that x ∈ X and x ∈ Z. Axioms are statements that are assumed to be true. What are parallel edges? If the coin is flipped 10 times, what is the probability of at most five heads given at least one head? 121. Implement the methods of computing the largest and smallest elements in an array (see Exercises 39–49, Section 3) and compare the times needed to execute each. What is the probability of getting a sum of 6 given that at least one die shows 2? {(x, y) | x is taller than y} 13. Notice that in the digraph of this relation (see Figure 3.2) there are two directed edges between b and c. Discussion We let n be an arbitrary integer. The set {y | (x, y) ∈ f } (which is a subset of the codomain Y) is called the range of f . 2 + 4 + 6 + · · · + 2n 8. , sr }, to find the following string t1 . 2 ➔ More Methods of Proof In this section, we discuss several more methods of proof: proof by contradiction, proof by contrapositive, proof by cases, proofs of equivalence, and existence proofs. Such representations facilitate the computer evaluation of expressions. Region 2 represents elements in X, but in neither Y nor Z. If, for example, q = 3, the remainder would be n − dq = 74 − 13 · 3 = 35, which is too large. We turn next to existentially quantified statements. We also assume that pn = kqn+1 , where k is another positive parameter. ´ The Tower of Hanoi puzzle was invented by the French mathematician Edouard Lucas in the late nineteenth century. First we solve the recurrence relation (3.2) in case n is a power of 2. We call a network such as that of Figure 4.3 a matching network. If A and B are assigned a job, none remains for D. ■ It often helps to write out carefully what is to be counted. Merging sequences 14. Under these assumptions, t(n) grows like n2 as n increases. ; there are at most k − 1 values x ∈ X with f(x) = ym . AB+C− 18. Using the Multiplication Principle, we find that we can select two books, one from computer science and one from mathematics, in 5 · 3 = 15 ways. (a) There is no solution. Then 3 ai = a1 + a2 + a3 = 2 + 4 + 6 = 12, i=1 3 ai = a1 · a2 · a3 = 2 · 4 · 6 = 48. We can also quickly determine whether a relation R is antisymmetric by examining the matrix of R (relative to some ordering) (see Exercise 11). Put an edge between two vertices if their labels (i.e., subsets) have no elements in common. Do you want fries or a salad with your burger? Example 3.13 Find a theta notation in terms of n for the number of times the statement x = x + 1 is executed. , wn of distinct words and the length n of the sequence Output: A binary search tree T make bin search tree(w, n) { let T be the tree with one vertex, root store w1 in root for i = 2 to n { v = root search = true // find spot for wi while (search) { s = word in v if (wi < s) if (v has no left child) { add a left child l to v store wi in l search = false // end search } else v = left child of v else // wi > s 538 Trees } T* Figure 5.6 Expanding a binary search tree to a full binary tree. (X Y is the symmetric difference of X and Y . If either tree is empty, four comparisons are required at lines 1 and 3, after which the procedure returns. (A deal consists of partitioning a 52-card deck into four hands, each containing 13 cards.) Counting Methods and the Pigeonhole Principle 34. Coloring the map G is equivalent to coloring the vertices of the dual graph G . State the absorption laws for sets. A good start is achieved by writing out the hypotheses and conclusion so that we are clear where we start and where we are headed. Suppose that we are given a set X of n elements, some labeled "red" and some labeled "black," and we want to find the number of subsets of X that contain at least one red item. To obtain the matrix of a relation from X to Y , we label the rows with the elements of X and the columns with the elements of Y . 6a. Notice that when (3.1) is applied to clauses, the result q ∨ r is a clause. 6 ➔ Nested Quantifiers Consider writing the statement The sum of any two positive real numbers is positive, symbolically. Find the set [x2] of all elements related to x2 . We summarize the method of Examples 2.1 and 2.2 as a theorem. To determine whether a graph is bipartite, try to separate the vertices into two disjoint sets V1 and V2 so that each edge is incident on one vertex in one set and one vertex in the other set. G is not a simple graph. 55. We can begin to fill in the gap by using the definitions of "odd" and "even" to obtain m is odd and n is even. ∀x∀y P(x, y) 81. The other edge incident on B is 3, so make cube 3's front face blue. The digraph of R shown in Figure 3.3 has at most one directed edge between each pair of distinct vertices. In the Google search engine, and is the default operator so that, for example, entering discrete mathematics produces a list of pages containing both of the words "discrete" and "mathematics." The or operator is OR, and the not operator is the minus sign −. (3.9) Since the sequence a is nondecreasing (see Exercise 25), a2k−1 ≤ an ≤ a2k . If v is the first vertex examined by Prim's Algorithm, the edge will be in the minimal spanning tree constructed by the algorithm. There must be some situations in which a recursive function does not invoke itself; otherwise, it would invoke itself forever. 248 Section 1 2. In words, a relation is reflexive if every element in X is related to itself. ¬p → ¬q 15. On the other hand, if the items are to be selected with repetition, the formula C(k + t − 1, t − 1) may be used. 9 9 13 3 9 1 9 6 7 16 4 8 4 7 16 9 14 6 8 10 8 16 12 13 20 25. Since f(n) is an odd integer for all n, we may choose for y any positive, even integer, for example, y = 2. A proof by contradiction can be devised (see Exercise 1), but here a proof by contrapositive is requested. The domain of discourse is Z+ . Find all subgraphs of the graph of Exercise 29 satisfying properties (8.1) and (8.2). Then x ∈ A or x ∈ ∅. For example, in the graph of Figure 7.2, face A is bounded by the cycle (5, 2, 3, 4, 5) and face C is bounded by the cycle (1, 2, 5, 1). The set A ∪ B is shown in Figure 1.4, and Figure 1.5 represents the set A − B. The problem is to determine for which n a knight's tour exists. To find the second-largest value, first pick a value v less than all the items in the sequence. C = (v0 , . (1.4) and To prove equation (1.3), let x ∈ X ∩ (Y − Z). Use the ordinary algorithm for adding decimal numbers to add hexadecimal numbers—except replace the decimal addition table by the hexadecimal addition table. Define internal vertex in a rooted tree. All rights reserved. As shown, we sometimes indicate a cut by drawing a dashed line to partition the vertices. What is the probability of at least one boy and at least one girl given that there is at least one girl? When you write up your proof, begin by writing out the statement to be proved. Therefore, the number of eight-bit strings that start 10 or end 011 or both is equal to |X ∪ Y | = |X| + |Y | − |X ∩ Y | = 26 + 25 − 23 . Now assume that if a sequence of length n, where n satisfies 2m−1 < n ≤ 2m , m ≥ 2, is input to the while loop, the loop iterates at most m−1 times. Because of the nested loops, it runs in time (n3). We have omitted symmetric positions. In this case, at most three weighings are required. How do we do that? Let R = {(5, 2), (6, 3)} and S = {(1, 5), (2, 6)}. Describe the shuffle algorithm in words. Of the 33 students taking calculus and computer science, 8 are also taking psychology; thus 25 are taking calculus and computer science but not psychology. If there is no child, create one, put an edge incident on it and the last vertex visited, and store the data item in the added vertex. The proof by contradiction is complete. A binary tree is a rooted tree in which each vertex has either no children, one child, or two children. Example 1.10 If the input to Algorithm 1.8 is n = 1274, the algorithm returns the prime 2 because 2 divides 1274, specifically 1274 = 2 · 637. Suppose that x ∈ A. The converse of Theorem 2.17 is also true. a 1 12. 72 · 132 4. (An example of such an eight-bit string is 01111110. If we are told that the domain is the set of all real numbers and the codomain is the set of all nonnegative real numbers, in ordered-pair notation, we would have f = {(x, x2) | x is a real number}. (3.4) 99 Proofs In this case, we may replace the single hypothesis a ∨ bc by the two hypotheses a ∨ b and a ∨ c. 119. x becomes a16 , and n becomes 1. (b) Find an explicit formula for pm . What is a simple path? (For Exercise 19) False; Pat and Sandy are distinct persons, but Pat is not taller than Sandy. ■ Each cube represents an edge exactly once in each graph. Trace Algorithm 2.4 for the input 5 51 2 44 96. 293 Introduction to Number Theory Section Review Exercises 1. C(1000, 5) 1000 · 999 · 998 · 997 · 996 Example 5.5 In the Illinois state lottery Lotto game, to win the grand prize the contestant must match six distinct numbers, in any order, among the numbers 1 through 52 randomly drawn by a lottery representative. 3 3 3 4 1 R 4 4 W 2 4 2 1 1 2 G 3 3 B 4 3 2 W G 1 10. (Each di is one of 0–9.) 2. Assume that bk = k − 1 for all k < n. If s1 = key, line 3 is executed once. , and 2n the ai are positive numbers n 16. This compound action is called composition. 110111011011 34. The intended meaning is: Some circumstance is not right for a formal investigation. Which input produces the worst-case behavior for Algorithm 3.14? Suppose that G contains a cycle. 2 ➔ Sequences and Strings Blue Taxi Inc. The two forms of mathematical induction are logically equivalent (see Exercise 33). If x ∈ X, then x ∈ f −1 (f({x})). What is the intended meaning? Let us determine the members of the equivalence classes. A relation R on a set X is transitive if for all x, y, z ∈ X, if (x, y) and (y, z) ∈ R, then (x, z) ∈ R. 6 + 12 + 9 4. Choose a vertex v0 . As an example, consider the erroneous "proof" that for all integers m and n, if m and m + n are even, then n is even: Let m = 2k1 and n = 2k2 . Another example of a selection is CS Physics |× × × ×| History × × which denotes the selection consisting of no computer science books, four physics books, and two history books. Thus, by Theorem 5.4, n + i = 2i + 1, so that i = n − 1. Reduce the case n < 0 to the case n > 0. If three points are not collinear, then there is exactly one plane that contains them. The following are equivalent. (c) At line 2, since 2 = 0, we proceed to line 4. (b) Show, by consulting 2 3the figure,nthat e 45. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, 13, ... The truth table of a proposition P made up of the individual propositions p1 , . Comment on the following argument: 26. Since lg n! ≤ h and lg n! = (n lg n), worst-case sorting requires at least (n lg n) comparisons. On the other hand, a function may be defined by a formula. Then an ≤ an/2 + a(n+1)/2 + 2 lg n ≤ bn/2 + b(n+1)/2 + 2 lg n = bn . 1 ➔ Introduction Consider the following instructions for generating a sequence: 1. The graph of Exercise 8 has the Euler cycle (1, 2, 3, 1). Again G has n edges; hence, x G G Figure 7.10 The proof of Theorem 7.9 for the case that G has a cycle. (3.5) It then sets a = 110 and b = 53. Therefore, K3,3 is not planar. Since a solution, if one exists, is found at a terminal vertex, by moving to the terminal vertices as rapidly as possible, in general we can avoid generating some unnecessary vertices. Furthermore, q and r are unique. No one attended Professor Sandwich's office hours. If the Chargers can beat the Broncos, then the Chargers can beat the Jets. Then P(E1 ∪ E2) = P(E1) + P(E2) − P(E1 ∩ E2). Definition 1.48 Example 1.49 A function from X to X is called a unary operator on X. (For Exercise 42) Since both quantifiers are ∀, Farley chooses values for both x and y. Since we want the next smallest permutation, the next digit is 4, the smallest available digit. Show that some pair has a midpoint that has integer coordinates. Consider the situation from the second (circle) player's point of view. When we want to explicitly state the initial index k of an infinite sequence s, we can write {sn }∞ n=k . The relation τ(R) is called the transitive closure of R. Define string. (4.6) Because of (4.5) and (4.6), we have Sn+1 = Sn + (n + 1) = n(n + 1) + (n + 1). i (1.1) i [In a sum such as (1.1), unless specified otherwise, the sum is assumed to be taken over all vertices i. It follows that there are 120 − 24 = 96 strings that do not begin with the letter B. ∀x∀y((x < y) → (x2 55. Divide the quotient by 16. But this last condition holds precisely when A is symmetric about the main diagonal. The Euclidean algorithm next computes r = 4 mod 1 = 0. (b) Find the worst-case running time of binary search2. We next show that our solution is optimal. The algorithm uses the divide-and-conquer approach. Two sequences are equal if the two sets of ordered pairs are equal. (Exercise 6 asks for a formal statement of this process.) The time spent searching for an item in a binary search tree is longest when the item is not present and we follow a longest path from the root. Therefore, ai − aj = 9 for some i and j, as desired. Since k2 + k is an integer, set {m, m + 1, . 449 Graph Theory The solution to the existence of Euler cycles is nicely stated by introducing the degree of a vertex. If 0 ≤ j ≤ p − 1, then −t(p − 1) + jt ≥ −t(p − 1) + t(p − 1) = 0. In this chapter we begin by giving the requisite terminology. 24 = 16 ≥ 16 = 42 Inductive Step Assume true for n. Explain how to find a maximal flow in a network. Show that the bound am−1 + am in Exercise 29 cannot be lowered. P is a simple path. The proof that there are constants b and d such that U0 = C0 and U1 = C1 is similar to the argument given in Theorem 2.11 and is left as an exercise (Exercise 48). What is the relation between flows and matchings? The columns of an n-ary relation are called attributes. We must prove that X has a least element. 36 Counting Methods and the Pigeonhole Principle By Theorem 6.9, the probability of getting doubles or a sum of 6 is P(E1 ∪ E2) = P(E1) + P(E2) − P(E1 ∩ E2) 1 5 1 5 = + − = . We then invoke this algorithm with i = 1, j = 2 to search for key in s1 = 'C', s2 = 'G'. satisfies the recurrence relation Pn = n−1 8. Thus (n + 2)kCn = (4n + 2)Cn . A television commercial for a popular beverage showed the following Venn diagram Great Taste 62. Why is this true in general? If X is a finite set, what is |X|? Give an example of a second-order, linear homogeneous recurrence relation with constant coefficients. k=0 16. We close this section by proving a special result. In the second case, if the student stays with A, the answer will be incorrect; but if the student switches to B, the answer will be correct. (c) [First solution] Arguing as in part (a), if Egbert is chairperson, we have 20 ways to select the remaining officers. Let X be the set of strings over {a, b} of length 4 and let Y be the set of strings over {a, b} of length 3. Give an example of a set X and a relation R on X that is not reflexive, symmetric, and not transitive, but for which the collection of pseudo equivalence classes (see Example 4.15) partitions X. But if x ∈ A ∩ B, x ∈ A by the definition of intersection. According to definition (4.3), Sn+1 = 1 + 2 + · · · + n + (n + 1). At the Inductive Step when the (n + 1)st line is added, because of the assumptions, the line will intersect each of the other n lines. 100 b ∨ d. Example 1.7 Let f be the function defined by the rule f(x) = x2 . Usually we are less interested in the exact best-case or worst-case time required for an algorithm to execute than we are in how the best-case or worst-case time grows as the size of the input increases. A track enthusiast decides to try to complete the East-SouthEast Marathon. Show, by an example, that Algorithm 3.7 can produce identical spanning trees for a connected graph G from two distinct vertex orderings of G. Number the squares as shown: Sn = Sn−1 + 2n 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 Notice that each tromino covers exactly one 1, one 2, and one 3. Then x ∈ A ∪ B or x ∈ C. Write a program to determine whether one string is a substring of another string. Find the names of all teams that have players aged 30 years or older. A B C D E F H G 503 Graph Theory Section 8 22. 2 Combining these last inequalities, we obtain n+1 3 < m. We first show that if n ≥ 9, there is an optimal solution containing at least one 7. a ∨ c 3. Let z ∈ Z. The graph of Exercise 3 10. Given an (n−1)-vertex binary tree T , n ≥ 2, construct a binary tree from T by adding a left child to every vertex in T that does not have a left child, and adding a right child to every vertex in T that does not have a right child. Letter Frequency A B C D E F 5 8 5 12 20 10 Section 2 5. Index n − m + 1 is the last possible value for i since, at this point, the string tn−m+1 tn−m+2 · · · tm has length exactly m. . We then set the entry in row x and column y to 1 if xRy and to 0 otherwise. Therefore the loop iterates a total of m times. Elements related to one another may be thought of as equivalent. 106 Proofs Theorem 4.6 If |X| = n, then |P (X)| = 2n (4.13) for all n ≥ 0. We add z to U so that U = {z, e}. (a) Show that R is an equivalence relation. Determine whether each statement in Exercises 42–52 is true or false. The line "Play it again, Sam" occurs in the movie Casablanca. ¬d Applying (3.1) to expressions 1 and 2, we derive 6. Such an edge is called a loop. 114. Similarly, the third person's birthday can occur on any one of 363 days. Since m/b = 2/2 = 1, m is set to 1. Since there are 28 eight-bit strings, there are 28 − [1 + 8 + C(7, 2) + C(6, 3) + C(5, 4)] eight-bit strings that contain at least two 0's in a row. Suppose that the roots are t1 and t2 and that t1 = t2 . If v and w are vertices corresponding to two programs, s(v, w) is a measure of how dissimilar the programs are. Contrapositive of p → q: ¬q → ¬p 35. List all strings over X = {0, 1} of length 2 or less. Then x = p/q for some integers p and q. (3.10) k − 1 < lg n ≤ k. i=1 yi i=3 4 zi . Example 1.8 2 −n Internet Addresses The Internet is a network of interconnected computers. We are supposing that s1 = 2, s2 = 3, s3 = 4, s4 = 6, s5 = 7. If i = 0, T0 consists of a single vertex. Review the tiling problem (Example 4.7), which provides a striking example of case n "within" case n + 1. 53 Sets and Logic Example 6.8 Suppose that P is a propositional function with domain of discourse {d1 , . List all strings over X = {0, 1} of length 3 or less. FROM SHANE In many discrete problems, we are confronted with the problem of counting. Show that a dag has at least one vertex with no out edges [i.e., there is at least one vertex v such that there are no edges of the form (v, w)]. A flow F in G assigns each directed edge (i, j) a nonnegative number Fij such that: (a) Fij ≤ Cij . Instead of dealing directly with the input, we use parameters that characterize the size of the input. The last step results from the formula for the geometric sum. In this case, leave an even number of piles. What is a definition? Therefore q3 is even and, thus, q is even. Which properties are lacking? † Exercise 21. If they are consecutive, we may also write (∗) as n= m Chapter Self-Test 1. Thus a decision tree of height 2 or less can account for at most nine outcomes. 76 Definition 1.7 An integer n is even if there exists an integer k such that n = 2k. c l 3. Brit attended someone's office hours. (1.3) Then (x, y) ∈ f . Show that Kruskal's Algorithm is correct; that is, at the termination of Kruskal's Algorithm, T is a minimal spanning tree. The first bits are 10 (to identify it as a Class B address). 2n+1 < 1 + (n + 1)2n Section 5 17. Definition 2.1 Let v0 and vn be vertices in a graph. By Theorem 3.5, a pair a, b, a > b, would require n modulus operations when input to the Euclidean Algorithm only if a ≥ fn+2 and b ≥ fn+1 . Find the internal vertices. The image of T is T . This implies that (i, k) ∈ R2 ◦ R1 . By Exercise 64, J(100,000) = J(n) = 2j + 1 = 2 · 34,464 + 1 = 68,929. Since the flow out of v is also 1, the only edge of the form (v, x) having flow 1 is (v, w). In other words, circle moves to a position represented by a child with the minimum value. Suppose that the ijth entry of A is 1. Since a R a, a ∈ [a]. For example, if the statement to prove involves the absolute value of x, you may want to consider the cases x ≥ 0 and x < 0 because |x| is itself defined by the cases x ≥ 0 and x < 0. Recursively move n−1 disks on top of the largest disk. Given the matrix of a relation, how can we determine whether the relation is reflexive? If S and T are solutions of (2.13), then U = bS + dT is also a solution of (2.13). Thus x ∈ ∅ and U ⊆ ∅. Also, write a sequence of operations that could be used to answer the query. What is a domain of discourse? Consider the k + t − 1 slots — — —···— — and k + t − 1 symbols consisting of k ×'s and t − 1 |'s. Ordered pair: (x, y) 26. In a balanced binary tree of height 1, the root must have at least one child. Find a formula for the sequence c defined by cn = n i . , pn /∴ q is valid provided that if p1 and p2 and . return 1.12 ∗ compound interest(n − 1) 5. Exercises Exercises 1–3 refer to Example 6.2 in which a die is loaded so that the numbers 2 through 6 are equally likely to appear, but 1 is three times as likely as any other number to appear. We are given that P(d) is true for any d in the domain of discourse D. , dn } × {d1 , . 1 4 c 8 e 8 d 6 3 f 6 h v1 v2 a e3 6. Now suppose that x ∈ A. zn = 6zn−1 − 9zn−2 , ui . Let R be a reflexive and transitive relation on X. i=1 163 Functions, Sequences, and Relations 69. [Inductive Step only] Assume that the theorem is true for n. A ∪ (A ∩ B) = A, A ∩ (A ∪ B) = A B 37. Resolution proof; uses: if p ∨ q and ¬p ∨ r are both true, then q ∨ r is true. Then there are km edges from the set S. All but {a, b, c} are proper subsets of A. 611 Network Models Suppose that the edge (v, w), v ∈ V, w ∈ W, has flow 1. (p ∨ ¬q) → ¬rs ≡ ¬(p ∨ ¬q) ∨ ¬rs ≡ ¬pq ∨ ¬rs ≡ (¬p ∨ ¬r)(¬p ∨ s)(q ∨ ¬r)(q ∨ s) 11. Seven days after Wednesday, it is Wednesday again; 14 days after Wednesday, it is Wednesday again; and in general, if n is a positive integer, 7n days after Wednesday, it is Wednesday again. Therefore we can make postage for n + j, 0 ≤ j ≤ p − 1, by using js p-cent stamps and −t(p − 1) + jt q-cent stamps. 0 1 1 a A2 = b The product of these matrices is ⎛ 1 A1 A2 = ⎝0 1 1 1 2 ⎞ 0 1⎠ . 53 − 43 52. We have shown that (i, k) ∈ R2 ◦R1 if and only if the ikth entry in A1 A2 is nonzero; thus A1 A2 is "almost" the matrix of the relation R2 ◦ R1 . Let T be an n-node tree. The inverse of R, denoted R−1 , is the relation from Y to X defined by R−1 = {(y, x) | (x, y) ∈ R}. The isomorphism is f(vi) = wi for i = 1, . Select the first vertex a and label it the root. Write the converse of each proposition in Exercises 1–10. C(11, 7)x4 y7 9. small = si 26. Algorithm Polynomial Evaluation This algorithm evaluates the polynomial p(x) = n ck xn−k k=0 C(n − 1, k)Pk . A relation is symmetric if and only if its matrix A satisfies the following: For all i and j, the ijth entry of A is equal to the jith entry of A. ∀y∃x A(x, y) 37. We must show that if n satisfies 2k ≤ n < 2k+1 , the statement x = x + 1 is executed k + 1 times. A string s is related to 01111010 if the first four bits of 01111010 and s coincide. Prove that n n 118. There is no restriction on the remaining entries; each can be 0 or 1. Therefore, n is 1. See Exercises 43–45 for an alternative way to construct the n-cube. There are 10 copies of one book and one copy each of 10 other books. x bn = (n + 1)bn . The graph of the edges of the dodecahedron is given in Figure 3.2. We can solve Hamilton's puzzle if we can find a cycle in the graph of Figure 3.2 that contains each vertex exactly once (except for the starting and ending vertex that appears twice). Since no edges can be added to the single vertex h on level 4, the procedure ends. In this case, i < j, so the condition in line 2 is false. We can choose three cards of one suit in C(13, 3) ways and we can choose three cards of the other suit in C(13, 3) ways. The Euclidean algorithm is quite efficient. ∀x (P(x) ∨ Q(x)) 40. Let dn denote the deer population at time n. If X and Y are n-element sets, how many one-to-one, onto functions are there from X to Y ? Let A be a matrix whose ijth entry is 1 if the ijth entries of both A1 and A2 are 1. For example, at one time, the following ordinance was in effect in Naperville, Illinois: "It shall be unlawful for any person to keep more than three dogs and three cats upon his property within the city." Was one of the citizens, who owned five dogs and no cats, in violation of the ordinance? The null string is the string with no elements. 19. Proof The proof of Theorem 2.11 shows that the sequence r n , n = 0, 1, . We are given that a is a least upper bound for X. To prove that a function f from X to Y is not onto, find y ∈ Y such that f(x) = y for all x ∈ X. For example, walk(4) = walk(3) + walk(2) = 3 + 2 = 5. (b) List one member of each equivalence class. ABFGCDE 24. 1170– 1250), an Italian merchant and mathematician. 2 k 2k 2 k 2k Figure 4.6 Using mathematical induction to tile a 2k+1 × 2k+1 deficient board with trominoes. In the following graph the vertices represent cities and the numbers on the edges represent the costs of building the indicated roads. Does the graph of Exercise 8 have a Hamiltonian path? Applying (3.1) to expressions 3 and 4, we derive b ∨ d, 5. Figure 1.7 shows the Windows Explorer display of folders on the left and files on the right on a particular computer. In case (a), both edges are properly oriented. Proof by contradiction Indirect proof Proof by contrapositive Proof by cases Exhaustive proof Proving an if-and-only-if statement Proving several statements are equivalent Existence proof Constructive existence proof Nonconstructive existence proof Section 3 25. Similarly, we must have 3n2 ≤ 40. For example, the ordered trees 28. Then there are four edges incident on v and, because of Theorem 2.3, these are all the edges. Then cube 1 has a red right face. Notice the difference between the polynomial ak nk + ak−1 nk−1 + · · · + a1 n + a0 in Theorem 3.4 and the expression 1 k + 2 k + · · · + nk in Example 3.8. A polynomial has a fixed number of terms, whereas the number of terms in the expression in Example 3.8 is dependent on the value of n. (p ∧ q) ∧ ¬p 53. Write an algorithm that generates all r-permutations of an n-element set. Since S contains at most two 1's (since S is optimal), S contains at least three 1's. We derive a formula for the number of binary trees with n vertices. P(H) = 5/6, P(T) = 1/6 22. Inclusive-or: The meeting will be canceled if fewer than 10 persons sign up or at least 3 inches of snow falls or both. f(x) = 6x − 9 24. Let α = s1 · · · sp and β = t1 · · · tq be strings over {1, 2, . For example, to search for pages containing the keywords "Alfred Hitchcock" and (Herrmann or Waxman) and (not tv), 18 Sets and Logic Figure 2.1 The Google search engine, which allows the user to use and (space), or (OR), and not (−) operators to combine keywords. Pattern recognition places items into various classes based on features of the items. At this point, a1 , a2 , a4 , a5 is sorted. n−2 If v is a vertex in V , the path consisting of v and no edges is a path from v to v; thus vRv for every vertex v in V . The proposition is true. The entry in this matrix in row i, column j, i = j, is the number of edges incident on i and j. , m. 23 · 8!/(3! 1! 4!) 27. If for every vertex v in T the data item in v is greater than the data item in the left child of v and the data item in v is less than the data item in the right b d g e b f d h g c f e h Trees In Exercises 22–24, Nh is defined as the minimum number of vertices in a balanced binary tree of height h and f1 , f2 , . Prove that 2 x − 1 ≤ 2x ≤ 2 x for all real numbers x. y, z ≥ 0, y, z ≥ 0, y, z ≥ 0, x, y ≥ 0, 52. Example 3.2 The expression xy ∨ w ∨ ¬z is not a clause even though the terms are separated by or's, since the term xy consists of two variables—not a single variable. The intended meaning is: Some man does not cheat on his wife. Show that |X2 | = (n + 1)Cn for all n ≥ 1. What is the probability that the wheel lands on a black number? Use Tables 6.1 and 6.2. 21. Write a program to represent the sets X ∪ Y, X ∩ Y, X − Y , and X Y , given the arrays representing X and Y . P(2) ∧ P(3) ∧ P(4) 28. To derive a theta notation, you must derive both big oh and omega notations. v4 v2 e1 v5 e4 e2 e5 v3 v6 v1 Figure 1.5 A graph with parallel edges and loops. Since n = 7 > 0, the body of the while loop executes. By considering the number of n-bit strings that do not contain the pattern 010 that have no leading 0's (i.e., that begin with 1); that have one leading 0 (i.e., that begin 01); that have two leading 0's; and so on, derive the recurrence relation 6. Restate the proposition "A necessary condition for Leah to get an A in discrete mathematics is to study hard" in the form of a conditional proposition. Now (vn , . The probability of completing the marathon in one attempt is 1/3. Write a program that tests whether A is increasing. Input: PT , the root of a nonempty tree Output: 582 Each vertex of the tree has a field in cover that is set to true if that vertex is in the vertex cover or to false if that vertex is not in the vertex cover. In particular, since cn is defined in terms of cn/2 , in the Inductive Step we assume the truth of the statement for k = n/2. Exercises Use the Multiplication Principle to solve Exercises 1–9. A greedy algorithm optimizes the choice at each iteration. Thus, if we set s = −64 and t = 93105, 7. What is a trace of an algorithm? 17. Books specifically on graph theory are [Berge; Bondy; Chartrand; Deo; Even, 1979; Gibbons; Harary; K¨onig; Ore; West; and Wilson]. Conclusion: I will get a soda. 52 27. Finding explicit formulas from recurrence relations is the topic of Section 2. An analysis of the argument must take into account the fact that "nothing

proposition in Exercises 37–40 is true or false. We can then modify this recursive algorithm to obtain a recursive algorithm to compute s and t. As an example of a partition of X, we could let Xi be the set of five-bit strings that contain exactly i zeros. In this case, there is one subset of {1} with an even number of elements, namely, ∅. 2 W B WR G W (b) Find a solution to the puzzle. Derive a recurrence relation for the sequence of Schr¨oder numbers. It follows from Theorem 6.9 that if E1 and E2 are mutually exclusive, P(E1 ∪ E2) = P(E1) + P(E2). (1.7) Letting m = 11 and n = 4 and noting that m + n = 15 is odd does not constitute a proof that (1.7) is true for all integers m and n, it merely proves that (1.7) is true for the specific values m = 11 and n = 4. ABC+− 13. If T is connected and the removal of any edge (but no vertices) from T disconnects T, then T contains no cycles. Then R and S are transitive, but R ∪ S is not transitive; (1, 2), (2, 3) ∈ R ∪ S, but (1, 3) ∈ R ∪ S. Taking multi-vitamins is sufficient for being healthy. 92 Proofs The proof in Example 2.12 is nonconstructive; it does not exhibit an i for which si ≥ A. This groups the data into n − k classes. Give examples of distinct binary trees, B1 and B2 , each with two vertices, with the preorder vertex listing of B1 equal to the preorder listing of B2 and the postorder vertex listing of B1 equal to the postorder listing of B2 . Place the words FOUR SCORE AND SEVEN YEARS AGO OUR FOREFATHERS BROUGHT FORTH, in the order in which they appear, in a binary search tree. We let P(E | F) denote the probability of E given F . List all finite subsets of S and their sums. We show that the set of common divisors of a and b is equal to the set of common divisors of b and r, thus proving the theorem. If n > 1, we solve the (n − 1)-disk problem twice and we explicitly move one disk. Thus the number of ways of ordering BBBBSSMM is the number of ways to distribute the books. Pheasants Skyscrapers Tuna 17. By the inductive assumption, the function correctly computes the value of (n − 1)!. A function is a very general concept. Suppose that sj = sk for some j = k. These colorful pairs determine a rectangle whose four corner squares are red. d1 d0 ? (Set F0 = 1.) Show that the sequence F0 , F1 , . At line 4, the function correctly computes the value of 1!: n · (n − 1)! = 1 · 0! = 1 · 1 = 1. (For Exercise 1) False; Garth is not taller than Garth. It was said that before the monks finished moving the tower, the tower would collapse and the world would end in a clap of thunder. To see this, suppose that T contains a cycle C. (a) In how many ways can the months of the birthdays of five people be distinct? Since there are equal numbers of outside and inside squares, vertices vi where i is odd correspond to outside squares, and vertices vi where i is even correspond to Graph Theory inside squares. In how many ways can 15 identical mathematics books be distributed among six students? Thus n is either 3 or 4. 6 ➜ Discrete Probability Theory† In Section 5, we assume that all outcomes are equally likely; that is, if there are n possible outcomes, the probability of each outcome is 1/n. If x ∈ B − A, then x ∈ B and x ∈ / A. Assume that X and Y are arbitrary subsets of the universal set U. Before reading on, you might think about why we might divide the routes into classes. We count the number of rise/fall permutations of 1, . It follows that an even number of edges must touch Wor. But this last condition holds precisely when the main diagonal consists of 1's. Example 3.11 (3.14) Let n = 110 and φ = 273. its columns indexed by the edges of G. Working upward from the terminal vertices and using the minimax procedure, we can assign values to all of the vertices in the game tree (see Figure 9.2). If the terminal vertex is a circle, since the first player lost, this position is worthless to the first player and we assign it the value 0 (see Figure 9.2). (V , E) is a subgraph of G if V ⊆ V , E ⊆ E, and, for every edge e ∈ E , if e is incident on v and w , then v , w ∈ V . Insertion sort begins by inserting s2 into the subsequence s1 . As you may learn, the bit string 101111 represents the number 47. (a) ¬(∀x P(x)); ∃x ¬P(x) (b) ¬(∃x P(x)); ∀x ¬P(x) Proof We prove only part (a) and leave the proof of part (b) to the reader (Exercise 68). If key is found at the ith position, line 3 is executed i times; if key is not in the sequence, line 3 is executed n times. An algorithm by Coppersmith and Winograd (see [Coppersmith]) runs in time (n2.376) and, so, is even asymptotically faster than Strassen's algorithm. , m = 1, 2, . †A starred 30 7 d 9 e 12 8 20 3. ∃x∃y ¬P(x, y) 72. The conjunction of propositions p and q is the proposition p and q. When a teller looks for a record in a bank, a computer program searches for the record. AB+C− 12. Is w decreasing? The proof hinges on the fact that Dijkstra's algorithm finds the lengths of shortest paths from a in a nondecreasing order. Thus we assume the hypothesis n2 is even and that the conclusion is false n is odd. Any such string can be obtained by selecting n positions for the R's, without regard to the order of selection, among the n + m available positions in the string and then filling the remaining positions with U's. 125. In the worst case, the item will not be found at line 5, so the algorithm will be invoked at line 11. Then x ∈ A ∪ B and x ∈ / A ∩ B. C(11, 3) 31. Proof By the quotient-remainder theorem, there exist q and r satisfying a = bq + r, 0 ≤ r < b. Thus, 2n + 3 lg n = O(n). This gives cube 2 a blue back face. TABLE 6.5 ■ DEPARTMENT Dept Manager 23 04 96 66 Jones Yu Zamora Washington In Exercises 5–20, write a sequence of operations to answer the query. Let X and Y be sets. Thus (vx vy , wx wy) is an edge in the (i + j)-cube. Example 3.12 Show that, in Java, the expressions x < 10 || x > 20 and !(x >= 10 && x = means ≥, and = 10 and q denote the expression x = 10 && x 20, ¬p ∨ ¬q translates as x < 10 || x > 20. Rather, it applies to one set of inputs—namely, the empty set. At line 5, since c > large (0 > 12) is false, we move to line 7, where we return large (12), the maximum of the given values. Give an application of the shuffle algorithm. If each receives at least two compact discs, we must distribute the remaining six discs, and there are C(6 + 3 − 1, 3 − 1) ways to do this. n5 m3 (b) Divide the routes into classes based on when the route first meets the top edge to derive the formula n C(k + m − 1, k) = C(m + n, m). Join 54. (n + 1)(n + 3) n+2 12. Which of Exercises 86–88 must also be false? Does the graph of Figure 3.5 have a Hamiltonian path? There are C(52, 13) bridge hands. We must show that if β ∈ X∗ , there exists α ∈ X∗ such that f(α) = β. In general, there are C(k + m − 1, k) strings that end URn−k . Suppose that R1 and R2 are relations on a set X, A1 is the matrix of R1 relative to some ordering of X, and A2 is the matrix of R2 relative to the same ordering of X. v1 e7 v6 v2 e1 e5 e4 v7 e8 e6 e 11 e9 e2 v5 e 10 v3 v3 6 v4 e3 v4 7. Since MORE is greater than CLEAN, we go to the right child. The rooted trees T1 and T2 are isomorphic if there is a one-to-one, onto function f from the vertex set of T1 to the vertex set of T2 satisfying the following: (a) Vertices vi and vj are adjacent in T1 if and only if the vertices f(vi) and f(vj) are adjacent in T2 . Show that any coloring of the map of Exercise 22, excluding the unbounded region, requires at least four colors. Is b decreasing? Find qk . A committee that has at most one man has exactly one man or no men. f(x, y) = x + y, n2 4 2 3 4 5 6 Non-Leap Year January, October April, July September, December June February, March, November August May Leap Year January, April, July September, December June March, November February, August May October 107. Now the terminal vertices correspond to possible outcomes. Vertex e is the left child of vertex c; vertex c has no right child. 4 2 G 18. Then X ∩ Y = {2}, which is not a subset of X. This last condition holds if and only if −tan β < tan α. Suppose that vRw. Then there is a path (v0 , . (The symmetric difference is denoted .) 64 2. Exercises 45– 48 concern the Monty Hall problem, in which a contestant chooses one of three doors; behind one of the doors is a car, and behind the other two are goats. If Algorithm 1.8 returns a composite divisor of n, say a, 262 Introduction to Number Theory then a has a divisor a less than a. (A, B) ∈ R if for every a ∈ A and ε > 0, there exists b ∈ B with |a − b| < ε. , n} for some n ∈ Z+ . Algorithm 3.11 Computing an Exponential Input: 16. Because the relation is symmetric, once these values are assigned, the values below the main diagonal are determined. Prove that if a and b are real numbers with a < b, there exists an irrational number x satisfying a < x < b. p → q ¬p 12. Exercises In Exercises 1–5, find the length of a shortest path and a shortest path between each pair of vertices in the weighted graph. First, select one item of type 1, two items of type 2, and three items of type 3. 1 2 3 4 W 17. Algorithm 6.1 Preorder Traversal This recursive algorithm processes the vertices of a binary tree using preorder traversal. Thus the average number of times line 3 is executed is (1 + 2 + · · · + n) + n . We draw five balls and at least one is red, then replace them. Find the ancestors of c and of j. To verify that A is a subset of B, written A ⊆ B, show that for every x, if x ∈ A, then x ∈ B. The conditional probability of E given F is P(E | F) = Example 6.14 P(E ∩ F) . Edge (v3 , w2) is the only type II edge, and edge (w3 , z) is the only type III edge. Such strings are called palindromes.) In Exercises 38–43, a six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. Find the probability that both pass. For example, 00001 ∈ X2 ∩ X3 . ∃x (Q(x) → P(x)) 39. Since f(−1) = | − 1| = 1 = f(1), f is not one-to-one. Join The selection and projection operators manipulate a single relation; join manipulates two relations. What is a source in a network? √ 51. Exercise 8; a0 = a1 = 1 a0 = 2, a0 = 6, 35. Let g be the greatest common divisor of m and n. Write a program that, given a tree T , computes the eccentricity of each vertex in T and finds the center(s) of T . We say that the pair x = 1 and y = −1 is a counterexample. For each strategy, what is the probability of winning the car? , vn), v0 = a, vn = z, be a path from a to z in this undirected graph. Suppose, for example, that we want to convert the decimal number 91 to binary. We first evaluate ¬p, which is true. Example 3.8 Consider how the Euclidean algorithm computes gcd(273, 110). Determine whether the free trees are isomorphic. Let T1 be the left subtree of T , let T2 be the right subtree of T , and let r be the root of T . Mathematicians who co-authored a paper with Erd˝os are said to have Erd˝os number one. Binomial Theorem: (a + b)n = k=0 C(n, k)an−k bk 29. In how many ways can we draw two red, three green, and two purple balls if the balls are considered distinct? (a) A1 = (b) A2 = 1 1 1 0 1 0 c d ⎞ 0 1 0 1⎥ 1 1⎠ 0 1 0 1 1 1 0 1 1 2 1 (c) A1 A2 = 0 1 0 0 1 0 (d) We change each nonzero entry in part (c) to 1 to obtain A1 A2 = 1 0 0 1 1 1 1 0 . It is a function from X to Y ; domain = X, range = {a, b, c}; it is neither one-to-one nor onto. The number 57229 has 80 digits, so Theorem 2.17 indeed simplifies the computation. Region 4 represents elements in Y , but in neither X nor Z. |1| < 3 if −3 < 1 < 3. 414 Recurrence Relations In merge sort the sequence to be sorted, si , . If we roll two dice, the events "roll doubles" and "the sum is odd" are mutually exclusive. We consider two cases: p1 is true, p1 is false. Write an algorithm that returns the sum of the sequence of numbers s1 , . You got any? , n} and edge set {(1, i) | i = 2, . 6g. How many edges are incident on a vertex in an n-cube? Name and Position, in combination, could be used as a key for Table 6.1, since in our example a player is uniquely defined by a name and a position. At the for loop, either i = j or i = j + 1. There are two possibilities: Person 2 gets order 1, person 2 does not get order 1. Algorithm for generating permutations: Algorithm 4.14 19. Therefore, n < (log3/2 m) − 1 = log3/2 m − log3/2 3 2m = log3/2 . Then the capacity of (P, P) is greater than or equal to the value of F ; that is, Cij ≥ Fai . (a) We use the Multiplication Principle. The graph is a tree. The technique can be justified by appealing to Theorem 5.14. f(n) = |n| 15. Finding a Solution To show that R is an equivalence relation, we must show that R is reflexive, symmetric, and transitive (see Definition 4.3). a 1, a 3, a 2 No Yes a 3, a 1, a 2 a 2, a 3, a 1 No a 2 < a 3? Recurrence Relations and demand curves are tan α and tan β, respectively; thus in Figure 1.6, we have k = tan α = − tan β = b. A minimal spanning tree of G is a spanning tree of G with minimum weight. For example, "Find all persons who play outfield" is a meaningful query for the relation given by Table 6.1. We will discuss several operations on relations that are used to answer queries in the relational database model. Theorem 8.7 There are four nonisomorphic rooted trees with four vertices. Algorithm 3.13 Finding the Largest and Smallest Elements in a Sequence This recursive algorithm finds the largest and smallest elements in a sequence. , n = 1, 2, . Then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}. Let P1 and P2 be permutations of ABCDEF . The number of different possible activities is the product of the number of ways each step can be done. The connected graph shown has a unique simple path from any vertex to any other vertex, but it is not a tree. At line 7, key < sk ('S' < 'F ') is false, so at line 10, we set i to 4. Then R is reflexive, symmetric, and transitive. If n = 4, binary insertion sort sorts three items (three comparisons—worst case) and then inserts the fourth item in the sorted three-item list (two comparisons—worst case) for a total of five comparisons in the worst case. Write the binary number 10010110 in decimal. Summary of Problem-Solving Techniques ■ When trying to construct a proof, write out carefully what is assumed and what is to be proved. Draw a Venn diagram of three sets and identify the set represented by each region. For example, if f(n) = n and g(n) = 2n , then f(n) = O(g(n)), but g grows considerably faster than f . Exercise 7 shows that if the height of T is minimized, the worst case requires time equal to lg(n + 1) . Y − Z = (X ∪ Y) − (X ∪ Z) for all sets X, Y , and Z. We would then repeatedly compare D with the data item at the current vertex. If collisions occur infrequently, and if when one does occur it is resolved quickly, then hashing provides a very fast method of storing and retrieving data. State the distributive laws for sets. Use the result of Exercise 19 to find the matrix of the relation R2 ◦ R1 . 2563 32. How is the probability of an event defined? X 1 Y 3 2 6 5 4 7 8 Z Region 1 represents elements in none of X, Y , or Z. Now suppose that x R y and y R z. When we apply (3.1), p must be a single variable, but q and r can be expressions. Such edges are called parallel edges. Therefore, 2 is irrational. Many other problems, which on the surface seem not to be flow problems, can, in fact, be modeled as network flow problems. yn = 2n − 1, 3 yi i=1 3 10 wi . At the conclusion of the algorithm, large will then be equal to the largest of the three numbers. This is not always the case, however. Properties of sets (see Theorem 1.21) 23. , n? Hints/Solutions to Selected Exercises Section 1 Review 1. 535 Trees Our next result about binary trees relates the number of terminal vertices to the height. Answer Exercises 17–20 to give an argument that shows that if X is any (n + 2)-element subset of {1, 2, . We will first try to show that if x ∈ (A ∩ B) ∪ C, then x ∈ A ∩ (B ∪ C). 1 + 3 + 5 + · · · + (2n − 1) = n2 = n(n + 1)(n + 2) 2. Prove that {zn } satisfies n ≥ 2. This results in a beta value of 6 for F . Upon reflection, many mathematical statements have this property. Define quotient. ; thus the quotient is q = 36844/2427 = 15. Assume that the statement is true for k. Define nonincreasing sequence. Show that bn ≤ bn+1 for n = 1, 2, 3, . Input: n Output: All permutations of {1, 2, . We may use Theorem 1.7 to construct the following algorithm that tests whether a positive integer n > 1 is prime. Since (a + b)n = (a + b)(a + b) · · · (a + b), n factors (7.1) the expansion results from selecting either a or b from each of the n factors, multiplying the selections together, and then summing all such products obtained. If n is prime, the algorithm√returns 0. Now suppose that p occurs in t and its first occurrence is at index i in t.As noted in the previous paragraph, the algorithm correctly returns i, the smallest index in t where p occurs. A knight's tour of an n × n board begins at some square, visits each square exactly once making legal moves, and returns to the initial square. This notation will be used throughout this chapter. Sum or sigma notation: 23. ∃x P(x) 17. Draw the similarity graph that results from setting S = 40 in Example 1.7. How many classes are there? The other edge incident on W is 2, so make cube 2's front face white. In how many ways can 10 balls be selected if exactly one red ball must be selected? ∃x∀y(x2 + y2 ≥ 0) 56. Thus R is not transitive if ¬[∀x∀y∀z[(x, y) ∈ R ∧ (y, z) ∈ R] → [(x, z) ∈ R]]. Combining |x| = −x and d > −x gives |x| = −x < d. Therefore, any vertex in G is in the same component as either v1 or v2 . , vn−1 , en , vn), 0 0 2. If we delete e from this Euler cycle, we obtain a path with no repeated edges from v to w containing 452 Graph Theory all the edges and vertices of G. Events E and F are independent if P(E ∩ F) = P(E)P(F). The statement is true. Find t7 . This function is one-to-one because each element in Y has at most one arrow. Since all partitions of X can be generated in this way, we obtain the desired recurrence relation. The tree constructed is a full binary tree. We must show that T is a tree. What is the complete graph on n vertices? Exercises 40– 46 refer to Ackermann's function A(m, n). What is a planar graph? Explain how to use an arrow diagram to depict a function. Let x1 , . max3(a, b, c) { 2. The preceding equation shows that d divides r2 − r1 . If b > c, then min{min{a, b}, c} = min{b, c} = c = min{a, c} 1 2 n n+1 + + ··· + + 2! 3! (n + 1)! (n + 2)! =1− = min{a, min{b, c}}. The order of the subsets is not taken into account. (a) (b) (c) (d) (e) Figure 8.12 The five nonisomorphic binary trees with three vertices. Let P(x, y) denote the expression x ≤ y. In symbols, a relation R is transitive if ∀x∀y∀z[(x, y) ∈ R ∧ (y, z) ∈ R] → [(x, z) ∈ R]. ∴2=3 symbolically and determine whether the argument is valid. Assume that ∀x∀y P(x, y) is true and that the domain of discourse is nonempty. Postorder traversal processes the vertices of a binary tree by beginning at the root and recursively processing the vertex's left subtree, the vertex's right subtree, and then the current vertex. (1.5) The initial condition (1.5) and the recurrence relation (1.4) allow us to compute the value of An for any n. Therefore, 2e ≥ 4f . A binary tree T is balanced if for every vertex v in T, the heights of the left and right subtrees of v differ by at most 1. Is τ increasing? 66. The original invocation together with the invocations at line 11 gives all the invocations; thus we obtain the recurrence relation an = 1 + an/2 . We then suppose that the conclusion is false, that is, that ¬(x ≥ 1 ∨ y ≥ 1) is true. Similarly, we define the intersection of an arbitrary family S of sets to be those elements x belonging to every set X in S. p ∧ r 23. In either case, sort does produce as output a rearranged version of the input sequence in which all of the zeros precede all of the ones, and the Inductive Step is complete. Exercises 14 –20 refer to tournament sort. The increase in pairs of rabbits an − an−1 from month n − 1 to month n is due to each pair alive in month n − 2 producing an additional pair. , (1.11) (1.12) and the initial conditions A(0, n) = n + 1, n = 0, 1, . We call this time the best-case time. Since there are 3 groups of 2, there are 3 · 2 = 6 elements in X × Y (labeled at the bottom of the figure). Can a column of an incidence matrix consist only of zeros? (c) Scientist defeat Whales; Pilots defeat Scientists. The difference of two primes. Since there are 2n n rows, the function value assignments can be made in 22 n 2 ways. Using the definition of R, we may translate this condition to: If the first four bits of s1 and s2 coincide, then the first four bits of s2 and s1 coincide. Example 6.11 Consider the statement ∃x∃y((x > 1) ∧ (y > 1) ∧ (xy = 6)). We argue by contradiction to show that no algorithm that sorts three items has worst-case time less than 3. (0.2)(0.4) + (0.55)(0.25) + (0.3)(0.35) P(D | H) = 357 Counting Methods and the Pigeonhole Principle A similar computation using Bayes' Theorem gives P(R | H) = 0.426. This contradiction shows that if n ≥ 5, Cn is not prime. If Sets and Logic you can always win regardless of how Farley chooses values for the variables, the quantified propositional function is true, but if Farley can choose values for the variables so that you cannot win, the quantified propositional function is false. I get A's. Another important case is average-case time—the average time needed to execute the algorithm over some finite set of inputs all of size n. Figure 1.9 A similarity graph corresponding to the programs of Table 1.2 with S = 25. Since |x| = −x > 0 and 0 > x, |x| ≥ x. 1/52 17. You could go on forever, couldn't you? If there are no 5cent stamps among the stamps that make n cents postage, there are at least two 2-cent stamps (because n ≥ 4). 249 Algorithms 7. Similarly, we can select two books, one from computer science and one from art, in 5 · 2 = 10 ways, and we can select two books, one from mathematics and one from art, in 3 · 2 = 6 ways. 3 3 9 Similarly, P(E3) = 2 2 1 4 · · = . Counting Methods and the Pigeonhole Principle Example 4.4 Let α = 1324 and β = 1342 be strings over {1, 2, 3, 4}. Determine the number of comparisons used by binary insertion sort in the worst case for n = 4, 5, 6. This means that s must begin 0111 and the last four bits can be anything. Draw the similarity graph that results from setting S = 50 in Example 1.7. How many classes are there? R = {(x, y) | x < y}; ordering of X: 1, 2, 3, 4 (a) The matrix A1 of the relation R1 (relative to the given orderings). Multiplication Principle 2. The security of the RSA encryption system relies mainly on the fact that at present there is no efficient algorithm known for factoring integers. Problem-Solving Tips To construct a direct proof of a universally quantified statement, first write down the hypotheses (so you know what you are assuming), and then write down the conclusion (so you know what you must prove). There exists x such that for every y, P(x, y). What is the value of S(∅)? Example 2.2 b 3, 1 Consider the path from a to z in Figure 2.5. Edges (a, b), (c, d), and (d, z) are properly oriented and edge (c, b) is improperly oriented. We label the rows with the elements of X (in some arbitrary order), and we label the columns with the elements of Y (again, in some arbitrary order). 87. by the recurrence relation tn = tn−1 tn−2 , n ≥ 3, and initial conditions t1 = 1, for Stirling numbers of the second kind. You can always win the game, so the statement ∀x∃y(x + y = 0) is true. The circled vertices have permanent labels.) Figure 4.2 shows the result of executing lines 2–5. Theorem 5.6 Quotient-Remainder Theorem If d and n are integers, d > 0, there exist integers q (quotient) and r (remainder) satisfying n = dq + r 0 ≤ r < d. Now m + n = 2k1 + 2k2 = 2(k1 + k2). Algorithm 2.4 Finding a Maximal Flow in a Network This algorithm finds a maximal flow in a network. Therefore g = pc11 · · · pckk for some c1 , . In how many ways can these books be arranged on a shelf if all five computer science books are on the left and both art books are on the right? Let (x, y), (y, z) ∈ R−1 . a1 < a2 < a4 < a5 . If there is a path from v to w, there is a simple path from v to w. Show that if the conditions in Exercise 28 fail, then 16 divides aj + ak . We next evaluate p ∨ q. Snow 8. Draw all nonisomorphic simple graphs having three vertices. Write a nonrecursive version of the binary search algorithm. We construct the strings by a three-step process. The argument p1 , p2 , . [q → p.] Proof of q → p goes here. State the Multiplication Principle and give an example of its use. ∴ The bug is in module 17. Inductive Step We assume that we can make n cents postage, and we prove that we can make n + 1 cents postage. Informally, f can be thought of as a mapping of elements from X to Y . Prove the distributive laws for sets. 13 18. We will show shortly (Theorem 4.5) that Prim's Algorithm is correct—selecting an edge having minimum we do obtain a minimal spanning tree. Since either x ∈ / X or x ∈ / Y , either CX (x) = 0 or CY (x) = 0. 593 Network Models Example 1.4 The assignments, Fab = 2, Fdc = 1, Fbc = 2, Fde = 2, Fcz = 3, Fez = 2, Fad = 3, define a flow for the network of Figure 1.1. For example, the flow into vertex d, Fad = 3, is the same as the flow out of vertex d, Fdc + Fde = 1 + 2 = 3. Contestant 1 Contestant 2 Contestant 3 Contestant 4 Root Winner Contestant 5 Contestant 6 Contestant 7 Contestant m ⩾1 Contestant m Figure 5.2 The graph (full binary tree) of a single-elimination tournament. , sn in nonincreasing order. The 1-cube has two processors, labeled 0 and 1, and one edge. Because resolution depends on this single, simple rule, it is the basis of many computer programs that reason and prove theorems. Show that Nh = 1 + Nh−1 + Nh−2 , for h ≥ 0. q → r 38. Graph Theory 110 110 111 010 011 011 101 101 100 000 010 001 100 000 001 H2 H1 0110 0111 0010 0011 1110 1010 0001 1111 1011 1101 0101 0100 0000 111 1100 1000 1001 Figure 1.11 Combining two 3-cubes to obtain a 4-cube. As an example, if there are two choices for an appetizer and four choices for a main dish, the total number of dinners is 2 · 4 = 8. Since both p and q are true, the proposition p ↔ q is true. The pair 514229, 832040 itself requires 28 modulus operations. Explain how to convert from binary to decimal. Hint: Consider a preorder traversal of an ordered tree in which 1 means down and 0 means up. In this case we have 2 < n ≤ 4. Suppose that the n-cube has a simple cycle of length m. ∃x ¬(∀y P(x, y)) is not logically equivalent to ¬(∀x ∃y P(x, y)). p → q 6. Let g(x) = 2x and h(x) = sin x. ¬p ∨ q 7. Although P(E1) = 1/3, P(E2) = P(fail on attempt 1 and succeed on attempt 2) = P(fail on attempt 1)P(succeed on attempt 2) 2 2 1 = · = . f 20. Suppose that we choose c. Write a program to solve the n-queens problem. If α ∈ X∗ , let αR denote α written in reverse. Write the nonzero remainders as found by the Euclidean algorithm in the form 1 = 79 · 4 − (258 − 79 · 3)15 = 79 · 49 − 258 · 15. b r r g g b r b g r Figure 4.1 A set of colored balls. Use the Inclusion-Exclusion Principle for three finite sets (see Exercise 92) to compute the number of integers between 1 and 10,000, inclusive, that are multiples of 3 or 5 or 11 or any combination thereof. Give an example of a proof by contradiction. Suppose that we want to count the number of eight-bit strings that start 10 or end 011 or both. Therefore, x ∈ Y . Suppose further that no deaths occur. v1 562 18. In the system shown, we want to maximize the flow from a to z. Balanced 3. Find two graphs where B 2 W B G4 1 3 31. We let X∗ denote the set of all strings over X, including the null string, and we let X+ denote the set of all nonnull strings over X. 1 24 24 −1 24 = 1 15 thus, 28. Again using the definition of R, we may translate this condition to: If the first four bits of s1 and s2 coincide and the first four bits of s2 and s3 coincide, then the first four bits of s1 and s3 coincide. The idea of breadth-first search is to process all the vertices on a given level before moving to the next-higher level. We may use Theorem 3.5 to analyze the worst-case performance of the Euclidean algorithm. Example 1.7 The value of the flow in the network of Figure 1.2 is 5. , sj , and indexes i, m, and j The sequence ci , . v i1 v1 Properly oriented a v0 vn z v i+1 v2 v n1 vi Improperly oriented Path P Figure 2.1 Properly and improperly oriented edges. For example, 5/6 might be expressed as 5 1 1 = + . ■ If the problem is to count the number of items satisfying some property (e.g., in our problem we were asked to count the number of equivalence classes) and the number is sufficiently small, just list all the items and count them directly. Then x ∈ / ∅ (by the definition of "empty set"). The first division by 2 in (2.2) gives the 1's bit; the second division by 2 in (2.3) gives the 2's bit; and so on. Input: n, the number of years Output: The amount of money at the end of n years 1. Let the 13 persons be the pigeons and the 12 = 3 · 4 possible names be the pigeonholes. (a) Each solution of (3.1) is equivalent to selecting 29 items, xi of type i, i = 1, 2, 3, 4. The first condition ensures that every element of X is an element of Y , and the second condition ensures that every element of Y is an element of X. Use induction to show that when two k-vertex isomorphic binary trees are input to Algorithm 8.13, the number of comparisons with null is equal to 6k + 2. Relations can be used to order elements of a set. The statement x = x + 1 was executed four times. , m − 1} to X defined by 13. , wm) are distinct simple paths from a to b in a simple graph G, is (v0 , . Let (x, y) ∈ R∩S. Explain how Algorithm 3.13 finds the largest and smallest elements. It follows that c1 = 2r, Now c2 = −r 2 . The initial condition is c1 = 1. For these reasons and others, many compilers translate infix expressions to postfix form. If we consider the squares covered by T as missing, each of these quadrants is a 2k × 2k deficient board. The given algorithm is faster than the straightforward technique and is, therefore, preferred. Show that postage of 24 cents or more can be achieved by using only 5-cent and 7-cent stamps. If Pn denotes the number of permutations of n distinct objects, find a recurrence relation and an initial condition for the sequence P1 , P2 , . Given the matrix A1 of the relation R1 and the matrix A2 of the relation R2 , explain how to obtain the matrix of the relation R 2 ◦ R1 . We conclude that 1k + 2k + · · · + nk = (nk+1), and hence 1k + 2k + · · · + nk = (nk+1). 2 .. in decimal? m = 2n/2 8. 0110000100100001111 7. Show that s1 , . Proofs Richard Johnsonbaugh 73 3. Let Sn,k denote the number of ways to partition an n-element set into exactly k nonempty subsets. Next, we illustrate the Addition Principle by an example and then present the principle. We shall illustrate the first algorithm for finding a spanning tree by an example and then we will state the algorithm. An expression that consists of terms separated by or's, where each term consists of the and of several variables, may be replaced by an equivalent expression that consists of the and of clauses by using the equivalence a ∨ bc ≡ (a ∨ b)(a ∨ c). 1 8 2 4 16 20. FOUR Input: PT , the root of a binary tree Output: AND OUR BROUGHT FORTH 8. Assume that x ∈ X and x ∈ [a] ∩ [b]. First suppose that some pi is true. Then f(G) = n. an = an/2 + a(n+1)/2 + n − 1 11. In inequality (5.1), k indexes a statement S(k) that is an arbitrary predecessor of the statement S(n) (thus k < n), which we are to prove true. We use the Well-Ordering Property to prove something familiar from long division: When we divide an integer n by a positive integer d, we obtain a quotient q and a remainder r satisfying 0 ≤ r < d so that n = dq + r. a b c f g g h i m l 9. Then the general solution is of the form an = bt1n + dt2n , where b and d are constants. Thus Suppose that α is in L because of rule 1. , sr , the next r-combination (The first r-combination follows the last r-combination.) next comb(s, r, n) { s0 = n + 1 // dummy value m=r max val = n // loop test always fails if m = 0 while (sm == max val) { // find rightmost element not at its maximum value m=m−1 max val = max val − 1 } if (m == 0) // last r-combination detected s1 = 0 m=1 } // increment rightmost element sm = sm + 1 376 s1 , . A database management system responds to queries. If x = 0, |x| = 0 and sgn(x) = 0. For each false statement, provide a counterexample. Give an example of a real situation (like that of Figure 7.1) that can be modeled as a decision tree. Designate the processors 1, 2, . 490256 mod 337 = 258; 337 mod 258 = 79; 258 mod 79 = 21; 79 mod 21 = 16; 21 mod 16 = 5; 16 mod 5 = 1; 5 mod 1 = 0; so gcd(490256, 337) = 1. Thus the number of rise/fall permutations of 1, . Let Fn denote the number of functions f from X = {1, . Assuming that i = 1 and n = 2m+1 , in the while loop, k is first set to 2m . f(n) = n/2 14. We = n + n+1. This contradiction completes the proof that if there is a path from a to a vertex w whose length is less than L(v), then w is not in T . Basis Step (n = 1) If n = 1, 5n − 1 = 51 − 1 = 4, which is divisible by 4. At termination, what is L(z) if there is no path from a to z? 548 Trees C 1C 2 : C 3 C 4 edge does not exist 6 outcomes Figure 7.5 A four-coins puzzle algorithm that begins by comparing two coins against two coins. 370 3. Write a program that accepts as input the edges of a graph and then draws the graph using a computer graphics display. Notice that by rotating the cube, we can obtain this left/right orientation without changing the colors of the front and back (see Figure 8.4). be an increasing sequence whose values are in the set {m, m + 1, . fn−1 + for all n ≥ 3. 1 ➜ Divisors In this section, we give the basic definitions and terminology. Many practical problems are too difficult to be solved efficiently, and compromises either in generality or correctness are necessary. Yes 71 Sets and Logic 4. We delete edge x in a cycle. We will first find all nonisomorphic trees with five vertices in which the maximum vertex degree that occurs is 4. ∴ I get rich. Suppose that g is a function from X to Y and f is a function from Y to Z. Let P(x, y) be the propositional function x > y with domain of discourse R × R. Algorithm 3.1 Selection Sort This algorithm sorts the sequence s1 , s2 , . Draw all nonisomorphic binary trees having four vertices. Under Desktop are My Documents, My Computer, and others. (1.1) |Y | = |Y − X| + |X ∩ Y |. How to find a recurrence relation that describes the time required by a recursive algorithm 11. Multiply both sides of the inequality in Exercise 82 by lg e and use the change-of-base formula for logarithms. The output is the previous (in lexicographic order) permutation. Similarly, f(n) = (g(n)) if, except for a constant factor and a finite number of exceptions, f is bounded below by g. If person 2 does not get order 1, the possibilities are 2,3,4,1 and 2,4,1,3. But since Gn−1 is a Gray code, each consecutive pair of strings in Gn−1 differs in exactly one bit. n3 + n is even. Show that any algorithm that finds a minimal spanning tree in Kn , when all the weights are the same, must examine every edge in Kn . 101 Proofs 1 x 2 x 3 x 4 x 5 x Figure 4.1 Numbered blocks on a table. State the generalized De Morgan's laws for logic. Proof We let Ti denote the graph constructed by Algorithm 4.3 after the ith iteration of the for loop, lines 5–17. Note that s1 by itself is sorted! Now s1 , s2 is sorted. Is w nonincreasing? , a110 + 19 range from 1 to 219. Is the recurrence relation an = an−1 + an−3 a linear homogeneous recurrence relation with constant coefficients? 90 90 90 90 46. This minimum value (0) is assigned to the circle. C3 = 5, C4 = 14, C5 = 42 425 Recurrence Relations 28. [Deo; Even, 1979; and Gibbons] emphasize graph algorithms. Write a recursive algorithm that generates all r-combinations of the set {s1 , s2 , . 136. Solve the recurrence relation of Exercise 67 in case n is a power of 2, assuming that b1 = 0. Section 4 45. Find gcd(2 · 52 · 72 · 134 , 74 · 132 · 17). 62. By Theorem 1.17, gcd(82320, 950796) = 2min(4,2) · 3min(1,2) · 5min(1,0) · 7min(3,4) · 11min(0,1) = 22 · 31 · 50 · 73 · 110 = 4116. Amount 220 14 302 3 7720 200 296 39 10. Now mn = (2k1 + 1)(2k2 + 1) = 4k1 k2 + 2k1 + 2k2 + 1 = 2(2k1 k2 + k1 + k2) + 1. The ijth entry is 1 if edge j is in the ith fundamental cycle and 0 otherwise. Given a string s, which represents a permutation, to find the following string, find the rightmost digit d of s whose right neighbor exceeds d. 1 ➜ Introduction Figure 1.1 shows the highway system in Wyoming that a particular person is responsible for inspecting. Because k is an integer, we must have k ≤ 1 + lg n . The order of the subsets listed in line 1 depends on the method used to generate them, so the algorithm lacks the determinism property. George W. As in Example 2.3, the growth is exponential. Give an example (different from those of Example 1.2) of an axiom in the system of real numbers. Then for all y, x2 + y2 ≥ 0. Now execution proceeds as before; the statement x = x + 1 is executed four more times, for a total of five times. Although we frequently find functions defined in this way, the definition is incomplete since the domain and codomain are not specified. Find the prime factorization of each integer in Exercises 1–8. Express each binary number in Exercises 8–13 in octal. For example, {1, 2, 3, 4} is the set consisting of the integers 1, 2, 3, 4. Since the sets of strings of types (a), (b), and (c) are disjoint, by the Addition Principle Sn will equal the sum of the numbers of strings of types (a), (b), and (c). Suppose that we store data as described in Example 1.14 and retrieve data as described in Exercise 60. Example 2.2 We can solve the recurrence relation Sn = 2Sn−1 of Example 1.5, subject to the initial condition S0 = 1, by iteration: Sn = 2Sn−1 = 2(2Sn−2) = · · · = 2n S0 = 2n . Therefore |X ∩ Y | = 666, |X ∩ Z| = 303, |Y ∩ Z| = 181, and |X ∩ Y ∩ Z| = 60. It is possible, in fact, to sort five items using seven comparisons in the worst case. ■ Instead of proving p → q directly, consider proving its contrapositive ¬q → ¬p or a proof by contradiction. Table 3.1 shows that n ≥ 4 and that a = b. Therefore, ∀x P(x) is true. , pn have the same truth value; each is false. Then, because vertices 2, 3, 4, and 5 all have degree 2, all the edges in the graph would have to be included in a Hamiltonian cycle. If the domain of discourse is Z × Z, ∃x ¬(∀y P(x, y)) is true; however, ¬ (∀x ∃y P(x, y)) is false. , pn , P and Q are logically equivalent provided that given any truth values of p1 , . 130. The output is a rooted tree with the vertices at the lowest levels labeled with the frequencies and with the edges labeled with bits as in Figure 1.10. †A starred 0≤a max) max = sumji } sumii = si if (sumii > max) max = sumii } return max } Since the nested for loops take time (n2), max sum2 's time is (n2). By Definition 6.13, P(E | F) = P(E ∩ F) . (For n = 1 and n = 16, the algorithms require equal numbers of comparisons.) 21. k=1 Example 1.8 Tower of Hanoi The Tower of Hanoi is a puzzle consisting of three pegs mounted on a board and n disks of various sizes with holes in their centers (see Figure 1.2). [Recall that a database is a collection of records that are manipulated by a computer.] The tree of Figure 1.9 might be used as a model for setting up a database to maintain records about books housed in several libraries. If C is not a simple cycle from v to v, we repeat the previous procedure. Then q must be true, for otherwise p → q would be false. X × Y × Z for some m ∈ C} 63. Input: a and b (nonnegative integers, not both zero) Output: Greatest common divisor of a and b 1. Suppose we construct an algorithm that examines all subsets of X and counts those that contain at least one red item and then implement this algorithm as a computer program. Someone is taller than someone. If x is not greater than sn , the maximum value in the sequence s1 , . q = 9, r = 2 18. h(x) = x mod 17; cells indexed 0 to 16; data: 714, 631, 26, 373, 775, 906, 509, 2032, 42, 4, 136, 1028 †A starred 59. Let P(x) denote the propositional function "x has a graphing calculator," and let Q(x) denote the propositional function "x understands the trigonometric functions." The hypotheses are ∀x P(x) and ∀x(P(x) → Q(x)). For example, baa = α1 β1 if we set α1 = b and β1 = aa. Since the statement is "if and only if," we must prove two statements: If T is a tree, then T is connected but the removal of any edge (but no vertices) from T disconnects T. Equation n is Sn = n(n + 1). True. Higher Erd˝os numbers are defined similarly. Therefore, gcd(105, 30) = gcd(30, 15) = gcd(15, 0) = 15. Give an example of something in the real world that can be modeled by an undirected graph. Output—the algorithm produces output. References on probability are [Billingsley; Ghahramani; Kelly; Ross; and Rozanov]. 472 ith row of An kth ⎛ column ⎞ of A t1 ⎜ t2 ⎜⎜ ⎜⎜ .. n = 50, φ = 231 40. We can arrange for nq + it ≥ 0 to hold by choosing nq ≥ −it. 1110 R 0001 (because 1110 and 0001 do not share a common substring of length 2). 3, 2 6,2 a 5, z 2, 2, 4, d 3. Pairwise disjoint family of sets 18. , cn } satisfying 3 8 5 4 7 3 8 4 2 9 7 2 j 8 i=1 cki = M. Example 4.4 The relation R of Example 4.2 is an equivalence relation on {1, 2, 3, 4, 5, 6} because of Theorem 4.1. We can also verify directly that R is reflexive, symmetric, and transitive. (A move is considered to be made when the move is made in either direction.) 50. Can the first player always win in Mu Torere? In the preceding chapters, we used some basic number theory definitions such as "divides" and "prime number." In Section 1, we review these basic definitions and extend the discussion to unique factorization, greatest common divisors, and least common multiples. Is the converse of Exercise 1 true or false? Write a program that tests if a graph is a tree. The relation {(1, 1), (3, 3), (1, 2), (2, 1)} is not an equivalence relation on {1, 2, 3}. .}? 5 Therefore, an explicit formula for the Fibonacci sequence is 1 fn = √ 5 √ n √ n 1+ 5 1 1− 5 −√ . If n > 0, let G be the graph with vertex set {1, 2, . Thus the algorithm is correct for i = 1. Therefore X ∩ Z ⊆ Y ∩ Z. nk = 2k − 1 86. First compute s, where s is 3 times the sum of every other number starting with the first plus the sum of every other number, except the check digit, starting with the second. If the terminal vertex is a box, since the first player won, this position 563 Trees 3 2 1 2 2 2 0 2 0 1 2 1 0 0 1 1 0 0 1 2 0 2 0 1 1 1 0 0 0 0 2 0 0 1 1 0 0 1 0 0 0 0 2 0 0 0 1 0 1 0 0 0 0 2 0 0 0 1 0 0 0 2 1 1 0 0 0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 3 1 0 0 2 1 1 1 0 0 1 1 0 1 1 0 0 1 0 0 0 0 2 0 0 0 1 0 0 0 0 1 1 0 0 1 1 0 0 0 0 3 0 0 0 2 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 3 0 2 0 1 0 1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 Figure 9.1 A game tree for nim. The first player is represented by a box and the second player is represented by a circle. Show that the maximum number of edges in a simple, bipartite graph with n vertices is n2 /4 . n+1 f2k = f2n+1 − 1. Professor Curly proposes the following version of binary search: binary search4(s, i, j, key) { if (i > j) return 0 419 Recurrence Relations k = (i + j)/2 if (key == sk) return k flag = binary search4(s, i, k − 1, key) if (flag == 0) return binary search4(s, k + 1, j, key) else return flag 21. The goal then is to compute |X ∪ Y |. All data structures and algorithms books have extended discussions of searching and sorting (see, e.g., [Brassard; Cormen; Johnsonbaugh; Knuth, 1998b; Kruse; and Nyhoff]). Multiply by t! We hope that the proof of part (b) proceeds in a similar way. Thus the Strong Form of Mathematical Induction allows us to give a correct proof based on our informal reasoning. The argument in Example 5.23 is called universal modus tollens. Sentence (c), which is another way to say that the number of primes is infinite, is true. Input: n, an integer greater than or equal to 0 Output: n! 1. X is a proper subset of Y if X ⊆ Y and X = Y . If S ⊆ V , we must have |S| ≤ |R(S)|. bn = b(1+n)/2 + bn/2 + 3 69. p ∨ q ∨ r In Exercises 44–48, formulate the symbolic expression in words using p : Today is Monday. , abm . If the Chargers can beat the Broncos, then the Chargers can beat the Dolphins. The propositional function ∀x P(x) is equivalent to P(−1) ∧ P(0) ∧ P(1). If f is not a binary operator, state why. Assume that the last player to take a token wins. Goldbach's conjecture states that every even number greater than 2 is the sum of two prime numbers. Theorem 1.7 A positive integer n√greater than 1 is composite if and only if n has a divisor d satisfying 2 ≤ d ≤ n. Someone who owns a Porsche is an accountant. What can you say about two vertices in a rooted tree that have the same ancestors? To put it another way, (a, b) = (c, d) precisely when a = c and b = d. Thus, d(q1 − q2) = 0; hence, q1 = q2 . Assume that each terminal vertex stores a character and its frequency. 362 (nine terms), 9! 7! 9! = = 1260. Isomorphic. Then 8 divides both 3(3n + 7n − 2) and 4(7n + 1), it divides their sum, which is 3n+1 + 7n+1 − 2. The matrix of R1 relative to the orderings 1, 2, 3 and a, b is ⎛a b⎞ 1 1 0 A1 = 2 ⎝0 1⎠ , 3 1 1 187 Functions, Sequences, and Relations and the matrix of R2 relative to the orderings a, b and x, y, z is x y z 1 1 0 . Find b3 . ABC∗∗ABC+−− 20. (p ∨ q) ∧ ¬p 25. ¬r Hypothesis 4. ■ For all real numbers x, y, and z, if x ≤ y and y ≤ z, then x ≤ z. Why is it useful to know or be able to estimate the time and space required by an algorithm? 496 4. How many edges are in an n-cube? 1 ➜ Functions If we travel for a certain amount of time at a constant rate, we know that distance = rate × time. An independent set in a graph G is a subset S of the vertices of G having the property that no two vertices in S are adjacent. To prove that X is a proper subset of Y , prove that X is a subset of Y and find x in Y such that x is not in X. To prove that ∀x∀y P(x, y) is false, find one value of x ∈ X and one value of y ∈ Y that make P(x, y) false, where the domain of discourse is X × Y. We have seen that it is possible to define a sequence by a recurrence relation together with certain initial conditions. If x ∈ B, this part of the proof is complete; so suppose that x ∈ A. v If this value is nonzero, then either su or tv is nonzero. The existence of a smallest, nonnegative remainder n−dq is guaranteed by the Well-Ordering Property. Definition 6.16 Events E and F are independent if P(E ∩ F) = P(E)P(F). State the modus ponens rule of inference. If true, prove it; otherwise, give a counterexample. A recursive algorithm uses smaller instances of the current input to process the current input. Unfortunately, no one knows a method that is much more practical for arbitrary graphs. The proof appears in [Even, 1979]. Such a value for x is called a counterexample. Finally, we place the remaining three letters (3! ways). P = (p → q) ∧ (q → r), Q = p → r (3.6) (b) Show that (3.6) remains true if we change the third row of imp2's truth table to F T F. Use the If S(n) is true, then S(n + 1) is true form of the Inductive Step to prove the statement in Example 5.1. 4. Not reflexive, symmetric, not antisymmetric, and transitive 53. 570 The following approach often leads to more pruning than pure alpha-beta minimax. Furthermore, enclosing a phrase, typically with embedded spaces, in double quotation marks causes the phrase to be treated as a single word. ¬p 34. 269 Introduction to Number Theory 2 ➜ Representations of Integers and Integer Algorithms A bit is a binary digit, that is, a 0 or a 1. Professor Larry proposes the following version of binary search: binary search3(s, i, j, key) { while (i ≤ j) { k = (i + j)/2 if (key == sk) return k if (key < sk) j=k else i=k } return 0 } Is the Professor's version correct (i.e., does it find key if it is present and return 0 if it is not present)? Given x ∈ X, we may apply g to determine a unique element y = g(x) ∈ Y . Let a be a lower bound for a set X of real numbers. Example 1.15 Pseudorandom Numbers Computers are often used to simulate random behavior. (w + x + y + z)12 12. (A ∗ B + C ∗ D) − (A/B − (D + E)) 9. List the order in which the vertices are processed using inorder traversal. 4/36 20. 383 Recurrence Relations If we denote the sequence (1.1) as a1 , a2 , . A polyomino of order s consists of s squares joined at the edges. G1 = G1 of Exercise 39; G2 = G1 of Exercise 38 41. Show that it is impossible to mark k squares in the upper-left, k × k subgrid and k squares in the lower-right, k × k subgrid so that no two marked squares are in the same row, column, or diagonal of the 2k × 2k grid. 6 · 9 · 7 + 6 · 9 · 4 + 6 · 7 · 4 + 9 · 7 · 4 3. (d) Is b decreasing? Let X = Y = {1} and U = {1, 2}. Step 6: We take our million bucks and we fly to Rio de Janeiro. FROM SUPERNATURAL This chapter discusses proofs. Since an odd number of edges touch some vertices (c and d), there is no path from a to a that passes through each edge exactly one time. Thus, if the Mathematics Department does not get an additional $60,000, the dean's statement must be true, regardless of whether the department hires an additional faculty member or not. B R 4 3 G W 4 B 4 R 1 W G W G2 2 B 3 R G 4 G3 4 R R ■ Each vertex has degree 2. Derive a recurrence relation for S(k, n), the number of ways of choosing k items, allowing repetitions, from n available types. 3, 6, 9, 15, 24, 39, . What is wrong with the following "proof" that cn ≤ 2n for all n ≥ 3? , ei , vi , ej+1 , vj+1 , . What conditions must A satisfy for f to be one-to-one? (c) When means the same as if; thus an equivalent formulation is If you sing, then my ears hurt. † 28. Now let x ∈ A. It follows that τ(R1) ∪ D ⊆ τ(R1 ∪ D). However, in Section 3, we will present the Euclidean algorithm, which does provide an efficient way to compute the greatest common divisor. 2n + 3(n − 1)n/2 70. How many selections are there in which both Ben and Francisco are officers? If p and q are propositions, the biconditional proposition is the proposition p if and only if q. We have verified the Basis Step. Similarly, there are 303 multiples of 3 and 11 between 1 and 10,000, there are 181 multiples of 5 and 11 between 1 and 10,000, and there are 60 multiples of 3, 5, and 11 between 1 and 10,000. e c d Figure 7.3 The nonplanar graph K5 . To prove that R ∪ S is transitive, we must show that if (x, y), (y, z) ∈ R ∪ S, then (x, z) ∈ R ∪ S. Write a program to list all rise/fall permutations of length n (see the definition before Exercise 72, Section 1). Suppose that the first player can always win in nim . A B 9 Figure 1.5 A Venn diagram of A − B. 355 Counting Methods and the Pigeonhole Principle Therefore, P(J ∪ A) = P(J) + P(A) − P(J ∩ A) = 0.70 + 0.95 − 0.665 = 0.985. Therefore, km ≤ jm. A weighted graph is a graph with numbers assigned to the edges. Suppose that f is a one-to-one, onto function from X to Y . Therefore, there are 8! − 8·7·6 3! 2! strings formed by ordering the letters ILLINOIS in which some I appears before some L. Define rooted tree. An analogous definition applies to the worst-case and average-case times of an algorithm. Therefore (A ∪ B) ∪ C = A ∪ (B ∪ C). f(n) = (g(n)) if f(n) = O(g(n)) and f(n) = (g(n)). n factorial: n! = n(n − 1) · · · 1, 0! = 1 31. , sn (Algorithm 1.2), it was helpful to assume that we had already found the largest element large in the subsequence s1 , . A simple, planar, connected graph with e edges and v vertices satisfies e ≤ 3v − 6 (see Exercise 13, Section 7). Therefore, E(P) = 2 − 1 = 1. If the earth has six moons, then 1 < 3. We must prove that the number of subsets of {1, . The Basis Steps are verified. v1 v2 w1 v3 v4 v6 v5 v7 T1 w2 w3 34. To apply the Pigeonhole Principle, we must decide which objects will play the roles of the pigeons and which objects will play the roles of the pigeonholes. In the subtree whose root is ∗ in Figure 6.7, the multiplication operator operates on the subtree headed by +, which itself represents an expression, and C. If a function from X to Y is onto Y , each element in Y in its arrow diagram will have at least one arrow pointing to it. I take hotel management and either I do not take recreation supervision or I take popular culture. by the equations c1 = c2 = 0, cn = cn/3 + n for all n > 2. Certain problems are so hard that they have no algorithms at all. 1 21. In practice, a pseudorandom number generator is used. We would add to T edges (a, b), (a, c), and (a, g). ∃x∀y(x2 = 9) 49. According to Exercise 65, Section 1, the number of strings that contain AB or BE = number of strings that contain AB + number of strings that contain BE − number of strings that contain AB and BE. 494 38. , n that order 1 is order 2. p ∨ ¬q 37. If f(n) = (g(n)), then lg f(n) = (lg g(n)). Compute T(n) for n = 1, . If n ≥ 2, dn = 4an−1 1 [4n−1 + (−1)n] =4 5 by Exercise 29 by Exercise 30. p → r r→q p ∴p 19. Hint: First, use proof by contradiction to prove that if n ≥ 5, Cn is not prime. Since 2 ∈ Z+ , 2n ∈ Z+ = B. C = max{C , f(1)/g(1), f(2)/g(2), . Moreover, it is assumed that the flow into a vertex v, which is neither the source nor the sink, is equal to the flow out of v. Also, notice the dramatic improvement that results when we move from n2 steps to n lg n steps. G2 G1 G2 a 4 c 4 G1 8. Given a winning strategy for a particular player for nim, describe a winning strategy for this player for nim . sn,k = n! for all n ≥ 1. (Taking the square root certainly does not work!) When a proof technique seems unpromising, try a different one. Each such route can be designated by a string of i X's, j Y 's, and k Z's, where an X means move one unit in the x-direction, a Y means move one unit in the y-direction, and a Z means move one unit in the z-direction. Free tree Rooted tree Level of a vertex in a rooted tree Height of a rooted tree Hierarchical definition tree Huffman code Section 2 7. 1⎠ 0 We see again that G1 and G2 are isomorphic. i=2 95. Example 2.10 Let a and b be real numbers with a < b. This function is not one-to-one because a has two arrows pointing to it. i=1 31. Show that the sequence P0 , P1 , . 123. symbolically, letting p : a < b, q : b < c, r : a < c. Functions are used extensively in discrete mathematics; for example, functions are used to analyze the time needed to execute algorithms. We let m = 2 · 3 · 5 · 7 · 11 + 1 = 2311. n!/(n1 ! · · · nt !). No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without either the prior written permission of the publisher or a licence permitting restricted copying in the United Kingdom issued by the Copyright Licensing Agency Ltd, Saffron House, 6–10 Kirby Street, London EC1N 8TS. v = c, w = d 29. A function f from X to Y can be thought of in many ways. One possible interpretation of this quotation is: Every object that glitters is not gold. C(50, 4) 63. To prove this, we use proof by contradiction. We obtain R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (1, 5), (2, 5), (3, 5), (4, 5)}. To solve this recurrence relation, we first solve the quadratic equation t 2 − 3t + 2 = 0 to obtain roots 1 and 2. Since 1111111 is not allowed as a netid, there are 27 − 1 netids. n ≥ 0. Show all solutions to the four-queens problem. s is the inverse of n mod z if ns mod z = 1. Now β = ab ∈ L and the first rule states that aabb = aβb ∈ L. For example, the path (2.1) may also be written (1, 2, 3, 4, 2). These techniques can be used to derive the binomial theorem. When a grandchild w of v has a value of at least x, the subtree whose root is the parent of w is deleted. 52 26. Consider the statement ∀x∃y(x + y = 0). 19. Thus if person 1 gets order 3, the only possibility is 3, 1, 2. If (j, k) is in T , then Ti+1 is contained in the minimal spanning tree T . 462 The Knight's Tour In chess, the knight's move consists of moving two squares horizontally or vertically and then moving one square in the perpendicular direction. There are 18 of these. Tile the two 6×4 subboards using the result of Exercise 34, Section 4. Show how a binary tree with n edges can be encoded as a string of n + 1 ones and n + 1 zeros where, reading from left to right, the number of zeros never exceeds the number of ones. Introduction to Number Theory 7. √ 3 7. In statement (a), for each positive integer n, we obtain a proposition; therefore, statement (a) is a propositional function. For the graph of Figure 2.1, we have the following information. A simple graph G is self-complementary if G and G are isomorphic. First, suppose that p1 is true. The largest index m satisfying sm < sm+1 is 3. ■ Construct additional examples—especially small examples (e.g., for sets look at some small finite sets). Proof Let an denote the number of binary trees with n vertices. Example 2.1 Consider the sequence s 2, 4, 6, . Substitute the third-to-last formula for the remainder into E2 . Com," let N denote the event "the computer was purchased from Nuclear," and let B denote the event "the computer was defective." 46. Because this latter set is an intersection, we immediately deduce that x ∈ X and x ∈ Y − Z. In Definition 4.1, case (a) corresponds to possibility 1 of (4.2) and case (b) corresponds to possibility 2 of (4.2). Having made these assignments, we may assign positions to the n2 items of type 2 in C(n − n1 , n2) ways, and so on. We must show that T is a tree, that is, that T is a simple graph and that T has a unique simple path from any vertex to any other vertex. We next restate the Pigeonhole Principle in an alternative form. The remainder gives the number of 16's. Define onto function. The complement obviously depends on the universe in which we are working. f(x) = 1 + x2 29. Determine whether each proposition is true or false. for all k = 1, 2, . By Exercise 23, Section 3, there exists an edge y in T2 that is not in T1 such that T3 = (T1 − {x}) ∪ {y} and T4 = (T2 − {y}) ∪ {x} are spanning trees. In Exercises 22–26, A= s1 + s2 + · · · + sn n is the average of the real numbers s1 , . If we select the edge labeled 1 incident on R and W , we can then select the edge labeled 4 incident on W and G (see Figure 8.9). Figure 7.5 shows that if we begin by comparing two coins against two coins, the decision tree can account for at most six outcomes. Explain what it means for s to be the inverse of n modulo φ. Define d divides n. Let P (P) denote the set of labeled (unlabeled) vertices. e 11 f e3 Show that Dijkstra's algorithm for finding the minimum length of each path in the weighted graph G from a fixed vertex v to all other vertices and performing a breadth-first search in the unweighted graph G starting with vertex v are, in effect, the same process. Taking the logarithm of both sides, we must have n log 1.14 = log 2. We use the notation sn to denote the single element of the sequence s at index n. Let an be as in the proof of Theorem 3.10. Since there are 12 possible outcomes, there is no such algorithm. Algorithm 3.10 Computing s and t of Theorem 3.7 This algorithm computes s and t satisfying gcd(a, b) = sa + tb, where a and b are nonnegative integers not both zero, and returns gcd(a, b). If i ≤ n (so that the loop body executes again), pow becomes ai−1 pow ∗ a = ai−1 ∗ a = ai , and i becomes i+1. Not every x satisfies P(x). a ∨ b 2. 467 Graph Theory Inductive Step Assume that for all k < i, the kth time we arrive at line 9, L(v) is the length of a shortest path from a to v. or n. Recurrence relations are also called difference equations. Thus dec val becomes 5. Proofs 1 ➜ Mathematical Systems, Direct Proofs, and Counterexamples A mathematical system consists of axioms, definitions, and undefined terms. What does it mean for an edge to be incident on a vertex? Show that Algorithm 4.6 examines (n2) edges in the worst case. , xn : ordering of x1 , . x + 3 = x + 3 99. Note that the set of orders S that the n−1 persons (excluding p) possess includes all except q's (which p has). 234567 15. a 0 0⎥ b ⎜0 ⎜⎜ 0⎥ c ⎜0 ⎠ 0 d ⎝1 1 e 1 1 1 0 0 0 0 1 0 0 0 0 0 0 1 1 1 1 0 0 0 ⎞ 1 0⎥ ⎥1⎥ 0⎠ 0 26. {{1}, {2}, {3}, {4}} 18. The argument that [d] ⊆ [c] is the same as that just given, but with the roles of c and d interchanged. By the Pigeonhole Principle, some pigeonhole (suit) will contain at least two pigeons (cards), that is, at least two cards are of the same suit. Thus the total number of multiplications is (t − 1) + (n − t − 1) + 1 = n − 1. Is x nondecreasing? For the same reason, we cannot take the attribute Position or Age as a key. In words, for every real number x, there is a number that when added to x makes the sum zero. The intersection of S is the set of elements that belong to every set in S. An inventory consists of a list of 115 items, each marked "available" or "unavailable." There are 60 available items. 150 g(n) = 3n − 1. {(a, a), (b, b)} 5. = 68. Maximizing the flow in a network is a problem that belongs both to graph theory and to operations research. +1 In Example 5.11 we showed that ∃x P(x) is false by verifying that ∀x ¬P(x) (5.2) is true. Thus the number of paths of length 2 from c to c is 3, the degree of c. A counterexample is f(n) = 1 and g(n) = 1/n. If the items to be ordered are distinct, the permutation formula may be used. ∀x∃y¬L(x, y) 33. For each pair of propositions P and Q in Exercises 60–69, state whether or not P ≡ Q. (c) There are 16 equivalence classes. For example, position P of Figure 9.5 has NX = 2, since X might complete the column or the diagonal, and NO = 1, since O can complete only a column. If P is a probability function and all outcomes are equally likely, what is the value of P(x), where x is an outcome? Show that if f is a homomorphism of G1 to G2 and f is one-to-one and onto, G1 and G2 are isomorphic. {. Explain how to find a recurrence relation that describes the time a recursive algorithm requires. But (i, j) is in R if and only if entry i, j in A is nonzero. Suppose that p → q and q are true. The symbol ∀ means "for every." Thus the statement for every x, P(x) may be written ∀x P(x). By using an argument like one shown in Examples 3.7–3.9 the n or otherwise, prove that i=1 i lg i = (n2 lg n). We then enter the while loop (line 11). We regard the routes that first meet the diagonal at (k, k) as constructed by a two-step process: First, construct the part from (0, 0) to (k, k). At line 5, since key ('G') is not equal to s3 ('J'), we proceed to line 7. If i < n is true (so that the loop body executes again), i becomes i + 1 and fact becomes fact ∗ (i + 1) = i! ∗ (i + 1) = (i + 1)!. Example 2.12 Section 2 1. The good routes that first meet the diagonal at (k, k) are distinct from those that first meet the diagonal at (k , k), k = k . A collection of sets S is said to be pairwise disjoint if, whenever X and Y are distinct sets in S, X and Y are disjoint. The next example shows how we sometimes use the f(x) notation to define a function. If every even number greater than 2 is the sum of two prime numbers, lines 2 and 3 become an infinite loop—in this case, the algorithm will not terminate and, therefore, will not produce output. In our case, the third digit, 3, has this property. 66. Let x and y be vertices in G − {e}. We first compare MORE with the word WORD in the root. w1 v2 w4 w4 w3 v5 w5 T1 T2 24. Define increasing sequence. For example, each character might be represented as a number. Section 4 Section 6 41. Next, the algorithm swaps a2 and arand(2,n) . T1 3 3 7 6 8 1 3 7 3 2 589 590 Network Models c 2009 by From Discrete Mathematics, Seventh Edition, Richard Johnsonbaugh. By inspection, we find the initial conditions S1 = 2, Example 1.7 386 S2 = 4, S3 = 7. The graph is not connected. Therefore x ∈ / A. Proof by contradiction may be justified by noting that the propositions p→q and (p ∧ ¬q) → (r ∧ ¬r) are equivalent. Using Algorithm 1.8, we find that 2311 is prime. This follows from the fact (Theorem 2.21) that the sum of all of the degrees in a graph is an even number. Since r is a root of (2.14), r 2 = c1 r + c2 . (f ◦ f)(x) = 22x, (g ◦ g)(x) = x4 , (f ◦ g)(x) = 2x2 , (g ◦ f)(x) = 2x2 42. The domain of discourse of the two-variable propositional function P is R × R, which means that 51 Sets and Logic each variable x and y must belong to the set of real numbers. Show that g is the greatest common divisor of a and b. We see that sometimes an explicit formula can be derived from a recurrence relation and initial conditions. If you are healthy and run 10 laps daily, then you do not take multi-vitamins. Problem-Solving Tips To solve a recurrence relation in which an is defined in terms of its immediate predecessor an−1 an = . fore pow = ai−1 is an invariant for the while loop. To meet the initial conditions (2.15), we must have U0 = b + d = C0 , U1 = br1 + dr2 = C1 . [Problem 1.] Our previous example, X = {1, 2, 3, 4}, will suffice. Choose the smallest positive integer n with 1/n ≤ p/q. We must prove that any 2n+1 × 2n+1 deficient board can be tiled with trominoes. Since n mod 2 is equal to 1, result becomes result ∗ x = a6 ∗ a8 = a14 . Suppose that S contains no 7's. If the index of the first term is 1, then b1 = d, b2 = b4 = i, and b7 = l. The given proposition can be written symbolically as p → q. (a) Use the formulas sn = sn−1 + n for all n ≥ 2, to write a recursive algorithm that computes sn = 2 + 4 + 6 + · · · + 2n. Now R◦S = {(1, 2), (2, 3)} is not transitive; (1, 2), (2, 3) ∈ R ◦ S, but (1, 3) ∈ R ◦ S. while (i ≤ m − 1) { 10. By replacing n by n − 1 in (1.14), write a formula for Sn−1 . ∀x(P(x) → Q(x)) 47. The algorithm does not receive input (but, logically, it needs none). The best an opponent can hope for is a draw. It is not clear how to get from n2 = 2k1 to n = 2k2 . Show that the following Instant Insanity puzzle has no solution by giving an argument to show that no subgraph satisfies properties (8.1) and (8.2). ⎛ 1 2 v1 0 1 v2 ⎜ 1 0 ⎜ v3 ⎜ 1 1 ⎜ v4 ⎜ 0 0 v5 ⎝ 0 0 v6 0 0 x x 7. , 8 may be verified directly. Describe best-case input for Algorithm 2.1. 20. No matter what the second player does, the first player can always move to a vertex having value 1. The set of all finite subsets of {1, 2, 3, . An undefined term is a term that is not explicitly defined but rather is implicitly defined by the axioms. Then bn is equal to the sum of the times required by the various steps. Find the equivalence relation (as a set of ordered pairs) on {a, b, c, d, e} whose equivalence classes are {a}, {b, d, e}, {c}. What is the probability that the sum of the numbers on the dice is odd? For example if α = abb, αR = bba. The two-dimensional mesh model for parallel computation when described as a graph consists of a rectangular array of vertices connected as shown (see Figure 6.2). Move the largest disk to the remaining empty peg. Start with a flow (e.g., the flow in which the flow in each edge is 0). Write its negation symbolically and in words. If j ≥ 1, we will execute x = x + 1 at most n/2 additional times in the next iteration of the while loop, and so on. 87 Proofs m n 1 2 3 1 2 3 4 5 14 29 11 20 35 21 30 45 35 44 59 40 for m = 1, 2, 3, 4 and n = 1, 2, 3, and 2m2 + 3n2 > 40 for Since 2m2 + 3n2 = m > 4 or n > 3, we conclude that 2m2 + 3n2 = 40 has no solution in positive integers. Balanced 25. Example 5.1 Use mathematical induction to show that postage of four cents or more can be achieved by using only 2-cent and 5-cent stamps. Thus s = s mod 425 = 206 mod 425 = 206. Exercises In Exercises 1–8, trace Algorithm 1.8 for the given input. T2 has a simple path of length 2 from a vertex of degree 1 to a vertex of degree 1, but T1 does not. Headline in the New York Times: A World Where All Is Not Sweetness and Light. Hint: Consider the set of integer upper bounds for X. Assume that the result holds for a binary tree whose height is less than h. 1 3 5 3 6 2 2 4 2 5 4 1 3 6 6 G Figure 4.1 Six cities 1–6 and the costs of building roads between certain pairs of them. The set X is called the domain of f and the set Y is called the codomain of f . An eight-bit encoding can be constructed in eight successive steps: Select the first bit; select the second bit; . For example, 2 and 3 are incomparable (since 2 does not divide 3 and 3 does not divide 2), but 3 and 6 are comparable (since 3 divides 6). A recurrence relation that defines a sequence can be directly converted to an algorithm to compute the sequence. The best-case time for input of size n of an algorithm is the minimum time needed to execute the algorithm among all inputs of size n. Corollary 3.7 The n-cube has a Hamiltonian cycle for every positive integer n ≥ 2. P(n, r): number of r-permutations of an n-element set; P(n, r) = n(n − 1) · · · (n − r + 1) 8. Thus x ∈ B. Let an denote the number of comparisons required by merge sort in the worst case. 470 Graph Theory Example 5.1 a Adjacency Matrix Consider the graph of Figure 5.1. To obtain the adjacency matrix of this graph, we first select an ordering of the vertices, say a, b, c, d, e. We need counterexamples for parts (a) and (b). For example, the function √ f(x) = sin 2x can be decomposed into the functions √ g(x) = x, h(x) = sin x, w(x) = 2x. S3 = 1/2, S4 = 3/4 58. If we list the terms of the sequence, we obtain 5, 8, 11, 14, 17, ... Suppose that (x, y), (y, z) ∈ R ◦ S. As an example of the general approach, consider a version of the game of nim. By the Pigeonhole Principle, some name (pigeonhole) is assigned to at least two persons (pigeons). The next theorem gives the details. We select the first vertex a and call it the root (see Figure 3.2). Define acyclic graph. Since R is transitive, (b, b) ∈ R, which is a contradiction. If a position is shown in a square, it is the 2 first player's move; if a position is shown in a circle, it is the second player's move. By the inductive assumption, we can make postage of n − 2 cents. ABAB∗+∗D∗ 21. ∃x∀y¬P(x, y) 9. Explain how the max flow, min cut theorem proves that the algorithm of Section 2 correctly finds a maximal flow in a network. By using mathematical induction or otherwise, show that gn = 2fn+1 − 1, n ≥ 1, (a) Find the values of L3 , L4 , and L5 . (p ∨ q) ∧ (¬p ∨ q) ∧ (p ∨ ¬q) ∧ (¬p ∨ ¬q) 29. What is an nth-order, linear homogeneous recurrence relation with constant coefficients? The only Fibonacci numbers in the sum (∗) that might be consecutive are fk1 and fk2 . Twelve coins are identical in appearance, but one coin is either heavier or lighter than the others, which all weigh the same. , n + p − 1 using only p-cent and q-cent stamps. The array row is replaced by the array p, which is the permutation. We argue by contradiction to show that no algorithm that solves the five-coins puzzle has worst-case time less than 3. We write 1 and carry 1. How do we prove that a set, X ∩ Y in this case, is the empty set? Consider the proof using mathematical induction that, given an n × n deficient board (a board with one square removed), where n is a power of 2, we can tile the board with right trominoes (three squares that form an "L"). (A, B) ∈ R if for every ε > 0, there exists a ∈ A and b ∈ B with |a − b| < ε. Since 60n2 + 5n + 1 = O(n2) and 60n2 + 5n + 1 = (n2), 60n2 + 5n + 1 = (n2). This formula follows from the formula of Exercise 7 because P(E1 ∩ E2) = 0. 94. We can formalize the repeated squaring technique if, in addition to repeated squaring, we simultaneously determine the binary expansion of the exponent. Since the sizes of the left and right sides of the original sequence are (n − 1)/2 and n/2 and the worst case occurs with the larger sequence, the total number of invocations at line 11 will be an/2 . Proof If suffices to show that if p is a prime, there is a prime larger than p. Before proving the theorem, we first illustrate the proof with a specific example. The first zero identifies this as an ordinary grocery item, the next five digits 54400 identify the manufacturer Nabisco Foods, and the next five digits 00800 identify the product as a package of 10 Ortega taco shells. P ≡ Q 66. (sr 's maximum value is n, sr−1 's maximum value is n − 1, etc.) Then set ti = si for i = 1, . Thus x ∈ A−B; hence x ∈ (A−B)∪(B−A). p → q q→r ∴p→r Sets and Logic 13. What is the probability that the student answers exactly one question correctly? Express the relation given by Table 6.7 as a set of n-tuples. An or statement often lends itself to a proof by cases. At this point s1 = 2, s2 = 3, s3 = 5, s4 = 6, s5 = 7. A vertex that is not incident on any edge is called an isolated vertex. It follows that Un = an , n = 0, 1, . temp1 := player [Position = p] temp2 := temp1 [ID Number = PID] assignment temp2 [Team] Mutts, Jackalopes 24. For example, if x = 2, we obtain the true proposition 2 2 = . In Section 4, we discuss the general problem of finding shortest 438 Graph Theory Ellen Barkin Kevin Bacon Dennis Quaid Charles Siebert Robert Wagner Emmett Vogan Bela Lugosi ... Exercises In Exercises 1–6, write the adjacency matrix of each graph. Cartesian product of X and Y : X ×Y = {(x, y) | x ∈ X, y ∈ Y} 27. The radius is the eccentricity of a center. Holton and J. Now 1 ,b1) min(a2 ,b2) pmin(a p2 · · · pkmin(ak ,bk) 1 divides both m and n and if any exponent, min(ai , bi), is increased, the resulting integer will fail to divide m or n (or both). Definition 1.1 A (free) tree T is a simple graph satisfying the following: If v and w are vertices in T, there is a unique simple path from v to w. Hint: Assume that g does not divide a. {2, 3, 5} 10. This result was first given by the English mathematician Philip Hall and is known as Hall's Marriage Theorem, since if V is a set of men and W is a set of women and edges exist from v ∈ V to w ∈ W if v and w are compatible, the theorem gives a condition under which each man can marry a compatible woman. Basis Steps (n = 7; 11) Exercise 35, Section 4, gives a solution if n = 7. Then Z− , the complement of the set of negative integers, is Znonneg , the set of nonnegative integers. We illustrate by giving two proofs of the statement X ∪ (Y − X) = X ∪ Y for all sets X and Y . Checked vertices are roots of subtrees that are pruned. In insertion sort (Algorithm 2.3), it was helpful to assume that the subsequence s1 , . Show that if M is a p1 × p2 × · · · × pk mesh, where pi ≤ 2ti for i = 1, . If for some di , P(di , dj) is false for every j, then, for this i, P(di , dj) is false for every j. Example 2.3 Population Growth Assume that the deer population of Rustic County is 1000 at time n = 0 and that the increase from time n − 1 to time n is 10 percent of the size at time n − 1. Example 4.15 } } while (sm > sk) // find the rightmost element sk with sm < sk k =k−1 swap(sm , sk) p=m+1 q=n while (p < q) { // swap sm+1 and sn , swap sm+2 and sn−1 , and so on swap(sp , sq) p=p+1 q=q−1 } println(s1 , . Exercise 21 29. Give an intuitive interpretation of how f and g are related if f(n) = O(g(n)). It is denoted αβ. The proof is a combinatorial argument. , n, n + 1} i2 − 2 129. Let T be a binary tree. Now x = k < x + 1. Tips specific to the present section follow. n+1 4 Therefore the average-case time of Algorithm 3.17 is (n). (c) Describe the relation R in familiar terms. b 3, 2 a c 2, 2 (a, 1) (d, 2) 4, 2 (−1,) 5, 0 2, 0 (a, 5) d (c, 2) (d, 2) 2, 0 z 4, 0 e Figure 2.8 After increasing the flow on path (a, b, c, z) by 2 and the second labeling. Edge (vi , vi+1) is improperly oriented because it is not in the a-to-z direction. 516 Trees P v Figure 2.4 The proof of Theorem 2.3 [if (b), then (c)]. Let P be the portion of the path P starting at v whose edges are also in G . Thus in the cycle C, each vertex corresponding to an outside square must be preceded and

followed by a vertex corresponding to an inside square. If e = 31 and v = 12, the inequality is not satisfied, so such a graph cannot be planar. 407 Recurrence Relations Problem-Solving Corner Recurrence Relations Problem (a) Several persons order compact discs. TABLE 3.1 ■ Time to execute an algorithm if one step takes 1 microsecond to execute. 21–22]). There are many answers to this question. Proofs 9. for i = 1 to n for j = 1 to n x=x+1 247 Algorithms 12. As before, np +is ≥ 0 holds because np ≥ 0 and s > 0. Therefore, τ(R1) ⊆ τ(R1 ∪ D). Therefore, working from the right, we must find the first digit d whose right neighbor r satisfies d < r. 1 = 1. A relation is symmetric if and only if its matrix is symmetric (i.e., entry i, j always equals entry j, i). (Such an equation is said to be inhomogeneous. √ 7. Find a shortest path from a to z. Prove that if a planar graph has an Euler cycle, it has an Euler cycle with no crossings. Example 2.5 If x is the sequence defined by xn = 1 , 2n −1 ≤ n ≤ 4, the elements of x are 2, Example 2.6 1, 1/2, 1/4, 1/8, 1/16. p → (q → r) 19. We then compare A and A2 . (∗) i=1 We next show that fk1 > fk2 , so that, in particular, n is the sum of distinct Fibonacci numbers. Prove that X ∩ Y ⊆ X for all sets X and Y . a b c d e ⎛ 1 ⎜0 ⎜ ⎜1 ⎝0 0 0 1 0 1 0 0 1 0 0 1 0 0 1 1 0 ⎞ 0 1 0 0 1 ⎟ 1 25. q : The bug is in module 81. (We define a partition of a set X to be a collection S of nonempty subsets of X such that every element in X belongs to exactly one member of S.) A partition can be used to define a relation. If G is a connected graph and T is a spanning tree for G, there is an ordering of the vertices of G such that Algorithm 3.7 produces T as a spanning tree. Find all department numbers of departments that supply parts to Danny's. The numbers S0 , S1 , . To disprove the universally quantified statement ∀xP(x), we need to find one member x in the domain of discourse that makes P(x) false. Prove that the following are equivalent for sets A, B, and C: (a) A ∪ B = U (b) A ∩ B = ∅ (c) A ⊆ B, where U is a universal set. If we define the worst-case time to solve a coin-weighing problem to be the number of weighings required in the worst case, it is easy to determine the worst-case time from the decision tree; the worst case time is equal to the height of the tree. (f) Prove that {sn } satisfies sn = 5sn−1 − 6sn−2 for all n ≥ 2. Determine whether the statement p→q∨r p ∨ ¬q r∨q ∴q 16. Therefore b/t is an upper bound for X. Therefore, there exists i such that si ≤ A. X ∩ Y ⊆ X for all sets X and Y . Since 3d is odd, we have a contradiction. 21000 7. In words, a relation is symmetric if whenever x is related to y, then y is related to x. Project Whereas the selection operator chooses rows of a relation, the projection operator chooses columns. 6. Finally, by the definition of intersection, since x ∈ X and x ∈ Y − Z, x ∈ X ∩ (Y − Z). large = b 5. The last string will be 34567. We obtain an n-cube (Exercise 39). We may now cancel 2a from both sides of 2a 3b = 2c 3d to obtain 3b = 3d . Give an example of consecutive primes p1 = 2, p2 , . P(11) 28. Isomorphic rooted trees 45. q ↔ (p ∧ r) 9. We have obtained one of the graphs. When n is not a power of 2, n lies between two powers of 2, say 2k−1 and 2k , and an lies between a2k−1 and a2k . X1 × X2 × · · · × Xn = {(a1 , a2 , . C(10, 4)C(12, 3)C(4, 2) 42. State the Addition Principle and give an example of its use. Together with the first iteration, we conclude that if n satisfies 2m < n ≤ 2m+1 , the while loop iterates at most m times. Show that the number of simple paths from v to w is A vertex v in a connected graph G is an articulation point if the removal of v and all edges incident on v disconnects G. Example 5.5 Consider the universally quantified statement ∀x(x2 ≥ 0). A serial algorithm executes one instruction at a time. .}, we write S= ∞ Ai , S= i=1 Example 1.22 ∞ Ai . (named after Edouard Lucas, the inventor of the Tower of Hanoi puzzle) is defined by the recurrence relation Ln = Ln−1 + Ln−2 , n ≥ 3, n ≥ 3, and the initial conditions and initial conditions g1 = 1, L1 = 1, g2 = 3. Audrey Meadows was the original "Alice" in "The Honeymooners." p : You play football. A proof of (2.4) that does not exhibit an element a of the domain of discourse that makes P(a) true, but rather proves (2.4) some other way (e.g., using proof by contradiction), is called a nonconstructive proof. How many relations are there on an n-element set? Find i=1 3 7 9. Problem-Solving Tips The Euclidean algorithm for computing the greatest common divisor of nonnegative integers a and b, not both zero, is based on the equation gcd(a, b) = gcd(b, r), where r = a mod b. 1 44. Given a 2n+1 × 2n+1 deficient board, we divide the board into four 2n × 2n subboards as shown in Figure 4.6. By the inductive assumption, we can tile the subboard containing the missing square. Note that A = B ∩ C = {2, 4}. Theorem 2.3 Let T be a graph with n vertices. Temporarily assign ownership of p's order to q. We define the worst-case time required by binary search to be the number of times the algorithm is invoked in the worst case for a sequence containing n items. [Problem 3.] Let ε > 0. Let X and Y be nonempty sets of real numbers such that X ⊆ Y and Y is bounded above. A function f from X = {1, . The matrix of the relation R2 ◦ R1 with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product A1 A2 by 1. , the ijth entry in the matrix An is equal to the number of paths from i to j of length n. Strong Form of Mathematical Induction 34. State the Well-Ordering Property. (A look ahead at the next Problem-Solving Corner may help in understanding the following exercises.) 90. Sudoku is a puzzle in which the goal is to fill in a 9 × 9 grid so that each of the numbers 1 through 9 appears in each column, each row, and each 3 × 3 box delineated by the heavy lines: 5 1 6 2 42. Show that 4 n−1 [4 + (−1)n]. Explain how to solve a recurrence relation by iteration. When these forwarded copies were received and opened, the macro again forwarded the e-mail message and the attached word processor document, and so on. |P(X)| = 2|X| 12. (p → q) ∧ (q → r) 17. To prove that a relation R on a set X is not transitive, find x, y, z ∈ X such that (x, y) and (y, z) are in R, but (x, z) is not in R. Now tj is either 0 or 1. There are Sn−3 such (n − 3)-bit strings. You will be healthy if and only if you run 10 laps daily and take multi-vitamins. Since R is reflexive, symmetric, and transitive, R is an equivalence relation on {1, 2, 3, 4, 5}. Define a relation R on the set V of vertices of G as vRw if there is a path from v to w. ■ All newborn children at St. Rocco's Hospital. Thus the worst-case time of Algorithm 8.13 is (n). , n with n in the fourth position is C(n − 1, 3)E3 En−4 . We use strong induction to prove the result. For example, there are 11 ways to spend $4: MB, BM, OOM, OOB, OMO, OBO, MOO, BOO, OOOO, MM, BB. Contrapositive: If Leah does not study hard, then Leah does not get an A in discrete mathematics. Given a description of a set X such as (1.1) or (1.2) and an element x, we can determine whether or not x belongs to X. Now b2k = 2b2k−1 + 2 = 2[2b2k−2 + 2] + 2 = 22 b2k−2 + 22 + 2 = · · · = 2k−1 b21 + 2k−1 + 2k−2 + · · · + 2 = 2k−1 + 2k−1 + · · · + 2 = 2k−1 + 2k − 2 n 3n = n−2+ = − 2. Since α = βγ, α also has an equal number of = |x − xi | + (xn − x1). The hostid is given by the remaining 16 bits, which identifies the computer interface. is the base of the natural logarithm.) 68. Is b nondecreasing? Draw the subtree rooted at e. ¬r 128 1 + · · · + (2n − 1) + (2n + 1) = n2 + 2n + 1 = (n + 1)2 4. An edge exists from v ∈ V to w ∈ W if v and w are compatible. Section 4 Review 1. Write the negation of each proposition in Exercises 19–22 in words and symbolically. Thus, "equivalence relation" and "partition of a set" are different views of the same situation. It is denoted ∪ S. 412 binary search(s, i, j, key) { if (i > j) // not found return 0 Recurrence Relations 4. The same argument applied to + 2(n − 1)] + 2n = [Sn−2 + 2n + 2(n − 1) = Sn−2 + 2n + 2(n − 1) + 2(n − 2) = Sn−3 .. Write each proposition in Exercises 33–36 symbolically. If R and S are antisymmetric, then R ∪ S is antisymmetric. 311 Counting Methods and the Pigeonhole Principle Inclusion-Exclusion Principle U X Y X⋡Y Y⋡X X✕Y Figure 1.3 |X| counts the number of elements in X − Y and X ∩ Y, and |Y | counts the number of elements in Y − X and X ∩ Y . i∈P j∈P Example 3.5 The capacity of the cut of Figure 3.1 is Cbc + Cde = 8. Exercises How many bits are needed to represent each integer in Exercises 1–7? The output is the previous (in lexicographic order) r-combination. A route can first meet the top edge at any one of n + 1 positions. P(F) This discussion motivates the following definition. The check character is used to validate an ISBN. Subsets of the universal set are drawn as circles. We prove the equivalent contrapositive If X ∩ Y = ∅, then CX∪Y (x) = CX (x) + CY (x) for at least one x ∈ U. We can choose positions for N, O, and S in 8 · 7 · 6 ways. (All paths in this section are with reference to the underlying undirected graph.) If an edge e in P is directed from vi−1 to vi , we say that e is properly oriented (with respect to P); otherwise, we say that e is improperly oriented (with respect to P) (see Figure 2.1). We prove only A ∪ U = U. i∈P j∈P 18. Therefore, Gn is a Gray code. Now let's turn to part (c). Now suppose that x ∈ B. for i = 1 to m for j = 1 to n println(i, j) 15. Use the code developed in Exercise 25 to encode the following words (which have frequencies consistent with the table of Exercise 25): bus, cups, mush, puss, sip, cuss, hip, pup, pups, hips. Such relations are called partial orders. Prove that CX∩Y (x) = CX (x)CY (x) for all x ∈ U. p q r ¬(p ∧ q) ∨ (p ∨ ¬r) T T T T F F F T T F F T T T F F T F T F T F T F F T T T T T T T T 15. For some positive integer n, n and n + 2 are prime. Once the positions for the P's have been selected, there are C(9, 4) ways to choose positions for the four S's. How many outcomes are possible? 210 Algorithms Section Review Exercises † 1. Thus the conclusion p follows from the hypotheses and the argument is valid. If we take the natural logarithm (logarithm to the base e) of both sides of this equation, we obtain ln 2 = n ln(1 + i). Prove that the function f from P(U) to the set of characteristic functions in U defined by f(X) = CX A binary operator f on a set X is commutative if f(x, y) = f(y, x) for all x, y ∈ X. f1 = 1, f2 = 1, fn = fn−1 + fn−2 for n ≥ 3 Section 4 Taking logs to the base 2, we obtain 2 = lg 4 ≥ lg 6. When d = 7, 539 mod d equals zero, so the algorithm returns d = 7 to indicate that 539 is composite and 7 is a divisor of 539. We will solve this recurrence relation in Section 2. For example, in base 2, 101101 = 1 · 25 + 0 · 24 + 1 · 23 + 1 · 22 + 0 · 21 + 1 · 20 (see Figure 2.2). p ∨ q 45. If we let We have proved that CX∪Y (x) = CX (x) + CY (x) for at least one x ∈ U. Thus 4. The program is readable only if it is well structured. Answer 35–38 for the sequence Answer 67–73 for the sequence x defined by ϒ2 = 5. Find the probability of obtaining no defective microprocessors. Given a flow F in a network, the flow out of the source a equals the flow into the sink z; that is, Fai = Fiz . Prove that for every x, if x is in X, then x is in Y , and if x is in Y , then x is in X. If an edge is added to T , the resulting graph T is connected. The complement of a simple graph G is the simple graph G with the same vertices as G. Give an example different from Example 1.7 of a hierarchical definition tree. ∃x∃y P(x, y) 74. w7 w6 w4 T1 w4 w3 w5 v 10 v9 v5 w1 w2 v4 w3 w2 v4 T2 v3 v3 v2 w 12 v1 w1 w4 w5 T1 v2 v1 w6 v 12 6. In general, the symbol in position n (with the rightmost symbol being in position 0) represents the number of 8n 's. For example, wine might be placed into the classes premium, table wine, and swill based on features such as acidity and bouquet. Input: s1 , s2 , . 603 Network Models b 3, 2 a (a, 1) 2, 2 5, 2 (a, 3) (e, 2) (d, 2) 2, 0 5, 4 4, 0 z 2, 2 (a, 1) d e Figure 2.9 After increasing the flow on path (a, d, c, z) by 2 and the third labeling. The Euclidean algorithm is an old, famous, and efficient algorithm for finding the greatest common divisor of two integers. By definition, since m is odd, there exists an integer k1 such that m = 2k1 + 1. Sentence (b) is false. Let b be an upper bound for tX. 127 2. If the items are to be selected without repetition, the combination formula may be used. Using the definition of f , this latter equation translates as 2n1 + 1 = 2n2 + 1. Use Exercise 22 to find the matrix of the relation R1 ∩ R2 relative to the ordering 1, 2, 3 for the relations of Exercise 23. , xn when n is odd, and any value between the two middle values of x1 , . Example 1.7 Similarity Graphs This example deals with the problem of grouping "like" objects into classes based on properties of the objects. R = {(a, 6), (b, 2), (a, 1), (c, 1)} 6. (b) βα (c) αα (d) ββ (e) |αβ| (f) |βα| (g) |αα| (h) |ββ| (i) αλ (j) λβ (k) αβγ (l) ββγα 128. If a3 is greater than the second item, at most one additional comparison is needed to locate the correct position for a3 . The Inductive Step will assume the truth of the statement involving cn/3 . The justification for this technique is given in our next theorem. If the first element is a one, it is swapped with the last element. √ Proof We use proof by contradiction and assume that 2 is rational. Approximately 15 percent of the patients at one clinic have the HIV virus. 20, 40 6. We have our contradiction: n2 is even (hypothesis) and n2 is odd. P(1) 9. Problem-Solving Tips To prove that ∀x∀y P(x, y) is true, where the domain of discourse is X × Y , you must show that P(x, y) is true for all values of x ∈ X and y ∈ Y . U 60. , bk−1 is the sequence {aj } defined by aj = ai+j bi . Prove that gn = g1 fn−2 + g2 fn−1 24. The sum on the left is called the geometric sum. How many round-trips are there of the form Oz–Mid Earth– Fantasy Island–Mid Earth–Oz? 11 P(F) 11 36 Example 6.15 354 Weather records show that the probability of high barometric pressure is 0.80, and the probability of rain and high barometric pressure is 0.10. A list of votes where Wright is never behind Upshaw and each receives r votes is a string of r W's and r U's where, reading the string from left to right, the number of W's is always greater than or equal to the number of U's. , i=0 2. Since {an } satisfies the same recurrence relation as {fn }, a1 = f2 , and a2 = f3 , an = fn+1 , n ≥ 1. There are n(n − 1) · · · (n − r + 1) r-permutations of an n-element set. Discrete mathematics is concerned with objects such as graphs (sets of vertices and edges) and Boolean algebras (sets with certain operations defined on them). What is a relational database? v4 v7 v2 v5 v3 v4 v2 v1 v6 v7 v6 v5 v3 Figure 1.2 The tournament of Figure 1.1 as a tree. We begin with some of the terminology. Corresponding to the four possible suits, there are four ways for each pattern to occur. Thus there are 2n−1 + 2n−1 = 2n subsets of {1, . In this problem, there's not much choice; we have persons and dissension. Define length of path in a weighted graph. What is a recursive algorithm? Give an example to show that if the input to Algorithm 3.2 is not in nondecreasing order, Algorithm 3.2 may not find key even if it is present. q : Lee takes mathematics. a 18. Since X ∩ Y = ∅, either x ∈ X or x ∈ Y but not both. 502 are not isomorphic; the first graph has two vertices, but the second graph has three vertices. 0, 17 13. f is not one-to-one. The vertical bar "|" is read "such that." Equation (1.2) would be read "B equals the set of all x such that x is a positive, even integer." Here the property necessary for membership is "is a positive, even integer." Note that the property appears after the vertical bar. What is an evaluation function? We count the number of n × n matrices that represent symmetric relations on an n-element set. Write f 9 and f 623 as sets of ordered pairs. Prefix form of an expression (Polish notation) Infix form of an expression Postfix form of an expression (reverse Polish notation) Tree representation of an expression Section 7 40. , w3 , v, w4 , . , vn = z, and n Output: A maximal flow F 1. Each subset of class 2 consists of a (k − 1)-element subset of X together with a and there are C(n, k − 1) of these. By Theorem 3.9, the flow at the termination of Algorithm 2.4 is maximal and the cut (P, P) is minimal. Since the given statement is true, its negation is false. Therefore, if E is the set of edges of the form (v, w) having flow 1, the members of E have no vertices in common; thus E is a matching for G. We first discuss an algorithm to compute a power an (without dealing with mod z). The symbol ∃ may be read "there exists," "for some," or "for at least one." Example 5.13 Consider the existentially quantified statement for some n, if n is prime, then n + 1, n + 2, n + 3, and n + 4 are not prime. Two fair dice are rolled. A ∪ A = U, A ∩ A = ∅ 44. Let R1 and R2 be the relations on {1, 2, 3, 4} given by R1 = {(1, 1), (1, 2), (3, 4), (4, 2)} R2 = {(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)}. By the second form of the Pigeonhole Principle, ai = aj , for some i = j, as desired. This is exactly what occurs in the third line of the truth table of Definition 3.3. If x = 0, then P(0) and Q(0) are both false (0 > 0 and 02 > 0 are both false). Example 5.5 A single-elimination tournament is a tournament in which a contestant is eliminated after one loss. The teams are the vertices. Exercises 74–76 deal with Class B and C addresses. gcd(a, b) { // make a largest if (a < b) swap(a, b) while (b ¬= 0) { r = a mod b a=b b=r } return a } We note that the while loop in the Euclidean algorithm (lines 5–9) always terminates since at the bottom of the loop (lines 7 and 8), the values of a and b are updated to smaller values. Section 4 1. How many symmetric relations are there on an n-element set? Then v is incident on three edges, as shown in Figure 8.4. The fourth edge cannot be incident on v since then v would have degree 4. Give an algorithm for constructing a full binary tree with n > 1 terminal vertices. , ik } of {1, 2, . This can be done in C(5 + 3 − 1, 3 − 1) = C(7, 2) = 21 ways. 4 ➔ Equivalence Relations Suppose that we have a set X of 10 balls, each of which is either red, blue, or green (see Figure 4.1). s = 91 24 = 84 − 60 · 1 15. a2k = 3a2k−1 + 2k , An arbitrary value of n falls between two powers of m, say so mk−1 < n ≤ mk . 2 Thus the solution is of the form √ n √ n 1+ 5 1− 5 fn = b +d . , 2n, . Define a relation R on X × X by (a, b) R (c, d) if ad = bc. Show that any algorithm that finds the largest value among n items requires at least n − 1 comparisons. 3 ➔ Spanning Trees In this section we consider the problem of finding a subgraph T of a graph G such that T is a tree containing all of the vertices of G. Show that when b < k in Example 1.9, the price tends to that given by the intersection of the supply and demand curves. Character ASCII Code A B C 1 2 ! * 100 0001 100 0010 100 0011 011 0001 011 0010 010 0001 010 1010 Root 1 0 0 1 A O 1 0 R 1 T 0 S Figure 1.10 A Huffman code. (7.2) lg n! = (n lg n); thus, for some positive constant C, Cn lg n ≤ lg n! (7.3) for all but finitely many integers n. At this point the computation is + 1 10011011 1011011 0 Next, we add 1 and 1 and 1, which is 112 . The preceding arguments may be generalized using mathematical induction to prove that Algorithm 4.2 correctly returns the value of n! for any nonnegative integer n. Since i is labeled, we must have Fji = 0; otherwise, we would have labeled j at lines 25 and 26. This statement is false. 60, 90 15. Functions involving the modulus operator play an important role in mathematics and computer science. an = 2an−1 + 8an−2 ; ixi−1 = i=1 3. Exercises 22–29 refer to a family with four children. Section 7 In Exercises 25 and 26, determine whether the graph is planar. If f and g are one-to-one and onto, then f ◦ g is one-to-one and onto. Suppose that the greedy algorithm is optimal for all denominations less than am−1 + am . We next swap ai and aj , where i = 2 and j = rand(2, 5) = 3. (x + y + z)10 1 · 2 + 2 · 3 + · · · + (n − 1)n. The Basis Steps (n = 1, 2) are readily verified. An event is an outcome or combination of outcomes from an experiment. , and n − r + 1 ways to choose the rth element. Write the negation of each proposition in Exercises 1–4 in words and symbolically. The Basis Step is proved. Exercises In Exercises 1– 4, suppose that a coin is flipped and a die is rolled. If the item is found at the dividing point (line 5), the algorithm terminates. Give an example of an experiment different from those in this section. Disjoint sets X and Y : X ∩ Y = ∅ 17. Then there exist k1 and k2 such that m = 2k1 + 1 and n = 2k2 + 1. The idea is that as the price increases, the consumers demand less of the product. ¬r → ¬p r 10. Theorem 3.10 Merge sort (Algorithm 3.8) is (n lg n) in the worst case. Thus the number of selections in which Alice is an officer is 3 · 5 · 4 = 60, that is, |X| = 60. For example, 3854 = 3 · 103 + 8 · 102 + 5 · 101 + 4 · 100 (see Figure 2.1). Which requirements of an algorithm as stated in Section 1 does a randomized algorithm violate? Golomb in 1954 (see [Golomb, 1954]), they have been a favorite topic in recreational mathematics. The formula computes the next pseudorandom number using its immediate predecessor. Therefore, the total number of comparisons is bn = n − 1 + bn−1 , which yields the desired recurrence relation. The values in the denominator give the numbers of duplicates of each letter. Many problems can be solved much faster using parallel computers rather than serial computers. (By assumption, such an edge exists.) Continue to follow an edge out of vi to a vertex vi+1 . w2 x3 y2 z5 ; (2w + x + 3y + z)12 C(n, k) = C(n, n − k). The publisher code 59448 identifies the book as 138 Functions, Sequences, and Relations one published by Riverhead Books, Penguin Group. Example 2.9 10. (H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6) 4. i=0 73. Love Actually is good, but it is too long. (a) Associative laws: (A ∪ B) ∪ C = A ∪ (B ∪ C), (A ∩ B) ∩ C = A ∩ (B ∩ C) (b) Commutative laws: A ∪ B = B ∪ A, A ∩ B = B ∩ A (c) Distributive laws: A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C), A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) (d) Identity laws: A ∪ ∅ = A, A ∩ U = A (e) Complement laws: A ∪ A = U, A ∩ A = ∅ (f) Idempotent laws: A ∪ A = A, A ∩ A = A (g) Bound laws: A ∪ U = U, A ∩ ∅ = ∅ (h) Absorption laws: A ∪ (A ∩ B) = A, A ∩ (A ∪ B) = A (i) Involution law: A=A 8 Sets and Logic (j) 0/1 laws: ∅ = U, U = ∅ (k) De Morgan's laws for sets: (A ∪ B) = A ∩ B, (A ∩ B) = A ∪ B We define the union of an arbitrary family S of sets to be those elements x belonging to at least one set X in S. Therefore, R is an equivalence relation. † Exercise g 9. More precisely, the edge set of Ti is the set E constructed after the ith iteration of the for loop, lines 5–17, and the vertex set of Ti is the set of vertices on which the edges in E are incident. f(n) = n + 1 11. Such propositions are said to be logically equivalent. Example 3.7 suggests that 1 + 2 + · · · + n = An2 + Bn + C for all n, for some constants A, B, and C. Discussion In a direct proof, we assume the hypotheses and derive the conclusion. Then 0G, 1GR describes a simple cycle of length m in the n-cube. , xr } = {s1 , . When n = 3, we obtain 6 = 9A + 3B + C. It is interesting that most actors, even actors who died many years ago, have Bacon numbers of three or less. Since Dale made 40 percent of the calls, P(D) = 0.4. Similarly, from the table we obtain P(R) = 0.25, P(L) = 0.35. Make up at least three n-ary relations with artificial data that might be used in a medical database. Show by using the rules of inference (see Figure 4.1) that the conclusion, the Chargers can beat the Jets and the Chargers can beat the Dolphins, follows from the hypotheses. What is the cardinality of {a, b, a, c}? Therefore 1 ,b1) max(a2 ,b2) k ,bk) pmax(a p2 · · · pmax(a 1 k is the least common multiple of m and n. Since b divides c, there exists q2 such that c = bq2 . As another example, aabb is in L, for if we take α = ab, then α ∈ L; by the first rule, aabb = aαb ∈ L. Yes, (i, j, g, h, e, d, c, b, a, f) 34. We denote the graph as GKn . † Exercise is not prime. An example is B B B S M B M S. Let S and R be as in Theorem 4.1. If S ∈ S, we can regard the members of S as equivalent in the sense of the relation R, which motivates calling relations that are reflexive, symmetric, and transitive equivalence relations. In general, is "is similar to" an equivalence relation? A proof of ∃x P(x) is called an existence proof. In this case n = 4. What is the relation between the capacity of any cut and the value of any flow? We then assume that S(n) is true and prove that S(n+1) is true (Inductive Step). Now solve for A, B, and C. Example 8.5 A useful feature of black-and-white pictures is the average brightness of the picture. The relation of Exercise 4 on {a, b, c} R = {(1, 2), (2, 1), (3, 3), (1, 1), (2, 2)} 21. Write a program to determine whether a relation is transitive. 479 Graph Theory It would be easy to test whether a pair of graphs is isomorphic if we could find a small number of easily checked invariants that isomorphic graphs and only isomorphic graphs share. Show that the first player can always win nim if and only if the first player can always win nim . Write a program that lists all simple paths between two given vertices. If x ∈ (X ∩ Y) − (X ∩ Z), then x ∈ X ∩ (Y − Z). Guess a formula for Sn and use induction to show that it is correct. From the definition of "union," we conclude that x ∈ X ∪ Y . The algorithm is invoked with input 2. J1 A B C D J2 J3 J4 J5 Figure 4.1 Applicants (A, B, C, D) and jobs (J1 , J2 , J3 , J4 , J5). A sequence named, say, s is denoted s or {sn }. The number of paths of length 3 from v2 to v3 20. For part (a), since z = tx (x ∈ X), we must show that tx ≤ ta for all x ∈ X. σ(ρ(R1)) = ρ(σ(R1)) 63. If Rn is the number of ways of spending all the money, show that Rn = Rn−1 + 2Rn−2 . Therefore, |x| = x = 1 · x = sgn(x)x. Thus exactly half of the subsets of X contain x, and exactly half of the subsets of X do not contain x. , a110 ; = 1 − 0.05 = 0.95. Therefore, G − {e} is connected. In Definition 5.1, the domain of discourse specifies the allowable values for x. Therefore, 42. The inductive step will use induction on n. Input: PT, the root of a binary tree, or the special value null to indicate that no tree is input Output: Dependent on how "process" is interpreted in line 7 1. Give an example (different from those of Example 1.1) of a definition in Euclidean geometry. What is the least upper bound of a nonempty finite set of real numbers? However, we just saw that when no tree is input to preorder, nothing is processed. Show that if n ≥ 3, the complete graph on n vertices Kn contains a Hamiltonian cycle. , n is C(n, n/2). Describe the Instant Insanity puzzle. In Figure 1.13, we arbitrarily chose one of the vertices labeled 12. The winning strategy is to first take one token; then, whatever the second player does, leave one token. Definition 3.17 A relation R on a set X is transitive if for all x, y, z ∈ X, if (x, y) and (y, z) ∈ R, then (x, z) ∈ R. By Definition 5.4, the proposition ∀x P(x) is false precisely when P(x) is false for at least one x in the domain of discourse. a2 x3 ; (a + ax + x)(a + x)4 √ 9. In this section we deal with the problem of estimating the time required to execute algorithms. Now An+1 = An A so that the ikth entry in An+1 is obtained by multiplying pairwise the elements in the ith row of An by the elements in the kth column of A and summing: j i k Figure 5.3 The proof of Theorem 5.3. A path from i to k of length n + 1 whose next-to-last vertex is j consists of a path of length n from i to j followed by edge (j, k). When a tree is used in this way, if vertex a is on a level one less than the level of vertex b and a and b are adjacent, then a is "just above" b and a logical relationship exists between a and b: a dominates b or b is subordinate to a in some way. Evaluate all the vertices using the evaluation function of Example 9.1. Where will O move? Prove that if X ⊆ Y , then Z − Y ⊆ Z − X for all sets X, Y , and Z. Example 1.15 Since Q ⊆ R, R∪Q = R R∩Q = Q Q − R = ∅. Example 6.6 Consider the statement ∀x∃y(x + y = 0). Letter Frequency Letter Frequency a b c 2 3 5 d e f 8 13 21 Decode each bit string using the Huffman code given. As in the previous problem, we must count the number of such strings. In how many ways can 12 distinct books be divided among four students if each student gets three books? , xn denote the n points, and let Xi be the circle of radius 1 centered at xi . 2 Let sn denote the number of paths of length n from v1 to v1 . Definition 1.45 Example 1.46 A function from X × X to X is called a binary operator on X. Contradiction. Trees 1 2 3 4 5 6 7 8 9 Introduction Terminology and Characterizations of Trees Problem-Solving Corner: Trees Spanning Trees Minimal Spanning Trees Binary Trees Tree Traversals Decision Trees and the Minimum Time for Sorting Isomorphisms of Trees Game Trees Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises I talk to the trees, but they don't listen to me. 34 72 101 34 i 11. Directly checking n = 0, 1, 2, 3, 4 shows that only C2 = 2 and C3 = 5 are prime. v1 27. (A ∩ B) − C 15. } 20. Algorithm 4.14 Generating Permutations This algorithm lists all permutations of {1, 2, . A domain of discourse for a propositional function P is a set D such that P(x) is defined for every x in D. Suppose that |X| = n and |Y | = m. Input: Output: The sequence of coefficients c0 , c1 , . Solving for x, we obtain x = 1 or x = −2/3. Thus there are 4 · 13 · C(13, 4)3 hands with a 4–4–4–1 distribution. In how many ways can three teams containing four, two, and two persons be selected from a group of eight persons? 4 ➔ The RSA Public-Key Cryptosystem Cryptology is the study of systems, called cryptosystems, for secure communications. Show that the recurrence relation (3.12) holds for n > 2. We leave the Inclusion-Exclusion Principle for three or more sets to the exercises (see Exercises 92–97). n−2 (n − k − 2)r k+2 k=−1 206 7. Insert the next data items in the tree according to the following steps. We use the first of De Morgan's laws (3.3) to replace hypothesis 2 with the two hypotheses ¬a ¬d. If we divide the balls into sets R, B, and G according to color, the family {R, B, G} is a partition of X. Explain why if sj = rj and sk = rk or sj = 16 − rj and sk = 16 − rk , then 16 divides aj − ak . Decision tree 41. Four, since the possibilities are (0, 0), (2, 1), (4, 2), and (6, 3), where the pair (r, g) designates r red and g green balls. Now we have an entry on our slate. Section 4 13. The requested inductive argument is attributed to J. A problem that has a worst-case polynomial-time algorithm is considered to have a "good" algorithm; the interpretation is that such a problem has an efficient solution. Prove Theorem 7.6 by using Theorem 2.17. If x has no children, x is a terminal vertex (or a leaf). Example 1.5 Let Sn denote the number of subsets of an n-element set. (b) Scientist defeat Whales; Scientists defeat Pilots. This is a constructive existence proof. Arbitrarily assign one of these colors, say red, to the front. If no integer in this list divides n, then n is prime. The following graphs d 16. Give an example of the use of the greedy method that does not lead to an optimal algorithm. Consider the sequence t a, b, a, a, b. B ⊆ A 91. We can also construct a stacking from a pair of graphs such as those in Figure 8.3, which represent a solution of the Instant Insanity puzzle. For example, the following figure is a 3 × 5 grid and one route is shown. defined by the equations c1 = 0, cn = 2cn/2 + n for all n > 1. Corollary 2.22 says that the number of vertices of odd degree is even. The following are equivalent: (a) G1 and G2 are isomorphic. The reason this statement is true can be seen by arguing by contradiction. v2 v3 v2 v3 v v1 v1 v4 v5 v4 v5 G G Figure 7.4 G is obtained from G by a series reduction. The resulting graph G is connected and every vertex has even degree. Therefore, h+2 n+1> 3 2 . Write the following argument symbolically and determine whether it is valid. Number of orderings of n items of t types with ni identical objects of type i = n!/[n1 ! · · · nt !] 11. For example, if the proposition, If the Mathematics Department gets an additional $60,000, then it will hire one new faculty member, is true because the Mathematics Department did not get an additional $60,000, we would say that the proposition is true by default or that it is vacuously true. Example 6.7 Birthday Problem Find the probability that among n persons, at least two people have birthdays on the same month and date (but not necessarily in the same year). Write a program that tests whether A is nondecreasing. To sort s1 , . Write a program that finds the inverse of a relation. Define an initial segment of a string to be the first i ≥ 1 characters for some i. Notice that the digraph of the relation R in the previous paragraph (see Figure 3.1) has at most one directed edge between each pair of vertices. (b) Prove that if there is a vertex of degree k > 1 and no vertex is adjacent to all other vertices, then the degree of every vertex is k. We have our counterexample. Use the following definitions in Exercises 35–40. [ ) 12. Prove that if v and w are vertices in G having odd parity, it is possible to change the orientation of certain edges in G so that v and w have even parity and the parity of all other vertices in G is unchanged. sn = 2n − 1, n ≥ 2. In words, for every x and for every y, if x > 0 and y > 0, then x + y > 0. Since 20 = 1, the symbol in position 0 represents the number of 20 's, or 1's; since 21 = 2, the symbol in position 1 represents the number of 21 's or 2's; since 22 = 4, the symbol in position 2 represents the number of 22 's or 4's; and so on. Problem-Solving Tips To convert the base b number cn bn + cn−1 bn−1 + · · · + c1 b1 + c0 b0 to decimal, carry out the indicated multiplications and additions in decimal. Exercises In a tournament, the Snow beat the Pheasants once, the Skyscrapers beat the Tuna once, the Snow beat the Skyscrapers twice, the Pheasants beat the Tuna once, and the Pheasants beat the Snow once. Suppose that f(X) = f(Y). Since f(m, n) = 5 for all m, n ∈ Z+ , f is not onto. But G is also greedy because the greedy solution begins by removing one am coin. , s2n , where each si is a string of n bits, satisfying the following: ■ Every n-bit string appears somewhere in the sequence. Complete Example 8.5 by showing that if the pairs (P1 , Pi), (P1 , Pj), (P1 , Pk) are dissimilar, there are three pictures that are mutually similar or mutually dissimilar. We let q denote the specific value of k for which r = n − dq. For example, if n = 2, and we choose ∗ and + as the operators and x, y, and z as the variables, some of the expressions are (x ∗ y) + z, x ∗ (y + z), x ∗ (z + y), x + (z ∗ y), z ∗ (y + x). I do not get a soda. Start with 5. a In Exercises 21 and 22, determine whether the graphs G1 and G2 are isomorphic. If x ∈ A, then x satisfies x2 − 4x + 4 = 1. Each row must contain exactly one 1 for the relation to be a function. , Cn . binary search nonrecurs(s, n, key) { i=1 j=n // the body of the loop executes only if the subsequence // si , . A major advantage of using a sorted sequence rather than an unsorted sequence is that it is much easier to find a particular item. Prove that the set Before beginning a proof, familiarize yourself with relevant definitions, theorems, examples, and so on. Exercises 5. .} 94. (Using the notation of Example 3.2, if p and q are both true, then p → q is true.) On the other hand, if the Mathematics Department gets an additional $60,000 and does not hire an additional faculty member, the dean is wrong—statement (3.1) is false. Full details of the RSA cryptosystem may be found in [Cormen]. In words, 7 is prime. There are nine consecutive patterns: A2345, 23456, 34567, 45678, 56789, 6789T, 789TJ, 89TJQ, 9TJQK. Therefore A ∪ ∅ = A. a3 x4 ; (a + ax + x)2 (a + x)5 20. , E2n−1 are the subsets of {1, . We assume that −d < x < d. Graph G = (V, E) (undirected and directed) Vertex Edge Edge e is incident on vertex v Vertex v is incident on edge e v and w are adjacent vertices Parallel edges Loop Isolated vertex Simple graph Weighted graph Weight of an edge Length of a path in a weighted graph Similarity graph Dissimilarity function n-cube (hypercube) Serial computer Serial algorithm Parallel computer Parallel algorithm Complete graph on n vertices, Kn Bipartite graph Complete bipartite graph on m and n vertices, Km,n Section 2 24. 3 i . Find a minimal spanning tree for the following graph. Now cn = cn/2 + n < 2n/2 + n ≤ 2(n/2) + n = 2n. We can show that it is not possible to match a job to each applicant by considering applicants A, B, and D, who are qualified for jobs J2 and J5 . Execution resumes in part (c) at line 4 after computing 1! (= 1). 55. Exercises 24–30 refer to the following situation. x mod y: remainder when x is divided by y 3. Prove that Algorithm 2.1 is correct. We can write a recursive algorithm to compute walk(n) by translating the equation walk(n) = walk(n − 1) + walk(n − 2) directly into an algorithm. The first way was to list all of the positive divisors of a and all of the positive divisors of b and then, among all of the common divisors, choose the largest. 1/210 34. (3.1) Now suppose that n > 1. 155 Functions, Sequences, and Relations Example 2.3 The sequence {un } defined by the rule un = n2 − 1, n ≥ 0, can be denoted {un }∞ n=0 . f(x) = sin x 27. Therefore, Algorithm 4.2 correctly returns the value of n! for every integer n ≥ 0. Place the words word processing produces clean manuscripts but not necessarily clear prose in the order in which they appear, in a binary search tree. Each is isomorphic to the tree T2 of Figure 8.3. v1 v2 v3 v5 w1 v4 w3 w2 w4 T1 w5 T2 Figure 8.8 Nonisomorphic rooted trees. A function that invokes itself 3. 427 Recurrence Relations 2. In calculus terminology, a nondecreasing sequence, which is bounded above, converges to the limit L, where L is the least upper bound of the set of elements of the sequence. e is the first vertex after d on P1 that is also on P2 . 13 + 23 + 33 + · · · + n3 = 110 1 1·3 1·3·5 1 · 3 · 5 · · · (2n − 1) + + + ··· + 2·4 2·4·6 2·4·6·8 2 · 4 · 6 · · · (2n + 2) 9. Applicant A is qualified for jobs J1 and J4 ; B is qualified for jobs J2 , J3 , and J6 ; C is qualified for jobs J1 , J3 , J5 , and J6 ; D is qualified for jobs J1 , J3 , and J4 ; and E is qualified for jobs J1 , J3 , and J6 . in which edge ei is incident on vertices vi−1 and vi for i = 1, . In other words, X is a subset of Y if for every x, if x ∈ X, then x ∈ Y . The code for a particular character is obtained by following the simple path from the root to that character. Let M be a mesh p vertices by q vertices, where p ≤ 2i and q ≤ 2j . Let G be a directed, bipartite graph with disjoint vertex sets V and W in which the edges are directed from vertices in V to vertices in W. ..., 40 30 30 50 40 1 12 40 3 9 50 9 35 50 Working left to right, create a parent for each pair and label it with the maximum of the children. A set of elements is determined by its members; order is irrelevant. We call s the inverse of n mod φ. What is a probability function? If S is a family of sets, how do we define the union of S? ⎭ tm = s1 t1 + s2 t2 + · · · + sj tj + · · · + sm tm = ikth entry in An+1 . Hypotheses: If there is gas in the car, then I will go to the store. At line 8, there are n − 1 comparisons (since line 7 causes line 8 to be executed n − 1 times). Example 3.3 } k = (i + j)/2 if (key == sk) // found return k if (key < sk) // search left half j =k−1 else // search right half i=k+1 return binary search(s, i, j, key) We illustrate Algorithm 3.2 for the input s1 = 'B', s2 = 'D', s3 = 'F ', s4 = 'S', and key = 'S'. Graphs are often convenient models to describe these machines. Therefore, f is one-to-one. Thus there are 8 · 7 · 6 strings in which no I appears before any L. The proof by induction is complete. Such a value, we recall, is called a counterexample to the universally quantified statement. Then x1 R1 x1 and x1 R1 x1 . Give an example of a graph that has a Hamiltonian cycle and an Euler cycle. Some theorems state that three or more statements are logically equivalent and, thus, have the form The following are equivalent: (a) — (b) — (c) — .. Show that the representation in Exercise 30 is unique if we do not allow f1 as a summand. Write an algorithm that tests whether two n × n matrices are equal and find a theta notation for its worst-case time. Suppose that f(n) = O(g(n)), and f(n) ≥ 0 and g(n) > 0 for all n ≥ 1. 6 4. Therefore, large is the largest value in the subsequence s1 , . Although Corriveau attributed his success to chaos theory, what in fact happened was that whenever power was cut, the random number generator started with the same seed, thus generating the same sequence of numbers. Definition 1.19 Let m and n be positive integers. , pn denote all of the distinct primes less than or equal to p. (d) e g 22. By the Multiplication Principle, the number of ways of ordering the letters is C(11, 2)C(9, 4)C(5, 4) = 11! 9! 5! 11! = = 34,650. We begin by recalling the definition of "divides," and we introduce some related terminology. For a history of logic, see [Kline]. It is denoted x . Example 3.12 Suppose that an algorithm is known to take 60n2 + 5n + 1 226 Algorithms units of time to terminate in the worst case for inputs of size n. In fact, Z is a proper subset of Q because, for example, 1/2 ∈ Q, but 1/2 ∈ Z. The shaded outcomes become the new sample space, and each shaded outcome is reassigned the value 1/11. To prove the recurrence relation, we must reduce the n-person problem to the (n−1)- and (n−2)person problems (since in the formula Dn is given in terms of Dn−1 and Dn−2). Therefore, the professor does not have a sorting algorithm that uses at most 100n comparisons in the worst case for all n ≥ 1. The use of any trademark in this text does not vest in the author or publisher any trademark ownership rights in such trademarks, nor does the use of such trademarks imply any affiliation with or endorsement of this book by such owners. for i = 1 to n for j = 1 to n for k = 1 to n x=x+1 22. True 16. We place an edge between each pair of vertices, one from H1 and one from H2 , provided that the vertices have identical labels. Some student is not taking a math course. Theorem 3.7 If a and b are nonnegative integers, not both zero, there exist integers s and t such that gcd(a, b) = sa + tb. Explain your answer. Recall (see Example 1.7) that we may label the vertices of the n-cube 0, 1, . Write a program that reads a logical expression in p, q, and r and prints the truth table of the expression. 6: (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), where (b, r) means the blue die shows b and the red die shows r. Let A be an adjacency matrix of a graph. (9.1) This lower bound for A is called an alpha value of A. Prove the Strong Form of Mathematical Induction. (d) Let Gn be the sequence consisting of Gn−1 followed by Gn−1 . Give an example of a symmetric relation. assignment [Team] 20. V1 = {v1 , v2 , v5 }, V2 = {v3 , v4 }. The truth table of an n-variable function has 2n rows since each variable can be either T or F. Thus, logb n = O(loga n). We show that for all x, if x ∈ X ∪ (Y − X), then x ∈ X ∪ Y , and if x ∈ X ∪ Y , then x ∈ X ∪ (Y − X). If A1 is the matrix of the relation R1 and A2 is the matrix of the relation R2 , the matrix of the relation R2 ◦R1 is obtained by replacing each nonzero term in the matrix product A1 A2 by 1. Write a program that finds the components of a graph. Now suppose that S does not use an am coin. In this case, 0 is the least element in X. Write an algorithm that, given the binary tree representation of an expression, outputs the fully parenthesized infix form of the expression. ai+j Bi , Section 8 i=0 1. ¬p ∧ ¬r ∧ ¬q 60. If the quantifier is ∀, Farley chooses a value for that variable; if the quantifier is ∃, you choose a value for that variable. 17 15. For this relation, equivalence is "has the same remainder when divided by 3." Example 4.15 180 We show that if a relation R on a set X is symmetric and transitive but not reflexive, the collection of sets [a], a ∈ X, defined in Theorem 4.8 does not partition X (see also Exercises 44–48). The value of the flow in this network can be increased by 1, as shown in Figure 2.3. 4, 1 2, 1 3, 1 a b c Figure 2.2 A path all of whose edges are properly oriented. Expand (2c − 3d)5 15. 245 Algorithms 16. 7. If the propositions P and Q are made up of the propositions p1 , . Since we added only terminal vertices, the original n−1 vertices in T are the only internal vertices in T ∗ . A database is a collection of records that are manipulated by a computer. If I study hard or I get rich, then I get A's. x becomes a8 , and n becomes 2. Therefore, R is transitive. In (a), edges e1 and e2 are deleted and edge e is added. Definition 3.6 Example 3.7 A relation R on a set X is reflexive if (x, x) ∈ R for every x ∈ X. Since the only permutation of the form 1635−− distinct from the given permutation is 163524, and 163524 is smaller than 163542, the permutation following the given permutation is not of the form 1635−− . The next smallest board that might have a knight's tour is the 4×4 board, although, as we shall show, it too does not have a knight's tour. 517 Trees P1 c e d b a P2 Figure 2.6 The proof of Theorem 2.3 [if (d), then (a)]. f(x) = x2 9. Give an example of functions f and g and their composition. v3 b a e2 d e3 G v4 2. Compute cs mod z. Find 3 i=1 xi . P(n, r) 6. The sequence d is of the form dn = b · 1n + c · 2n = b + c2n . or Xt is n1 + n2 + · · · + nt . Is bipartite 26. (4.1) Because of (4.1), the graph T = [T ∪{(j, k)}]−{(x, y)} has weight less than or equal to the weight of T . Write a backtracking algorithm that solves an arbitrary Sudoku puzzle. We can relate (4.10) and (4.11) by observing that (n + 1)! = (n + 1)(n!). Give an evaluation function for a tic-tac-toe position different from that of Example 9.1. Attempt to discriminate more among the positions than does the evaluation function of Example 9.1. 35. The value of the new flow is 1 greater than that of the original flow. , pn lists all possible combinations of truth values for p1 , . Capacities are shown on the edges. How many routes are there in the ordinary xyz-coordinate system from the origin to the point (i, j, k), where i, j, and k are positive integers, if we are limited to steps one unit in the positive x-direction, one unit in the positive y-direction, or one unit in the positive z-direction? The expression (6.1) would be represented as shown in Figure 6.7. In the binary tree representation of an expression, an operator operates on its left and right subtrees. 1 498 0 11 10 6. In words, we might describe this relation as "is divisible by." If we have a relation R1 from X to Y and a relation R2 from Y to Z, we can form the composition of the relations by applying first relation R1 and then relation R2 . 501 Graph Theory e1 ⎛1 1 2⎜ 3⎜ ⎜ 4⎜ ⎜ 5⎜ ⎝ 6 7 1 0 0 0 0 0 e2 0 1 1 0 0 0 0 e3 0 0 1 1 0 0 0 13. What is a subproof? (1, 2, 3, 1, 4, 5, 1) A graph is connected, if, given any vertices v and w, there is a path from v to w. If b is zero, it correctly returns a. A robot can take steps of 1 meter, 2 meters, or 3 meters. What is preorder traversal? Proceed as described in the solution to Exercise 11, only use the formula ab mod z = [(a mod z)(b mod z)] mod z. Since |β| < n, by the n n−1 inductive hypothesis β has an equal number of a's and b's. Thus there are less than n − (n − |V ∗ |) − |W1 | = |V ∗ | − |W1 | edges of type II in E. (e, d, c, b) 3. We consider three cases: x > 0, x = 0, and x < 0. P(Cj) and again by Definition 6.13, Combining these equations, we obtain P(Cj | F) = P(F | Cj)P(Cj) P(Cj ∩ F) = . We set to 3 [= val(b)] at line 16. Is f onto? Definition 2.4 446 A graph G is connected if given any vertices v and w in G, there is a path from v to w. Therefore, what we have just computed are the members of [01111010]. This is also certainly the case! For R to be transitive, for all eight-bit strings s1 , s2 , and s3 , if s1 R s2 and s2 R s3 , then s1 R s3 . (d) The restaurant rated over two stars in Chicago magazine (domain of discourse = restaurants rated in Chicago magazine). We let T0 be the graph constructed by Algorithm 4.3 just before the for loop at line 5 is entered for the first time; T0 consists of the single vertex s and no edges. That this relation is not reflexive can also be seen by looking at its digraph (see Figure 3.2); vertex b does not have a loop. 344 13. We can ask for the minimum time needed to execute the algorithm among all inputs of size n. For which values of m and n does the graph contain an Euler cycle? A Hamiltonian path in a graph G is a simple path that contains every vertex in G exactly once. A set is equivalent to itself by the identity function. If we use this collision resolution policy, we would store 257 at location 6 (see Figure 1.6). The statement is true if there is at least one x ∈ X such that P(x, y) is true for every y ∈ Y . In Exercises 9–11, draw a graph having the given properties or explain why no such graph exists. Let G be a connected graph. Here we can select one appetizer in 2 ways, one main course in 3 ways, and one beverage in 4 ways. By Theorem 4.7, there is not a complete matching for the graph of Figure 4.1. Example 4.9 There are n computers and n disk drives. In this case, all will be huge; so it is impractical to compute an and then compute the remainder when an is divided by z. The computation shows the successive divisions by 16 with the remainders recorded at the right. The proof is by induction on m. If E1 and E2 are events, what does the event E1 ∩ E2 represent? Show that Tmn = Tm + Tn for all m, n ≥ 1. Give an example of something in the real world that can be modeled by a weighted graph. Since 1 1 ≤D≤ , ni 1+n for i = 1, . The following exercises show how the proof begins. Use the Well-Ordering Property to show that any nonempty set X of nonnegative integers that has an upper bound contains a largest element. (c) Show that Sn,1 = 1 for all n ≥ 1. 1E9 28. By the Principle of Mathematical Induction, (4.13) holds for all n ≥ 0. Then a1 = 0. We must show that the inequality holds for k = n. Example 3.8 (a) How many solutions in nonnegative integers are there to the equation x1 + x2 + x3 + x4 = 29? Thus for every x, there exists y such that P(x, y) is false. 557 Trees If S is a set of trees of a particular type (e.g., S is a set of free trees or S is a set of rooted trees or S is a set of binary trees) and we define a relation R on S by the rule T1 R T2 if T1 and T2 are isomorphic, R is an equivalence relation. For our purposes, it suffices to think of the real numbers as points on a straight line. 474 11. A full binary tree has 15 internal vertices. Once the problem of computing 5! has been reduced to solving subproblems, the solution to the simplest subproblem can be used to solve the next simplest subproblem, and so on, until the original problem has been solved. Thus the number of strings that contain AB or BE is 4! + 4! − 3!. We next evaluate q ∧ r, which is false. We obtain the tiling: 14. The complete bipartite graph on m and n vertices has disjoint vertex sets V1 with m vertices and V2 with n vertices in which the edge set consists of all edges of the form (v1 , v2) with v1 ∈ V1 and v2 ∈ V2 . Since d is a divisor of n, d ≤ n. 64. If x ∈ Q, x corresponds to a point on the number line (see Figure 1.2) so x ∈ R. The input is a sequence s1 , . We consider two cases: q has p's order, and q does not have p's order. Example 2.6 Replace the zero flow in lines 1 and 2 of Algorithm 2.4 with the flow of Figure 2.11 and then find a maximal flow. This result shows that the worst-case 6 ➔ Tree Traversals Breadth-first search and depth-first search provide ways to "walk" a tree, that is, to traverse a tree in a systematic way so that each vertex is visited exactly once. r-permutation of x1 , . Another invariant that is sometimes useful is "has a simple cycle of length k." We leave the proof that this property is an invariant to the exercises (Exercise 17). Show that if p1 , p2 /∴ p and p, p3 , . There are C(n + m − 1, n) strings that end U, since we must choose n slots among the first n + m − 1 slots for the n R's. It is denoted X − Y. 282 Introduction to Number Theory The remainders give the base b representation of n. We divide this problem into counting the number of such strings with exactly eight 1's, with exactly seven 1's, and so on. We can use mathematical induction to prove that an invariant has the desired behavior. To prove that it does not have a Hamiltonian cycle, we argue by contradiction. In this case, leave an odd number of piles. If we add one am coin to the greedy solution for n − am , we obtain a solution G for n that uses the same number of coins as S. 17. For example, for the relation PLAYER given by Table 6.1, PLAYER [Name, Position] 192 Functions, Sequences, and Relations will select the tuples (Johnsonbaugh, c), (Homer, 1b), (Score, p), Example 6.5 (Glover, of), (Battey, p), (Cage, c), (Johnsonbaugh, of), (Singleton, 2b). If the first move is first column, second from top, there is no move possible in column two. Smoothing a by c to obtain a yields a j = = 1 (a + aj+1) 2 j 1 k−1 2n+1 = 1 ai+j Bi + i=0 = 1 aj B0 + 2n+1 = = aj B0 2n+1 1 2n+1 ai+j+1 Bi ai+j Bi + k−2 i=1 i=0 k−1 k−1 ai+j Bi + i=1 1 k−1 + ai+j+1 Bi + ak+j Bk−1 ai+j Bi−1 + ak+j Bk−1 i=1 k−1 ai+j Bi f(p1) = f(p2) = f(p3) = f(p4); that is, there are at least four persons in the same room. Find the level of each vertex in the tree shown. We prove the contrapositive of the given statement, which is if x is not irrational, then x2 is not irrational 86 Proofs or, equivalently, if x is rational, then x2 is rational. ¬a 4. Samuelson: Every environmental problem is not a tragedy. Regardless of the specific value of x, x + 1 > x. are not isomorphic. The next theorem shows that the capacity of any cut is always greater than or equal to the value of any flow. a 1, a 2, a 3 Yes a 2 < a 3? Prove that the coin cannot be found and identified as heavy or light in fewer than n weighings. Since doubles can be obtained in six ways, P(E1) = 6 1 = . c a b e g 454 In Exercises 24–27, find all subgraphs having at least one vertex of the graph given. Therefore, it is not possible in a department of 25 persons racked by dissension for each person to get along with exactly five others. Reading the preceding proof in the if-then direction, if n is odd, then n = 2k + 1 for some integer k; if n = 2k + 1 for some integer k, then n − 1 = 2k for some integer k; if n − 1 = 2k for some integer k, then n − 1 is even, proves that if n is odd, then n − 1 is even. By induction, sj gives the number of paths of length n from i to j in the graph G. Inductive Step for the Strong Form of Mathematical Induction: assume true for all instances less than n; then prove true for instance n. This method requires four integers: the modulus m, the multiplier a, the increment c, and a seed s satisfying 2 ≤ a < m, 0 ≤ c < m, 0 ≤ s < m. Show that τ(σ(ρ(R))) is the smallest equivalence relation on X containing R; that is, show that if R is an equivalence relation on X and R ⊇ R, then R ⊇ τ(σ(ρ(R))). {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (2, 1), (3, 2)} 40. Define a function from X∗ to X∗ as f(α) = αR . If the first index is 1, we have t1 = a, t2 = a, t3 = b, t4 = a, t5 = b. Let X denote the set of eight-bit strings that start 10 and Y denote the set of eight-bit strings that end 011. Finally, we again recursively invoke our algorithm to move the n − 1 disks on peg 2 to peg 3. What about n = 5? The left and right subtrees of a vertex correspond to the left and right operands or expressions. A binary operator on X is a function from X × X to X. Prove that the following graph is the Petersen graph; that is, prove that it is isomorphic to the graphs in Exercise 4. Give an example of a graph with six vertices that has no articulation points. In each exercise, G is a connected, weighted graph. Let α ∈ X∗ . If n = 2, we proceed to line 4 since n = 0. Since R is not reflexive, there exists b ∈ X such that (b, b) ∈ R. Implement Algorithm 2.4, shuffle, as a program. Step 4—move disk 1 from peg 1 to peg 3. The dashed line divides the vertices into sets P = {a, b, d} and P = {c, e, f, z} producing the cut (P, P). If v and w are vertices in G, the portion of the Euler cycle that takes us from v to w serves as a path from v to w. n2 30. n = 1, φ = 47 35. This leaves two edges—not enough for a Hamiltonian cycle. Give an intuitive interpretation of how f and g are related if f(n) = (g(n)). Define undirected graph. We can arrange for nq + 2t ≥ 0 to hold by choosing nq ≥ −2t. By Exercise 16, Tournament Sort requires 2k − 1 comparisons to find the largest element. n For the Inductive Step, let b0 = k=1 ak and bi = b0 + ai for 1 ≤ i ≤ n. Thus we may choose the vertex v in 2n − 1 ways and the vertex to mark (left or right) in 2 ways. 8 9 12, the output is Input: S (a set of integers), m (an integer) Output: All finite subsets of S that sum to m 44 rand(3, 5) = 5, Section 3 Select a theta notation from among (1), (n), (n2), (n3), (n4), (2n), or (n!) for each of the expressions in Exercises 9 and 10. Exercise 5, Section 1 12. , is a solution of (2.16). , ei , vi , ei+1 , . (3.4) An arbitrary value of n falls between two powers of 2, say 2k−1 < n ≤ 2k . The base cases n = 1 and n = 2 are left to the reader. [Pfleeger] is devoted to computer security. A common multiple of m and n is an integer that is divisible by both m and n. Use decision trees to explain why worst-case sorting requires at least (n lg n) comparisons. ¬p ∨ q ∨ r 2. An inventory consists of a list of 200 items, each marked "available" or "unavailable." There are 110 available items. Modify the proofs of Theorems 2.17 and 2.18. On the other hand, if there are duplicates among the items to be ordered, the formula n! n1 !n2 ! · · · nt ! may be used. Therefore, the function is onto. Graph Theory 16. a 6. Step 6—move disk 2 from peg 2 to peg 3. Many algorithms are based on the idea of repeated halving. Use induction on n to prove the Binomial Theorem. Solving for b and c, we find that b = 180 and c = 20. AB+CD∗AA/−−B∗ 19. Reflexive relation R on X: (x, x) ∈ R for all x ∈ X 35. The equation CX∪Y (x) = CX (x) + CY (x) is true if x ∈ X ∪ Y . Example 1.10 The complete graph on four vertices, K4 , is shown in Figure 1.12. Proof We have already shown that if G has a spanning tree, then G is connected. Show that fn+1 ≥ √ n−1 1+ 5 , 2 an = br n + dnr n , n ≥ 1, 49. Assign each sock (pigeon) to its type (pigeonhole). The least upper bound for X is 4. See Figures 5.4 and 5.5. n ≥ Nh = fh+3 − 1 > 10. If G is a graph (undirected or directed) with vertices V and edges E, we write G = (V, E). We must show that T is acyclic. Input: A graph G = (V, E) with n vertices Output: true if G is connected false if G is not connected is connected(V, E) { T = bfs(V, E) // T = (V , E) is the spanning tree returned by bfs if (|V | == n) return true else return false } 33. On the other hand, in 2002 Manindra Agarwal and two of his students, Nitin Saxena and Neeraj Kayal, discovered a polynomial-time algorithm that can determine whether or not a given integer is prime (see [Agarwal]). These numbers represent the value of the game, at any position, to the first player. Thus to list one member of each equivalence class, we simply need to append some four-bit string to each of the strings in the previous list: 00000000, 00010000, 00100000, 00110000, 01000000, 01010000, 01100000, 01110000, 10000000, 10010000, 10100000, 10110000, 11000000, 11010000, 11100000, 11110000. The great English mathematician, G. At lines 19 and 20 we label vertex b as (a, 1) and we label vertex d as (a, 1). It is apparent from the preceding discussion that a solution to the Instant Insanity puzzle can be obtained if we can find two graphs like those of Figure 8.3. The properties needed are ■ Each vertex should have degree 2. For example, when we compare C1 with C5 and the pans balance, we follow the edge to the terminal vertex labeled C2 , L, which tells us that the bad coin is C2 and that it is lighter than the others. Let x be an edge in a cycle (see Figure 7.10). (see Figure 4.4). each other or extend outside the figure. In general, the symbol in position n (with the rightmost symbol being in position 0) represents the number of 10n 's. Next, we move left and then right. We give several examples of theorems, corollaries, and lemmas in Euclidean geometry and in the system of real numbers. Definition 1.4 An integer greater than 1 whose only positive divisors are itself and 1 is called prime. 79 Proofs We prove that x ≤ d1 and x ≤ d2 . 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456 14. The solution to the exercise is that an eight-bit string with no two 0's in a row has exactly six 1's. The while loop then compares t1 · · · t4 = "bala" with p = "bala". 405 Recurrence Relations Section Review Exercises 1. We obtain B4F16 = 11 · 162 + 4 · 161 + 15 · 160 = 11 · 256 + 4 · 16 + 15 = 2816 + 64 + 15 = 289510 . ∃x∀y P(x, y) Assume that ∃x∀y P(x, y) is true and that the domain of discourse is nonempty. Proof (See Figures 2.2, 2.3, 2.5, and 2.6.) The argument that F ∗ is a flow is given just before Example 2.2. Since the edge (a, v) in P is increased by , the value of F ∗ is greater than the value of F . c e h f p t n m i q s r j c k l d Figure 3.3 Visiting each vertex once in the graph of Figure 3.2. In honor of Hamilton, we call a cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice, a Hamiltonian cycle. The ceiling of x is the least integer greater than or equal to x. The elements of X are listed consecutively in A starting in the first position and terminating with 0. (2.5) is a subsequence of the sequence 2, 4, 6, 8, 10, 12, 14, 16, . It only asserts the existence of a pigeonhole containing two or more pigeons. snk = 4k − 3 91. In this case, the for loop in exists dj(i) runs to termination and false is returned. Two proof-by-contradiction techniques were shown in Section 3. For the first-order recurrence relation (2.6), we found in Example 2.2 that the solution was of the form Sn = t n ; thus for our first attempt at finding a solution of the second-order recurrence relation (2.9), we will search for a solution of the form Vn = t n . Therefore, Dn − nDn−1 = (−1)n . Let u be the sequence defined by Answer 88–90 for the subsequence of t obtained by taking the first, second, fourth, seventh, eleventh, . Exercises 47–50 refer to the function AO and to Ackermann's function A. By universal instantiation, we have P(Lynn) ∨ Q(Lynn). How many are of the form xyz, where 0 = x < y and y > z? 292 Introduction to Number Theory Combining the previous equations, we have ns = ns − φnq = −t φ + 1 − φnq = φ(−t − nq) + 1. Thus, the 2n+1 × 2n+1 deficient board is tiled. Explain why the formula is not a recurrence relation. In the Inductive Step of the Strong Form of Mathematical Induction, we let n denote an arbitrary integer, n > n0 . f is not a binary operator since f(x, 0) is not defined. Thus G has one or two components. We let E denote the event "at least two persons have the same birthday." Then E is the event "no two persons have the same birthday." As we shall see, it is easier to compute P(E) than P(E). Prove that Algorithm 4.6 is correct. Thus it is possible to match each computer with a compatible disk drive. [Fukunaga; Gose; and Nadler] are texts on pattern recognition. First suppose that S uses at least one am coin. 6n6 + n + 4 + lg n)(n + 1) n + n2 13. c4 = c2 + 3 = 9. A decision tree that represents a sorting algorithm has n! terminal vertices corresponding to the n! possible permutations of input of size n. Derive a recurrence relation for C(n, k), the number of k-element subsets of an n-element set. , bk−1 is a (finite) k−1 isfying bi ≥ 0 for i = 0, . Using this analysis, the first move by the first player would be to the center square. There are three ways to assign Dolph. The solution is by induction on k. Therefore, T is connected, but the removal of any edge (but no vertices) from T disconnects T . At line 3, we execute the statement x = x + 1 the fourth time. Show that at least log3 (n − 1) weighings are necessary to determine the number of bad coins. 80 Proofs Example 1.15 If the statement (A ∩ B) ∪ C = A ∩ (B ∪ C), for all sets A, B, and C is true, prove it; otherwise, give a counterexample. for i = 2 to C(n, r) { is eliminated. Therefore, a2 = 2. If there is not a complete matching, explain why none exists. , xm and those with odd degree y1 , . P = {a, b, c, d} for Exercise 3, Section 1 In Exercises 4–16, find a minimal cut in each network. Show that if d | m, then d | mn. One way to get a sharper lower bound is to argue as in the previous paragraph, but first throw away the first half of the terms, to obtain 1 + 2 + · · · + n ≥ n/2 + · · · + (n − 1) + n ≥ n/2 + · · · + n/2 + n/2 = (n + 1)/2 n/2 ≥ (n/2)(n/2) = n2 4 (3.2) for all n ≥ 1. Compute un for n = 2, . Prove that if the greedy algorithm is optimal for all amounts of postage less than am−1 + am , then it is optimal for all amounts of postage. i∈V i∈V In light of Theorem 1.5, we can state the following definition. Therefore, p(n) = (nk). If the graph is bipartite, specify the disjoint vertex sets. j=0 S(n, k) = kn − 59. The relation is not an equivalence relation. We begin by evaluating vertices A, B, C, and D (see Figure 9.10). Define lexicographic order. Now, imagine traveling along the (n + 1)st line. Given the matrix of a relation, how can we determine whether the relation is symmetric? In the hexadecimal number system, to represent integers we use the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. Thus the fourth edge is incident on one of v1 , v2 , or v3 . Ideally, this statement tells us that the loop produces the expected result, that is, that the loop is correct. By construction, there is a path from a to v of length L(v), so this is a shortest path from a to v. Incidence matrix Section 6 47. Then n = dq + r. Also, n2 /4 + n (1 + 2 + · · · + n) + n ≥ n+1 n+1 2 ≥ by (3.2) n2 /4 + n/4 n = . Assume that the values of rand are rand(1, 5) = 5, rand(2, 5) = 5, rand(3, 5) = 4, rand(4, 5) = 4. Let A be the adjacency matrix for the graph of Exercise 1. As an example, here is a faulty "proof" that for all m and n, if m and n are even integers then mn is a square (i.e., mn = a2 for some integer a): Since m and n are even, m = 2k and n = 2k. Basis Step: prove true for the first instance. , i − 1. The "Fallacies, Flaws, and Flimflam" section of The College Mathematics Journal, published by the Mathematical Association of America, contains examples of mathematical mistakes, fallacious proofs, and faulty reasoning. Oil is unloaded at the dock a and pumped through the network to the refinery z. We conclude by showing that merge sort (Algorithm 3.8) is (n lg n) in the worst case. In Exercises 13–17, assume that a person invests $3000 at 12 percent annual interest compounded quarterly. Negation of conclusion 5. Prove or disprove: A ∩ (B C) = (A ∩ B) (A ∩ C) for all sets A, B, and C. Explain, in words, how we can obtain bounds on the worst-case time of merge sort for input of arbitrary size if we know the worst-case time when the input size is a power of two. Pseudorandom numbers are numbers that appear random even though they are generated by a program. Example 2.15 Solve the recurrence relation dn = 4(dn−1 − dn−2) (2.18) subject to the initial conditions d0 = 1 = d1 . Example 2.8 Prove that for all real numbers x and all positive real numbers d, |x| < d if and only if −d < x < d. Example 1.8 The set of rational numbers Q is a subset of the set of real numbers R. Here of direct relevance are the definition of a tree and Theorem 2.3, which gives equivalent conditions for a graph to be a tree. Since Kn has (n2) edges, the latter version is optimal. + 3 · 2k−1 + 2k = 3k + 2(3k − 2k) = 3 · 3k − 2 · 2k Now (logm n) = c + (−1 + logm n)d ≤ c + (k − 1)d (∗) = 3 · 3lg n − 2n. What is pattern recognition? The receiver computes cs mod z = 113569 mod 713 = 572 in order to decrypt the message. If we graph the price and quantity over time, the graph resembles a cobweb (see, e.g., Figure 1.5). List all strings over X = {0, 1} of length 3. The next theorem shows that it is always true that the flow out of the source equals the flow into the sink. (x, y) ∈ R if 2 divides x + y 10. Partial order: relation that is reflexive, antisymmetric, and transitive 39. Exercises 15–21 refer to piles of identical red, blue, and green balls where each pile contains at least 10 balls. 511 Trees 1 Input: A sequence of n frequencies, n ≥ 2 Output: A rooted tree that defines an optimal Huffman code 0 f1 f2 Figure 1.11 The case n = 2 for Algorithm 1.9. 1 0 fi fj Figure 1.12 The case n > 2 for Algorithm 1.9. Example 1.10 huffman(f, n) { if (n == 2) { let f1 and f2 denote the frequencies let T be as in Figure 1.11 return T } let fi and fj denote the smallest frequencies replace fi and fj in the list f by fi + fj T = huffman(f, n − 1) replace a vertex in T labeled fi + fj by the tree shown in Figure 1.12 to obtain the tree T return T } We show how Algorithm 1.9 constructs an optimal Huffman code using Table 1.2. TABLE 1.2 ■ Input for Example 1.10. This set U is called a universal set or a universe. Encrypt 144 using public key z, n. When does the complete bipartite graph Km,n contain an Euler cycle? min(s, n) { small = s1 for i = 2 to n if (si < small) // smaller value found small = si return small } 8. n1 ! n2 ! · · · nt ! Proof We assign positions to each of the n items to create an ordering of S. We next describe how the RSA system works, present a concrete example, and then discuss why it works. Example 1.14 Hash Functions Suppose that we have cells in a computer memory indexed from 0 to 10 (see Figure 1.6). Functions, Sequences, and Relations 57. Consider a "shortest-path algorithm" in which at each step we select an available edge having minimum weight incident on the most recently added vertex (see the discussion preceding Theorem 4.5). We discuss insertion sort, which is one of the fastest algorithms for sorting small sequences (less than 50 or so items). , n} altogether and 2n−1 contain an even number of elements, there are 2n − 2n−1 = 2n−1 subsets of {1, . Slater, Slater co- Graph Theory authored a paper with Frank Harary, and Harary co-authored a paper with Erd˝os. Thus, if we take s = 27 and t = −67, we obtain gcd(273, 110) = 1 = s · 273 + t · 110. If it finds a value di for which P(di) is true, the condition P(di) in the if statement is true; so the code returns true [to indicate that ∃x P(x) is true] and terminates. In this section we develop algorithms to generate combinations and permutations. Since z is not labeled and U is not empty, we move to line 14, where we choose vertex d in U and remove it from U at line 15. By properties of logarithms, we have lg n! = lg n + lg(n − 1) + · · · + lg 2 + lg 1 for all n ≥ 1. If some vertical line contains two or more points of some set, the domain point does not assign a unique codomain point and the set does not define a function (see Figure 1.5). n1 ! n2 ! · · · nt ! Next, we turn to the problem of counting unordered selections where repetitions are allowed. Let p : 4 megabytes is better than no memory at all. Use the Addition Principle to derive a formula like that proved in this section. The statement is true if, for every x ∈ X and for every y ∈ Y , P(x, y) is true. e1 and e6 are parallel edges. If we let P(x, y) denote the expression (x > 0) ∧ (y > 0) → (x + y > 0), the given statement can be written symbolically as ∀x∀y P(x, y). f(x, y) = x2 + y2 − xy, 152 X = {1, 2, . (3.7) Notice that (3.5) gives From (3.4), (3.6), and (3.7), we deduce that lg n < 1 + k = a2k−1 ≤ an ≤ a2k = 2 + k < 3 + lg n = O(lg n). a, z 6. 1/3 49. We have n = 74 = 13 · 5 + 9 = dq + r. Algorithm 2.3 Converting an Integer from Base b to Decimal This algorithm returns the decimal value of the base b integer cn cn−1 · · · c1 c0 . In a proof by contradiction, we assume the hypothesis (n2 is even) and the negation of the conclusion (n is not even, i.e., n is odd). v5 v6 w2 w3 w4 T2 v1 v5 v3 v4 v8 v9 10. Counting Methods and the Pigeonhole Principle C(n + 1, k) = C(n, k − 1) + C(n, k) Theorem 7.6 for 1 ≤ k ≤ n. The inverse is obtained by reversing all of the arrows in Figure 1.11. An nth-order, linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form an = c1 an−1 + c2 an−2 + · · · + ck an−k . Root T at some arbitrary vertex. Use equation (7.4) to derive a formula for 12 + 22 + · · · + n2 . (There are surely other ways to ensure that Maria visits France; for example, she could go to Lyon.) Thus, an equivalent formulation of the given statement is If Maria goes to the Eiffel Tower, then she visits France. The cell may be empty because an item was deleted. Throughout this section, G is a network with source a and sink z. Explain why if we allow cycles to repeat edges, a graph consisting of a single edge and two vertices is not acyclic. ¬c ∨ d ∴ d Applying (3.2) to expressions 1 and 2, we derive 4. The orderings v1 , v2 , v3 , v4 , v5 , v6 and w3 , w6 , w2 , w5 , w1 , w4 produce equal adjacency matrices. Since there are sn−1 such paths, there are sn−1 paths of length n from v1 to v1 that begin (v1 , v1 , . q : You are healthy. Example 6.3 Select The selection operator chooses certain n-tuples from a relation. 47 Sets and Logic Suppose that ∀xP(x) is true. State Bayes' Theorem. Solutions to subproblems of sizes m and k can be combined in time cm,k to solve the original problem. Since |X| + |Y | double-counts the elements in X ∩ Y , |X| + |Y | = |X ∪ Y | + |X ∩ Y |. Which are proper subsets of {a, b}? For example, if x = 1, the proposition 12 1 2 = +1 5 is false. We can represent these assignments as ordered pairs: (1, 55), (3.45, 189.75). C(13 + 2 − 1, 13) 28. If f(n) = (g(n)), then g(n) = (f(n)). Show that the vertices of a tree can be colored with two colors so that each edge is incident on vertices of different colors. How many outcomes have the blue die showing 2? A pattern is emerging. f(x) = 3x2 − 3x + 1 44. 121 Proofs 8. First, show that the graph T constructed is a tree. r : We will buy a new computer. At line 2, since n = 1, we proceed to line 4, where we invoke this algorithm with input n − 1. f 1, e 1, v 2 f 2, e 1, v 1 Figure 7.8 The Basis Step of Theorem 7.9. (7.3) Proof We will use induction on the number of edges. Therefore (X − Y) ∩ (Y − X) = ∅. [(b) → (c)] Assume that there exists k such that n = 2k − 1. A path from v0 to vn of length n is an alternating sequence of n+1 vertices and n edges beginning with vertex v0 and ending with vertex vn , (v0 , e1 , v1 , e2 , v2 , . Section 6 1. If q is true, then (2.1) is true. Prove that if n is an odd integer, 80. You heard either the "Flying Pigs" rock concert or the "Y2K" rock concert, but you do not have sore eardrums. For the situation of Exercise 31, find and prove an estimate for 17. Will any problem arise when retrieving data if we stop searching when we encounter an empty cell? If q is false, then (2.1) is false. i=1 Example 2.16 The geometric sum a + ar + ar 2 + · · · + ar n can be rewritten compactly using the sum

notation as n ar i . Just as in the case of the decision trees that solve coin puzzle problems, the height of a decision tree that solves a sorting problem is equal to the worst-case time. 5,987,520 3. 3 8 4 7 G2 19. temp1 := buyer [Name = JCN Electronics] temp2 := temp1 [Part No = Part No] supplier temp3 := temp2 [Dept = Dept] department temp4 := temp3 [Manager = Manager] employee temp4 [Name] Kaminski, Schmidt, Manacotti 22. Let bn be the recurrence relation obtained from Exercise 75 by replacing "≤" by "=." Assume that b1 = a1 = 0. Now this expression is supposed to be of the form (1.2). Major bridge tournaments use computer programs to shuffle the cards. (x − 1)2 Exercises 29–33 concern Toots and Sly, who flip fair pennies. p → (q → r) q → (p → r) 15. Order the vertices and edges and label the rows of a matrix with the vertices and the columns with the edges. Some heads were obtained. 6, 2 6, 1 1. 608 Network Models Theorem 3.11 At termination, Algorithm 2.4 produces a maximal flow. Exercises In Exercises 1–16, let the universe be the set U = {1, 2, 3, . The center of the circle Sets and Logic Section 2 Review 25. p ∨ r 45. Set: any collection of objects 2. ¬q Hypothesis 3. Thus the incidence matrix for the graph of Figure 5.4 is v1 e1 v4 e3 v2 e2 e4 v3 e5 e7 v5 Figure 5.4 The graph for Example 5.5. e6 ⎛e1 v1 1 v2 ⎜ ⎜0 v3 ⎜ ⎜0 v 4 ⎝1 v5 0 e2 1 0 0 1 0 e3 1 1 0 0 0 e4 0 1 0 1 0 e5 0 1 0 0 1 e6 0 0 1 0 1 e7⎞ 0 1⎟ ⎟ 0⎟ ⎟. A simple graph is a graph with neither loops nor parallel edges. The algorithm divides the board into four (n/2) × (n/2) subboards. Let Sn denote the number of n-bit strings that do not contain the pattern 000. ■ When trying to construct a proof, review the proofs of similar and related theorems. What is the value of pS+T ? We arbitrarily choose a vertex, say B, and choose two edges incident on vertex B. Define a function f from G to Znonneg by the rule f(G) = |E|. Suppose that the host forgets which door hides the car and, after the contestant chooses a door, picks a door at random. Boolean Algebras and Combinatorial Circuits Richard Johnsonbaugh 623 12. Minimal spanning tree 23. , sn }, invoke this function as r comb2(r, s, 1, n, λ), where λ is the null string.] r comb2(r, s, k, n, α) { if (r == 0) { println(α) return } if (k == n) { println(α, sn) return } β = α + " " + sk // print r-combinations containing sk r comb2(r − 1, s, k + 1, n, β) // print r-combinations not containing sk if (r ≤ n − k) r comb2(r, s, k + 1, n, α) } Section 5 Review 1. p ∨ q It follows that a = c and, hence, b = d. c Optimizing at each iteration does not necessarily give an optimal solution to the Figure 4.4 A graph that shows that original problem. , pn , are equivalent, prove p1 → p2 , p2 → p3 , . If we sum the products of all the selections, we obtain (a + b)3 = (a + b)(a + b)(a + b) = aaa + aab + aba + abb + baa + bab + bba + bbb = a3 + a2 b + a2 b + ab2 + a2 b + ab2 + ab2 + b3 = a3 + 3a2 b + 3ab2 + b3 . Example 3.6 We prove the following using resolution: 1. Example 3.2 If we define p : The Mathematics Department gets an additional $60,000, q : The Mathematics Department will hire one new faculty member, 21 Sets and Logic then proposition (3.1) assumes the form (3.2). ∀x∃y P(x, y) 78. By part (a), 56. Order the 2n items. Included in this column were an encoded message using the key z, n, where z was the product of 64- and 65-digit primes, and n = 9007, and an offer of $100 to the first person to crack the code. Write the negation of each proposition in Exercises 28–33 symbolically and in words. A conflict for p(k) now means that for some i < k, p(i) = p(k), that is, the value p(k) has already been assigned. After j is set to n, we arrive at the while loop for the first time. Since 27 is greater than 16, 27 moves one index to the right: 8 13 20 27 We next compare 16 with 20. Since we are given P(J) and P(A), we need only compute P(J ∩ A). e1 v1 v2 9. Letter Character 30. The statement is false. ∃x(x2 > x) x2 > x) 52. Define free tree. Given a (2i + 1)-vertex full binary tree T , if we eliminate all the terminal vertices, we obtain an i-vertex binary tree T . (For Exercise 8) 11 51. Spanning tree 18. Explain why, when testing whether an integer n > 1 is prime by looking for divisors, we need only check whether any of 2 √ to n divides n. Since Farley can choose values for the variables so that you cannot win, the quantified statement is false. Define terminal vertex in a rooted tree. Assume that the values of rand are 218 57 72 101 135. 14567 In Exercises 4–6, find the permutation that will be generated by Algorithm 4.14 after the permutation given. , sj // r is the position in the sequence ci , . By the inductive hypothesis, the component containing v1 has an Euler cycle C and the component containing v2 has an Euler cycle C beginning and ending at v2 . 348 37. At line 5, since key ('G') is equal to s2 ('G'), we return 2, the index of key in the sequence s. Twelve basketball players, whose uniforms are numbered 1 through 12, stand around the center ring on the court in an arbitrary arrangement. We illustrate with an example and then explain the technique in detail. Thus dec val becomes 13. By Theorem 3.7, there exist s and t such that 1 = sm + tn. swap(sm , sn) 19. Let v and w be distinct vertices in Kn . Draw a two-level game tree, with the root having an X in the center square. 5 < 9 and 9 < 7. For example, 4/6 is not in lowest terms because the greatest common divisor of 4 and 6 is 2, not 1. Does f(n) = (g(n)) define an equivalence relation on the set of real-valued functions on {1, 2, . If we select no edges, we may select one or both vertices yielding the subgraphs G1 , G2 , and G3 shown in Figure 2.6. If we select the one available edge e1 , we must select the two vertices on which e1 is incident. Mathematicians who did not co-author a paper with Erd˝os but who co-authored a paper with a mathematician whose Erd˝os number is one are said to have Erd˝os number two. If the computer has one gigabyte of memory, then it can run "Blast 'em." If the computer can run "Blast 'em," then the sonics will be impressive. Prove that there exists a real number x satisfying a < x < b. When m = n and n > 1 18. An Euler cycle in a graph G is a cycle that includes all of the edges and all of the vertices of G. The equation an = f(n)an−1 + g(n)an−2 (2.23) is called a second-order, linear homogeneous recurrence relation. p q r p∨q ¬p ∨ r q∨r T T T T F F F T T T F F T T F F T F T F T F T F T T T T T T T T F F T F T T T T T T T T T F T T T F 2. , n + k}, sm+i = tn+i for i = 0, . [Requires the integral] 43. We locate this quantity by moving horizontally to the supply curve (see Figure 1.5). an = an−1 + 1 + 2n−1 29. That is, an ≤ an/2 + a(n+1)/2 + n − 1. Example: If the sequence is 6.2 8.9 4.2 8.9, the algorithm returns the value 2. Codd, is based on the concept of an n-ary relation. Thus our result shows that the 5-cube contains a subgraph isomorphic to the graph in Figure 6.2.) 10 v y 11 v 01 00 000 001 010 110 111 v x 011 Figure 6.2 Mesh model for parallel computation. As an example, the strings related to 10110100 are 10110000, 10110001, 10110010, 10110011, 10110100, 10110101, 10110110, 10110111, 10111000, 10111001, 10111010, 10111011, 10111100, 10111101, 10111110, 10111111. Has n simple cycles of length k 481 Graph Theory 21. Thus, if a conclusion follows from a set of hypotheses, resolution will be able to derive a contradiction from the hypotheses and the negation of the conclusion. Any cycle C in the n-cube has even length since the vertices in C alternate between an even and an odd number of 1's. Suggest ways to make Algorithm 1.8 more efficient. Such programs generate numbers that appear random and are called pseudorandom numbers. Once—when we choose the five slots with 0's and 1's for the remaining slots. Adding an additional edge to C will give some vertex in H degree greater than 2. q2 = 12, 44. v1 13. Functions, Sequences, and Relations Hints/Solutions to Selected Exercises Section 1 Review If f is the function of Exercises 13 and 14, we have 1. Since n mod 1 is equal to 1, result is set to (result∗x) mod z = 16 mod 21 = 16. Which properties of an algorithm— input, output, precision, determinism, finiteness, correctness, generality—are present? By the definition of intersection, since x ∈ X ∩ Y , x ∈ X and x ∈ Y . Give a direct proof along the lines of the second proof in Example 1.13 of the statement X ∩ (Y − Z) = (X ∩ Y) − (X ∩ Z) for all sets X, Y , and Z. Show that R is an equivalence relation on X. Composition of functions: f ◦ g = {(x, z) | (x, y) ∈ g and (y, z) ∈ f } 13. Disprove the statement: For every positive integer n, n2 ≤ 2n . We count the number of comparisons at each line and then sum these numbers to obtain the total number of comparisons bn . 10-3]. If b/k > 1, (2.4) shows that the differences between successive prices increase. Vertices b, c, d, e, and f represent intermediate pumping stations. If a + b = a + c, then b = c. This time we must verify that the statement if x > 1, then x + 1 > 1 is true for every real number x. Input: a, n Output: a (shuffled) shuffle(a, n) { for i = 1 to n − 1 swap(ai , arand(i,n)) } Example 2.5 Suppose that the sequence a 17 9 5 23 21 is input to shuffle. Prove that if an is the number of times the binary search algorithm (Algorithm 3.2) is invoked in the worst case for a sequence containing n items, then an = 2 + lg n for every positive integer n. In addition, duplicates are eliminated. In general, we have an = an−k + k · 3. Suppose that R is a partial order on a set X. † 68. 213 Algorithms Sorting To sort a sequence is to put it in some specified order. 101 135. Write an algorithm that returns the index of the first occurrence of the value key in the sequence s1 , . Inclusive-or is the intended meaning. σ(τ(R1)) = τ(σ(R1)) 62. Determinism—the intermediate results of each step of execution are unique and are determined only by the inputs and the results of the preceding steps. Example 7.2 Show that the graph K3,3 of Figure 7.1 is not planar. We put (2, 6) in S and (6, 3) in R. That is, given a data item D, we can easily determine if D is in a binary search tree and, if it is present, where it is located. Similarly, if the proposition ¬(∀x P(x)) is false, the proposition ∃x ¬P(x) is false. 283 Introduction to Number Theory 55. Then x ∈ A or x ∈ B ∪ C. , in so that if i occurs, so do 1, 2, . 71. 14 Sets and Logic Definition 2.1 Let p and q be propositions. 4k2 107. Example 8.1 The function f from the vertex set of the tree T1 shown in Figure 8.1 to the vertex set of the tree T2 shown in Figure 8.2 defined by f(a) = 1, f(b) = 3, f(c) = 2, f(d) = 4, f(e) = 5 is a one-to-one, onto function that preserves the adjacency relation. If we apply this algorithm to the weighted graph of Figure 4.4 to find a shortest path. At line 7, key < sk ('G' < 'J') is true, so at line 8 we set j to 2. Must e be contained in every minimal spanning tree? The graph of Figure 2.2 5. Formally, we define a function to be a particular kind of set of ordered pairs. 112. The function of Exercises 13 and 14 is one-to-one and onto {0, 1, 3}. Then 2 + 4 = 6 start with either a or b. Exercises 45–48 refer to Algorithm 3.13, with the following inserted after line 6. This exercise shows another way to guess a formula for 1 + 2 + · · · + n. To shuffle s1 , . If every pair of elements in X is comparable, we call R a total order. In a planar graph, each edge belongs to at most two bounding cycles. Which of Exercises 77–79 must also be false? Inclusion-Exclusion Principle Section 2 16. If E and F are events and P(F) > 0, the conditional probability of E given F is P(E | F) = P(E ∩ F)/P(F). Definition 1.1 592 A transport network (or more simply network) is a simple, weighted, directed graph satisfying: Network Models b c 2 3 4 2 a = dock z = refinery 5 4 d e 2 Figure 1.1 A transport network. X × (Y − Z) = (X × Y) − (X × Z) for all sets X, Y , and Z. Not reflexive, not symmetric, and transitive 54. If I study hard, then I get A's or I get rich. ∃x∀yL(x, y). Unfortunately, resolution does not tell us which clauses to combine in order to deduce the contradiction. Product notation: 24. (a + b)n+1 = (a + b)(a + b)n = (a + b) n k=0 n C(n + 1, k)an+1−k bk x1 + x2 + · · · + xk+2 = n − k 1 . , si . If we delete an edge from T 's cycle, T will remain connected. The z's are thus found twice: once among the x's and again among the y's. Exercises 4 –13 refer to dice that are loaded so that the numbers 2, 4, and 6 are equally likely to appear. If it is acyclic, we stop. For example, if a vertex v has degree 2, both edges incident on v must be in the cycle. a b f c d e g v2 h i Show that each graph in Exercises 8–10 has a path from a to a that passes through each edge exactly one time by finding such a path by inspection. Breadth-first search can also be used to find minimum-length paths in an unweighted graph from a fixed vertex v to all other vertices (see Exercise 24). Let C1 = ak + ak−1 + · · · + a1 + a0 . Equal 83. [Photo by the author; pine cone courtesy of Andr´e Berthiaume and Sigrid (Anne) Settle.] For the basis step (n = 1), we must show that 1 fk = f3 − 1. Note that if β is a substring of α, γ is the part of α that precedes β (in α), and δ is the part of α that follows β (in α). We then get the next word in the list, compare it with the root, move left or right, compare it with the new vertex, move left or right, and so on, and eventually store it in the tree. Compute the squares of the adjacency matrices of K5 and the graphs of Exercises 1 and 3. E given F is the event E given that event F occurred. If we choose the other vertex labeled 12, we will obtain the tree of Figure 1.16. Suppose that for every pair of distinct vertices v1 and v2 in G, there is a unique vertex w in G such that v1 and w are adjacent and v2 and w are adjacent. The for loop in Algorithm 2.3 always executes n − 1 times, but the number of times that the while loop executes for a particular value of i depends on the input. Let v1 , . Example 5.3 The following are propositional functions. If a graph does not have a Hamiltonian cycle, you can sometimes use the fact that a Hamiltonian cycle in a graph containing n vertices has length n together with proof by contradiction to prove that it does not have a Hamiltonian cycle. Now n = 4 is the Basis Step. After introducing algorithms and providing several examples, we turn to the analysis of algorithms, which refers to the time and space required to execute algorithms. Theorem 2.17 If a graph G has an Euler cycle, then G is connected and every vertex has even degree. Place one tromino T in the center, as shown in Figure 4.6, so that each square of T is in each of the other quadrants. Definition 2.3 The truth value of the proposition p ∧ q is defined by the truth table p q p∧q T T F F T F T F T F F F 15 Sets and Logic Notice that, in the truth table in Definition 2.3, all four possible combinations of truth assignments for p and q are given. A trace of an algorithm is a simulation of execution of the algorithm. U 9. , n = n0 + p − 1. In Exercises 38–42, for each pair of graphs, give an example of a homomorphism from G1 to G2 . The value of m/b is the quotient when m is divided by b. Show that if dc | nc, then d | n. Since c R d and R is transitive, x R d. We must then show that 5n+1 − 1 is divisible by 4. Now AO(x + 1, 2, 3) − 3 = AO(x, 2, AO(x + 1, 2, 2)) − 3 by definition = AO(x, 2, 4) − 3 by Exercise 49 = A(x, 1) by the inductive assumption on x = A(x + 1, 0) by (1.11). Give an example of a connected graph. Basis Step (n = 1) 1 = f1 Inductive Step Suppose that n > 2 and that every positive integer less than n can be expressed as the sum of distinct Fibonacci numbers, no two of which are consecutive. To solve 100n(1.2n − 1) = 1.2 − 1 i=0 an = c1 an−1 + c2 an−2 , − 120 first solve the equation t 2 = c1 t + c2 n−1 i · 1.2i−1 i=1 (n − 1)1.2n − n1.2n−1 + 1 , (1.2 − 1)2 29. Define connected graph. According to Definition 3.2, if f(n) = O(g(n)), all that we can conclude is that, except for a constant factor and a finite number of exceptions, f is bounded above by g, so g grows at least as fast as f. Let ai denote the position of the ith unavailable item. At this point, we cannot add an edge of the form (h, x), so we backtrack to the parent f of h and try to add an edge of the form (f, x). The number of contestants is the same as the number of terminal vertices, and the number of matches i is the same as the number of internal vertices. Therefore, in the worst case, if we begin by weighing four coins against four coins, the twelve-coins puzzle requires at least four weighings. 420 1. (a) Show that C(n, k) < C(n, k+1) if and only if k < (n−1)/2. 510 Trees Book Author Location Publisher Availability Figure 1.9 A hierarchical definition tree. To show that two graphs are not isomorphic, find an invariant that one graph has and the other does not have. State De Morgan's laws for sets. Draw the decision tree. Write a recursive algorithm to find the minimum of a finite sequence of numbers. Suppose that the possible classes are C1 , . E = {e1 , e2 , e3 , e4 , e5 , e6 }. For n ≥ N, f(n) ≤ C g(n) ≤ Cg(n). T1 and T2 are isomorphic if and only if their left and right subtrees are isomorphic. Explain why the graph does not have a path from a to a that passes through each edge exactly one time. Similarly, if the prime pi is not a factor of n, we let bi = 0.) By Theorem 1.17, 1 ,b1) k ,bk) gcd(m, n) = pmin(a · · · pmin(a , 1 k and by Theorem 1.22, 1 ,b1) k ,bk) lcm(m, n) = pmax(a · · · pmax(a . 29. T. We first prove that if n is even, then n + 2 is even. This is a contradiction. k=0 P(at least one passes) = 1 − P(both fail) 24. To determine these constants, use the initial conditions, say a0 = C0 , a1 = C1 . However, C(13, 5) contain only spades and C(13, 5) contain only hearts. Step 3—move disk 3 from peg 3 to peg 2. 81. Since G is connected, there is a path P in G from v to w. (8.4) The 90 numbers in (8.3) and (8.4) have possible values only from 1 to 89. Continue until the sequence is ordered. dx dx dx Use mathematical induction to prove that dxn = nxn−1 for n = 1, 2, . Tile the deficient 7 × 7 subboard using the result of Exercise 35, Section 4. Person 2 can't get order 1 (for then person 3 would get the correct order); thus, person 2 gets order 3. Let L denote the sum of the numbers of vertices in the two left subtrees of the trees input and let R denote the sum of the numbers of vertices in the two right subtrees of the trees input. , Pn are replaced by an arbitrary family P(x), where x is in the domain of discourse, and (5.3) is replaced by ∀x P(x). Since m mod b = 1 mod 2 = 1, c3 is set to 1. The circuit must be capable of being in two states—one representing 1, the other 0. n factors Example 1.6 308 Let X be an n-element set. Thus the general solution of (2.18) is of the form U = aS + bT. Example 1.26 Prove that the function f(n) = 2n + 1 from the set of positive integers to the set of positive integers is one-to-one. (We assume now that a1 < a2 and a3 < a4 .) We then compare a2 and a4 . We must find positive integers n1 and n2 , n1 = n2 , such that f(n1) = f(n2). The product of n × n matrices A and B (i.e., A and B have n rows and n columns) is defined as the n × n matrix C, where n Cij = Aik Bkj , 1 ≤ i ≤ n, 1 ≤ j ≤ n. Here is a proposed algorithm that checks whether Goldbach's conjecture is true: 1. If x = −3, then x2 + x − 6 = (−3)2 + (−3) − 6 = 0. Theorem 3.7 states that there exist integers s and t such that gcd(a, b) = sa + tb. } Our algorithms consist of a title, a brief description of the algorithm, the input to and output from the algorithm, and the functions containing the instructions of the algorithm. Find the 5-combination that will be generated by Algorithm 4.9 after 12467 if n = 7. 4 6 a sequence so that adjacent permutations 2 26. Example 4.2 Suppose that we choose p = 23, q = 31, and n = 29. What is the probability that each person receives at least two compact discs? √ Thus, q ≤ n. A de Bruijn sequence for n (in 0's and 1's) is a sequence k=0 k! . The method of disproving the statement ∀x P(x) is quite different from the method used to prove that the statement is true. In words, there is a smallest positive integer (namely 1). (b) The statement means that in order for John to take calculus, he must have sophomore, junior, or senior standing. 165 Functions, Sequences, and Relations 3 → Relations A relation from one set to another can be thought of as a table that lists which elements of the first set relate to which elements of the second set (see Table 3.1). Suppose that it is possible to mark k squares in the upper-left k × k subgrid and k squares in the lower-right k × k subgrid so that no two marked squares are in the same row, column, or diagonal of the 2k × 2k grid. When does the complete bipartite graph Km,n contain a Hamiltonian cycle? an = −3an−1 2. However, the market forces drive the price down to p1 , as we can see by moving vertically to the demand curve. Write a program that determines whether a graph contains an Euler cycle. Show that X has a least element. Give an example of two graphs that are not isomorphic. For example, (2, 3) ∈ R, but (3, 2) ∈ / R. Therefore, the total number of hands is C(4, 2)C(13, 3)2 . Disprove that P(X ∪ Y) ⊆ P(X) ∪ P(Y) for all sets X and Y . Assume that (4.13) holds for n. 12 (A ∪ B) − (C − B) What is the cardinality of ∅? k=1 25. How is the union denoted? There is no edge between Georgia and New York, nor is there an edge between Utah and New Mexico. We can also use Theorem 3.5 to solve part (b) if we first select one ball of each color. However, at the second invocation we will have i > j and the algorithm will terminate unsuccessfully at line 3. = 10. Use mathematical induction to show that for all n ≥ 1, fn is even if and only if n is divisible by 3. If there is no child to move to, we create one, put in an edge incident on v and the new vertex, and place the word in the new vertex. Existence Proofs A proof of ∃x P(x) (2.4) is called an existence proof. ¬p ∨ ¬q 18. ∅, then 8. Since Y ⊂ Z, x ∈ Z. , n by 1 in the sum 1 + 2 + · · · + n to obtain 1 + 2 + ··· + n ≥ 1 + 1 + ··· + 1 = n for all n ≥ 1. If the coin is flipped 10 times, what is the probability of no heads? 417 Recurrence Relations We may solve this last equation by using iteration (see Section 2): a2k = 2a2k−1 + 2k − 1 = 2[2a2k−2 + 2k−1 − 1] + 2k − 1 = 22 a2k−2 + 2 · 2k − 1 − 2 = 22 [2a2k−3 + 2k−2 − 1] + 2 · 2k − 1 − 2 = 23 a2k−3 + 3 · 2k − 1 − 2 − 22 .. Either someone does not play soccer or some soccer player is a professional athlete. After introducing some context and terminology in Section 1, we devote the remainder of this chapter to various proof techniques. Substituting the next-to-last formula for 5 into the last equation yields Section 3 Review 1. , n} into X will be denoted (i1 , i2 , . What is the probability that it is the ace of spades? We can show that the road inspector cannot start in Greybull, travel each of the roads exactly once, and return to Greybull. The problem for a transport network G may be stated: Find a maximal flow in G; that is, among all possible flows in G, find a flow F so that the value of F is a maximum. It follows that G = gcd(a, b) is a loop invariant. Prove that the recursive construction in Example 1.8 actually yields an n-cube. 2 · 52 · 74 · 134 · 17 6. If n ≥ 4, then 2 ≤ n/2 and so 2 ≤ n/2. A permutation is an ordering of s1 , . (The set D1 is defined before Exercise 1.) 25. Therefore n ∈ Q and Z ⊆ Q. Assuming that this is true, plug in n = 1, 2, 3 to obtain three equations in the three unknowns A, B, and C. By the definition of logarithm, log2 y = x. exp2(a, n) { if (n == 1) return a m = n/2 power = exp2(a, m) power = power ∗ power if (n is even) return power else return power ∗ a } 34. The elements making up a set are assumed to be distinct, and although for some reason we may have duplicates in our list, only one occurrence of each element is in the set. For the function f , the element a in Y is used twice. 1 2 3 x y z a b c Y X g 1 2 3 x y z X Z f Z f g Figure 1.13 The composition of the functions of Example 1.41. Now bn = bn/2 + b(n+1)/2 + 1 n n + 1 −1+ −1+1 = 2 2 by the inductive assumption = n 2 + n + 1 2 − 1 = n − 1. , xn , n ≥ 2, are real numbers satisfying x1 < x2 < · · · < xn , and x is an arbitrary real number. We discuss one model for parallel computation known as the n-cube or hypercube. Therefore, an = (lg n), so binary search is (lg n) in the worst case. A bracket, [or], indicates that the point is to be included in the graph; a parenthesis, (or), indicates that the point is to be excluded from the graph. Similarly, there are C(6, 3) eight-bit strings with no two 0's in a row that have exactly five 1's and there are C(5, 4) eight-bit strings with no two 0's in a row that have exactly four 1's in a row. What is the worst-case time of merge sort? 22. ¬p ∧ ¬q In Exercises 39–43, formulate the symbolic expression in words using 6. Since m ≤ 2n + 1, 1 m ≤ n + < n + 1. Example 2.9 The recurrence relation an − an−1 = 2n is not a linear homogeneous recurrence relation with constant coefficients because the expression on the right side of the equation is not zero. , Xn is a partition of a set X and we define x R y if for some i, x and y both belong to Xi , then R is an equivalence relation on X. We turn to a proof by contradiction. The relation is transitive if and only if whenever entry i, j in A2 is nonzero, entry i, j in A is also nonzero. The inverse of the relation {(1, 2), (1, 3)} is {(2, 1), (3, 1)}. Compare the running times of insertion sort (Algorithm 2.3) and selection sort (see Exercise 20, Section 2) for several inputs of different sizes. Since g is the least element in X and 0 < r < g, we have a contradiction. Section 3 39. Substituting p = 2k into 2q2 = p2 gives 2q2 = (2k)2 = 4k2 . Input: A sequence w1 , . Define length of a string. After presenting ways of representing graphs, we study the question of when two graphs are essentially the same (i.e., when two graphs are isomorphic) and when a graph can be drawn in the plane without having any of its edges cross. Since an equivalence class is determined by an ordering of S, the number of orderings of S is equal to the number of equivalence classes. To prove that ∀x∃y P(x, y) is true, where the domain of discourse is X × Y , you must show that for all x ∈ X, there is at least one y ∈ Y such that P(x, y) is true. Therefore [b] = ∅. Show that graphs G1 and G2 are isomorphic if their vertices can be ordered so that their adjacency matrices are equal. Let X denote the set of selections in which Alice is an officer and let Y denote the set of selections in which Dolph is an officer. Exercises 55 and 56 refer to 10 identical compact discs that are randomly given to Mary, Ivan, and Juan. If the word to be added is less than (using lexicographic order) the word at the root, we move to the left child; if the word to be added is greater than the word at the root, we move to the right child. Suppose that we store data as described in Example 1.14 and that we never store more than 10 items. Assume that the number of subsets of {1, . A set may also be defined by listing a property necessary for membership. What is the mesh model for parallel computation? Explain how Prim's Algorithm finds a minimal spanning tree. We call P a probability function. If f(n) = O(g(n)), then g(n) = (f(n)). Using this equivalent definition of "antisymmetric," we again see that the relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is antisymmetric because for all x, y, if x = y, (x, y) ∈ R (i.e., x > y) or (y, x) ∈ R (i.e., y > x). What is a capacity in a network? Although it is easy to obtain an explicit formula from the recurrence relation and initial condition for the sequence of Example 1.3, it is not immediately apparent how to obtain an explicit formula for the Fibonacci sequence. ■ If you have trouble finding examples, look at existing examples (e.g., examples in this book). In Figure 3.1, we have drawn directed edges to represent the members of the relation R of Example 3.4. Notice that an element of the form (x, x) in a relation corresponds to a directed edge from x to x. Therefore n2 + 1 is even. In Section 4, we will need to compute an mod z for large values of a and n. Then x is in X and y is in X. The source is vertex a and the sink is vertex z. The Basis Step is n = 0. In case (b), if we increase the flow in e2 by , we must decrease the flow in e1 by so that the flow into x will still equal the flow out of x. , sm) ⎜ ⎜ tj ⎜⎜ ⎜⎜.⎜⎜ .. [We call CX the characteristic function of X (in U)]. If we think of the persons as pigeons and the names as pigeonholes, we can consider the assignment of names to people to be that of assigning pigeonholes to the pigeons. Thus binary insertion sort is optimal if n = 4. 20 24. Table 3.1 lists the number of modulus operations required for some small input values. Attacking the Problem Does this type of problem sound familiar? 7 3 1 9 7 4 16 4 4 6. Write a program that generates all r-combinations of the elements {1, . We will prove that for all i ≥ 1, the greedy algorithm is optimal for all amounts of postage n ≤ 6i. Suppose that G is connected. Exercises Select a theta notation from Table 3.3 for each expression in Exercises 1–12. 40. d f e g j i 2. Trace Algorithm 4.6 for n = 4. Since any cycle must alternate between a vertex in V1 and one in V2 , any cycle in GKn must have even length. An invalid argument is an argument that is not valid. Let (b) List one member of each equivalence class of X × X. If the data item to be added is less than the data item at the current vertex, move to the left child and repeat; otherwise, move to the right child and repeat. 309 Counting Methods and the Pigeonhole Principle Example 1.9 How many eight-bit strings begin either 101 or 111? Example 3.5 (3.2) We prove the following using resolution: 1. During this part of the algorithm, the kn bottom disks on peg 1 remain fixed. En = and p(i) > p(i + 1) 1, 3, 2, 4; 1, 4, 2, 3; 2, 3, 1, 4; 2, 4, 1, 3; 3, 4, 1, 2. The point midway between a and b suffices. R = {(x, x) | x ∈ X} 60. What is the matrix of a relation? Counting Methods and the Pigeonhole Principle 12. It must then be that one of (8.1) and one of (8.2) are equal. Throughout this chapter, if G is a network, we will denote the source by a and the sink by z. To what does "analysis of algorithms" refer? The objects are sometimes referred to as elements or members. Exclusive-or: The car comes with a cupholder that heats or cools your drink but not both. No 73. Section Review Exercises †1. 1 2 3 4. The result now follows because n−1 2 n+1 2 = = (2k + 1) − 1 2 (2k + 1) + 1 2 2k(2k + 2) 4 + 4k = k2 + k. The positions for the dots are selected from two vertical columns of three dots each. 9. , si will have been rearranged so that it is sorted. In the RSA system, each participant makes public an encryption key and hides a decryption key. takes to solve the n-disk puzzle. The following graphs a b 1 2 c d 3 4 are isomorphic. Explain how to increase the flow under the conditions of Exercise 4. In honor of Euler, a cycle in a graph G that includes all of the edges and all of the vertices of G is called an Euler cycle.† From the discussion of Section 1, we see that there is no Euler cycle in the graph of Figure 2.8 because the number of edges incident on vertex A is odd. Also, determine whether the trees are isomorphic as free trees. In this case, for the graph T , (5) is true, but (6) and (7) are false, which is a contradiction since either (5), (6), and (7) are all true (and the graph is a tree), or (5) , (6) , and (7) are all false (and the graph is a not a tree). f(x) = (cos 6x)3 43. a1 , . In Exercises 4–8, assume that a person invests $2000 at 14 percent interest compounded annually. Suppose that we can remove an edge from T to obtain T so that T is connected. Thus C(n, r) = P(n, r) n(n − 1) · · · (n − r + 1) = r! r! n! = . 2 · 4 10. Theorem 3.4 The worst-case time for binary search for input of size n is (lg n). The algorithm is explained in Example 2.18. The statement ∃x P(x) is true if P(x) is true for at least one x in D. Since fk1 is the largest Fibonacci number satisfying n ≥ fk1 , fk2 ≤ fk1 . Our last example of this section was suggested by Steve Jost. Define homeomorphic graphs. Then the vertex set can be partitioned into two subsets V1 and V2 such that each edge is incident on one vertex in V1 and one vertex in V2 . Since there are two ways to select each bit, by the Multiplication Principle the total number of eight-bit encodings is 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 = 28 = 256. Label the start vertex 0 and all other vertices ∞. Therefore, the 4-cube is not planar. There is an edge from vertex v to vertex w if the action modeled by v must occur before the action modeled by w. Prove that for all rational numbers x and y, xy is rational. In representing an integer, reading from the right, the first symbol represents the number of 1's, the next symbol the number of 2's, the next symbol the number of 4's, the next symbol the number of 8's, and so on. W G W Figure 8.8 Another attempt to find a subgraph of Figure 8.6 satisfying (8.1) and (8.2). v1 C D F G 21. Compute A(2, 2) and A(2, 3). Show that fn2 = fn−2 fn+2 + (−1)n . Implement preorder, postorder, and inorder tree traversals as programs. For this purpose, let us assume that we have a computer that can execute one step in 1 microsecond (10−6 sec). Define 1 if x ∈ X CX (x) = 0 if x ∈ / X. 257 258 Introduction to Number Theory 1 2 3 4 Divisors Representations of Integers and Integer Algorithms The Euclidean Algorithm Problem-Solving Corner: Making Postage The RSA Public-Key Cryptosystem Notes Chapter Review Chapter Self-Test Computer Exercises Hints/Solutions to Selected Exercises It's a digital camera. Show that simplification (see Figure 4.1) is valid. Show that the representation (5.3) is never unique. Show that at least two persons have the same first and last names. Let A be the event "the student takes art," and let C be the event "the student takes computer science." Then A ∪ C is the event "the student takes art or computer science or both," and A ∩ C is the event "the student takes art and computer science." The next theorem gives a formula for the probability of the union of two events. C(k + t − 1, t − 1). Show that 1 (−1)i (k − i)n C(k, i), k! k t2 = 2. Again, we cannot add an edge of the form (f, x), so we backtrack to the parent e of f . An n-tuple, written (a1 , a2 , . 2/38 45. How can we quickly determine whether a relation R is a function by examining the matrix of R (relative to some ordering)? For example, the string 0111 determines the equivalence class consisting of all eight-bit strings that begin 0111. Since adding y to T − {x} produces a connected graph, (T − {x}) ∪ {y} is a spanning tree. Write an algorithm that receives as input the matrix of a relation R and tests whether R is symmetric. i! 2! 3! 4! i=0 Thus for large n, the probability that no one receives the correct order is approximately e−1 = 0.368. Find the probability that a person has a headache given that the person has a fever. The entries in a book's index are sorted in increasing order, thus making it easy to quickly locate an entry in the index. , cn } of positive integers and a positive integer M, find all subsets {ck1 , . Divide the board into four 2k × 2k boards, as shown in Figure 4.6. Rotate the board so that the missing square is in the upper-left quadrant. A fine city. Information from the database is obtained by manipulating the tables. 56. Thus, if we set s = 17 and t = −14, s · 396 + t · 480 = gcd(396, 480) = 12. The literal meaning is: Everything is not sweetness and light. [Second solution] We can avoid writing the truth table by directly verifying that whenever the hypotheses are true, the conclusion is also true. Encrypt the message I AM NOT A CROOK using the key of Example 4.1. 4. A similar argument gives tilings for the remaining cases. Algorithm 3.3 correctly finds the greatest common divisor if lines 3 and 4 are omitted (see Exercise 13). Since T is a spanning tree, T is a minimal spanning tree. Preorder traversal processes the vertices of a binary tree by beginning at the root and recursively processing the current vertex, the vertex's left subtree, and then the vertex's right subtree. Since these three cases are pairwise disjoint, by the Addition Principle, there are 20 + 20 + 20 = 60 possibilities. Suppose that the weights of the edges incident on v are distinct. 1+ n (−1)k+1 k=1 fk fk+1 fk2 = fn fn+1 for all n ≥ 1. Let E denote the event "sum is 6," and let F denote the event "at least one die shows 2." Then P(E ∩ F) = P((2, 4)) + P((4, 2)) = 2 1 12 2 2 , 144 = and P(F) = P((1, 2)) + P((2, 1)) + P((2, 2)) + P((2, 3)) + P((2, 4)) + P((2, 5)) + P((2, 6)) + P((3, 2)) + P((4, 2)) + P((5, 2)) + P((6, 2)) = 3 12 + + + 1 12 1 12 3 12 1 12 2 + 3 12 1 12 + + 1 12 3 12 1 12 + 2 + + 1 12 1 12 3 12 1 12 2 1 12 = 23 . Since f(m, n) ≥ 0 for all m, n ∈ Z, f(m, n) = −1 for all m, n ∈ Z. Describe an optimal strategy for the winning player. Note that [1] = [4] = [7] = [10], [2] = [5] = [8], [3] = [6] = [9]. Your goal is to try to make P(x, y) true, and Farley's goal is to try to make P(x, y) false. If x ∈ A, then x is one of 3, 2, 1. 7b. Finally, the proof of Bayes' Theorem shows that P(H) = P(H | D)P(D) + P(H | R)P(R) + P(H | L)P(L) = (0.2)(0.4) + (0.55)(0.25) + (0.3)(0.35) = 0.3225. A sequence is a special kind of function. (For Exercise 8) 9 35. Example 4.6 Consider the order in which the 5-combinations of {1, 2, 3, 4, 5, 6, 7} will be listed. Example 2.4 6. If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. Step 5—move disk 3 from peg 2 to peg 1. Here, to be sure that Maria visits France, it suffices for her to go to the Eiffel Tower. If x ∈ Y − X, then x ∈ Y , so again x ∈ X ∪ Y . Note the similarity of the tiling algorithm to the inductive step of the proof that every n × n deficient board can be tiled with trominoes when n is a power of 2. It's very common in a graph model for the edges to indicate a relationship between the vertices. We turn next to an example that shows how a graph model can be used to analyze a manufacturing problem. Let x ∈ (A∪B)∪C. c0 = 0 70. Suppose that a relation R on a set X is symmetric and transitive but not reflexive. 424 4. We first prove that if n ≥ 5, then Cn is not prime. Figure 8.3 gives a solution to regular Instant Insanity for Figure 8.5. Chapter Self-Test 1. The method of the Euclidean algorithm can be used to prove Theorem 3.7 and to compute s and t. We assume that v1 corresponds to the upper-left square. Here, the given statement means that if the Cubs win the World Series, we can be sure that they signed a right-handed relief pitcher since, without such a signing, they would not have won the World Series. We show that a1 = 7 and a2 = 3 provide a solution. Does the conclusion to Example 8.5 necessarily follow if there are fewer than six pictures? Figure 1.2 20. Let R and S be relations on X. lg(nk 58. Then C(n+1, k) is the number of k-element subsets of Y = X ∪ {a}. After the contestant chooses a door, the host opens one of the other two doors that hides a goat. A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge e ∈ E is associated with an ordered pair of vertices. We label each terminal vertex with the value of the position to the first player. (Any vertex in G is either in V or in W.) We define the deficiency of G as δ(G) = max{|S| − |R(S)| | S ⊆ V }. Lines 2 and 3 correspond to initial condition (1.5), and line 4 corresponds to recurrence relation (1.4). Example 5.5 showed that n − 1 games are played. (4) si == key is false, but in the algorithm, the relation si == key evaluates to true.] Write an algorithm that uses at most 3 + 2lg n array comparisons to determine whether key is in the sequence. Isomorphic as rooted trees. S. Each placement of these symbols into the slots determines a selection. Since r is in X, r ≥ 0. Now X ∩ Y is the set of selections in which both Alice and Dolph are officers. We call the value on which the system is based (10 in the case of the decimal system) the base of the number system. Answer Fibonacci's question. , s10 range in value from 0 to 8. Suppose that at the beginning of the year, there is one pair of rabbits and that every month each pair produces a new pair that becomes productive after one month. Thus, no pair can require more than 28 modulus operations in the worst case because 29 modulus operations would require one member of the pair to exceed Introduction to Number Theory 1000000. If the condition does not hold, the outcome might be achieved in other ways or it might not be achieved at all; but if the condition does hold, the outcome is guaranteed. p ∨ ¬r C and D. Therefore, the algorithm returns 11 to indicate that 451 is composite and 11 divides 451. Describe the symmetric difference of sets A and B in words. Therefore f is onto. Therefore, the number s in equation (3.14) is unique. bn = 4n − 3 72. Since three edges touch Wor, we have a contradiction. Let us define the worst-case time to sort to be the number of comparisons in the worst case. Since there are n − 1 vertices remaining, if the left subtree has k vertices, the right subtree must have n − k − 1 vertices. d is the vertex preceding c on P1 . Since i = j − 1, the term ir n−i becomes (j − 1)r n−(j−1) = (j − 1)r n−j+1 . Since G is connected, there are vertices v1 , v2 , and v3 in G with edge e1 incident on v1 and v2 and edge e2 incident on v2 and v3 . Write a program that tiles with trominoes an n × n board with one square missing, provided that n = 5 and 3 does not divide n. Since the matrix is symmetric about the main diagonal (the elements on a line from the upper-left corner to the lower-right corner), the information, except that on the main diagonal, appears twice. What is a valid argument? To prove that ∃x∃y P(x, y) is true, where the domain of discourse is X × Y , find one value of x ∈ X and one value of y ∈ Y (two values suffice—one for x and one for y) that make P(x, y) true. A beta value for a vertex v is only an upper bound for the value of v. Therefore, (2.1) and (2.2) are equivalent. For this reason, we sometimes simply write log without specifying the base. if (v has no right child) { add a right child r to v store wi in r search = false // end search } else v = right child of v } // end while } // end for return T Binary search trees are useful for locating data. Example 2.19 v1 v2 v4 v3 v6 v5 v7 Figure 2.10 The graph for Example 2.19. Conversely, if we assign each element Counting Methods and the Pigeonhole Principle of X to one of the three sets A (and, by assumption, also to B and X), B − A (and, by assumption, also to X), or X − B, we obtain a unique ordered pair (A, B) satisfying A ⊆ B ⊆ X. ways to partition the (n + 2)-sided polygon is Pn = Pk−1 Pn−k . Input: A connected, weighted graph with vertices 1, . h(x) = x mod 11; cells indexed 0 to 10; data: 53, 13, 281, 743, 377, 20, 10, 796 56. 117. Show that (n/k)k ≤ C(n, k) ≤ nk /k!. Write a recursive algorithm that initializes each vertex of a binary tree to the number of its descendants. In either case, max{x, y} + min{x, y} = x + y. Prove that R is an equivalence relation on RR . If R is symmetric, then R−1 is symmetric. Compute the given quantity using the strings α = baab, Use Exercises 125 and 126 and mathematical induction to prove that the sum γ = bbab. We evaluate children left to right. What is the value of E3 ? How might the output be analyzed to determine if it is truly "random"? Trees 13. Is f onto X∗ ? (2.4) If we next divide 22 by 2, we find 22 = 2 · 11. Show that a1 = 1, a2 = 2, and an − an−1 = an−2 . An eight-bit string that begins 101 can be constructed in five successive steps: Select the fourth bit; select the fifth bit; . Therefore, CX (x) + CY (x) = 1 + 0 = 1 = CX∪Y (x) or CX (x) + CY (x) = 0 + 1 = 1 = CX∪Y (x). In words, x is related to y in R if and only if y is related to x in R−1 . Algorithms written for more than one processor, whether for a multiprocessor machine or for a distributed environment (such as the Internet), are rarely deterministic. probability of obtaining at least one defective microprocessor. Using a previous theorem, we may add these inequalities to obtain x + y < 1 + 1 = 2. First we show that if there is a path from a to a vertex w whose length is less than L(v), then w is not in T (i.e., w was previously selected at line 9). , f(vk)), which here we denote f(C), is a simple cycle of length k in G1 . When n = 1, the Gray code 0, 1 corresponds to the path (0, 1, 0), which is not a cycle because the edge (0, 1) is repeated. Instead of proving (p1 ∨ p2 ∨ · · · ∨ pn) → q, in proof by cases, we prove (p1 → q) ∧ (p2 → q) ∧ · · · ∧ (pn → q). Write a program that generates all r-permutations of the elements {1, . Functions, Sequences, and Relations Example 1.5 1 2 3 a b c X Y The set {(1, a), (2, b), (3, c), (1, b)} is not a function from X = {1, 2, 3} to Y = {a, b, c} because 1 is not assigned a unique element in Y (1 is assigned the values a and b). Find s4 . Notice that if n ≥ 2, n factorial can be written "in terms of itself" since, if we "peel off" n, the remaining product is simply (n − 1)!; that is, n! = n(n − 1)(n − 2) · · · 2 · 1 = n · (n − 1)!. By the Pigeonhole Principle, each column contains at least one colorful pair. Tile the 5 × 5 subboard with a corner square missing using the result of Exercise 32, Section 4. Use the evaluation function E, which assigns a position the value NX − NO P Figure 9.5 The value of position P is E(P) = NX−NO = 2−1 = 1. Use Exercise 32 to give another derivation of the explicit formula for the nth Catalan number Cn = C(2n, n)/(n + 1). Every discrete math student attended someone's office hours. Establish the recurrence relation sn+1,k = sn,k−1 + nsn,k for Stirling numbers of the first kind. Definition 3.3 The truth value of the conditional proposition p → q is defined by the following truth table: p q p→q T T F F T F T F T F T T For those who need additional evidence that we should define p → q to be true when p is false, we offer further justification. ¬p ∨ q ¬q ∨ ¬r p ∨ ¬r ∴ ¬r Computer Exercises 1. Thus fn+2 ≤ m. ■ Randomly selecting five microprocessors from a lot of 1000 microprocessors. If x = y or y = z, we need not explicitly verify that the condition if (x, y) and (y, z) ∈ R, then (x, z) ∈ R is satisfied since it will automatically be true. , xn) is false, (1.1) is vacuously true; thus, we need only consider the case that p(x1 , x2 , . E = {e1 , e2 , e3 }. Toots starts with T pennies, and Sly starts with S pennies. Let T be a tree. , n} and choose a subset so that the sum (4.1) does not exceed C and is as large as possible. At this point, the secondlargest value is found. The terminal vertices give the solution. For example, a vertex might model a statement in a computer program. How many 3-combinations are there of six objects? The 220 numbers a1 , . Therefore, there are C(26, 5) − 2C(13, 5) ways to select five cards containing cards in spades and hearts. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)}, [1] = [2] = {1, 2}, [3] = [4] = {3, 4} 18. The bottleneck in Algorithm 2.16 is the size of the numbers involved. Show that the dual of a planar map is a planar graph. Give an example of a function different from those presented in the text that is one-to-one but not onto, and prove that your function has the required properties. ∴ Floppy disk storage is better than a hard disk drive. Since x1 ≤ x ≤ xn , |x − x1 | = x − x1 and |x − xn | = xn − x. At each iteration, we add an edge e to T having minimum weight that does not complete a cycle in T. In the fourth case, if a3 is greater than a2 , we know that it goes between a2 and a4 . R × R 66. for i = 1 to n if (¬P(di)) return false else return true ∀x ((I(x) → P(x)) ∨ (P(x) → I(x))) What is the literal meaning of each statement in Exercises 57–66? To prove that the existentially quantified statement ∃x P(x) is true, find one value of x in the domain of discourse for which the proposition P(x) is true. For which n is algorithm B superior to algorithm A? Since G has one more face than G , one more edge than G , and the same number of vertices as G , it follows that (7.3) also holds for G. Notice that if person 1 gets order 2 and person 2 gets order 1, the number of ways for the other persons to get the wrong orders is Dn−2 (the remaining n − 2 persons must all get the wrong orders). Since T is not connected, k > 1. n1 !n2 ! · · · nt ! To determine whether one of these formulas may be relevant to a particular problem, first be sure that the problem asks for orderings. + 2n; thus (b) We must have Sn = Sn−1 39. A counterexample is n = 7. Before continuing, let's develop some notation for the distribution of the orders. Since A ∩ B = ∅, x ∈ / B. Vertex B is 2, the minimum of its children. c = an mod z = 10131 mod 391 = 186 12. Use resolution to prove 16. For each x ∈ T adjacent to v, relabel x with the minimum of its current label and the label of v + w(v, x), where w(v, x) is the weight of edge (v, x). The converse can be written symbolically as q → p. v3 e5 e6 v5 v6 Definition 1.1 allows distinct edges to be associated with the same pair of vertices. The method of Example 3.3 can be used to show that a polynomial in n of degree k with nonnegative coefficients is (nk). (If you choose x = 0, Farley can choose y = 1. Let R1 be a relation from X to Y and R2 be a relation from Y to Z. The relation R is symmetric if and only if for all i and j, the ijth entry of A is equal to the jith entry of A. , n, n must occur in position 2i, for some i. Since there are C(k + t − 1, t − 1) ways to select the positions of the |'s, there are also C(k + t − 1, t − 1) selections. Show that 2n = O(n!). (How does this complete the proof?) Is this proof constructive or nonconstructive? By Theorem 3.7, there exist integers s and t such that sp + tq = 1. 110, 273 3. mean For all x, ¬P(x). Subtracting 1 from both sides of the equation and then dividing both sides of the equation by 2 yields n1 = n2 . [Second solution] Let us consider the activity of assigning Egbert and two others to offices to be made up of three successive steps: Assign Egbert an office; fill the highest remaining office; fill the last office. an = 2n+1 − 4n 44. Each term is, however, a variable. Find z2 . test distinct(a, b, c) { if (a == b ∨ a == c ∨ b == c) return false return true } fki i=1 = fk1 + fk2 + m fki i=3 = fk1 + fk1 −1 + = fk1 +1 + m m fki i=3 fki . bn = n − 1 32. The value 1 is a counterexample to the statement ∀x(x2 − 1 > 0). Give an informal, intuitive description of the meaning of the event E given F . How do we denote the empty set? In how many ways can we place 10 identical balls in 12 boxes if each box can hold one ball? Taking n = 3 in Theorem 7.1, we obtain Example 7.2 (a + b)3 = C(3, 0)a3 b0 + C(3, 1)a2 b1 + C(3, 2)a1 b2 + C(3, 3)a0 b3 = a3 + 3a2 b + 3ab2 + b3 . Since the first element is a zero, all of the zeros precede all of the ones for all n elements. The Basis Step has been verified. Exercises In Exercises 1–3, find the capacity of the cut (P, P). 595 Network Models Example 1.9 A Traffic Flow Network It is possible to go from city A to city C directly or by going through city B. In a group of students, each student is taking a mathematics course or a computer science course or both. For example, in Java "Let's read Rolling Stone." denotes the string consisting of the sequence of characters Let's read Rolling Stone. C(12 + 4 − 1, 12) − [C(7 + 4 − 1, 7) + C(6 + 4 − 1, 6) + C(3 + 4 − 1, 3) + C(2 + 4 − 1, 2) − C(1 + 4 − 1, 1)] 36. Another algorithm to find a minimal spanning tree, known as Kruskal's Algorithm, is presented in the exercises (see Exercises 20–22). The argument is identical to that given in Theorem 2.11. Dividing by n gives s1 + s2 + · · · + sn < A, n which contradicts the hypothesis A= s1 + s 2 + · · · + s n . We try to construct a counterexample in which R and S are transitive but there exist (x, y) ∈ R and (y, z) ∈ S such that (x, z) ∈ R ∪ S. The length of a path in a weighted graph is the sum of the weights of its edges. A Venn diagram can suggest whether a statement about sets is true or false. Edge e4 is incident on Wor and Buf, and the vertices Wor and Buf are adjacent. Let P(x) be the propositional function "x flies." The given statement can be written symbolically as ∃x ¬P(x). To obtain the matrix of the relation R2 ◦ R1 , we need only change all nonzero entries in A1 A2 to 1. The minimum d of two numbers, d1 and d2 , is equal to one of the two numbers (namely, the smallest) and less than or equal to the other one (namely, the largest)—in symbols, d ≤ d1 and d ≤ d2 . If we take x to be the positive square root 1 x= √ , y 144 Functions, Sequences, and Relations √ then x ∈ X. Example 1.20 The least common multiple of 30 and 105, lcm(30, 105), is 210 because 210 is divisible by both 30 and 105 and, by inspection, no positive integer smaller than 210 is divisible by both 30 and 105. The domain of each function is the set of all integers. For example, for the function f = {(a, 1), (b, 3), (c, 2), (d, 1)}, we could write f(a) = 1, f(b) = 3, and so on. What is the height of a rooted tree? We'll need to come up with a counterexample. Write a recurrence relation for pn . What is the probability of getting a sum of 7? In general, the symbol in position n (with the rightmost symbol being in position 0) represents the number of 16n 's. Find the parent of Poseidon. Throughout this section, we assume that all sample spaces are finite. , bn . P(E) = x∈E Example 6.4 Given the assumptions of Example 6.2, the probability of an odd number is P(1) + P(3) + P(5) = 5 3 1 1 + + = . To satisfy the initial conditions (2.10), we must have 7 = U0 = b20 + d30 = b + d, 16 = U1 = b21 + d31 = 2b + 3d. (a) Input: Input: n Output: 2 + 4 + · · · + 2n Output: sum(s, n) { if (n == 1) return 2 return sum(n − 1) + 2n } 1. Therefore, no assignments exist for A, B, C, and D. At line 4, i is reset to n/2 and we return to line 2. However, this special result is also useful in other ways (see Exercises 25 and 27 and the following Problem-Solving Corner). Since n > 3, we proceed to line 3. We then argue that certain edges must be in the Hamiltonian cycle. Example 6.6 Five microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. G L F H I K J B M E D C 23. It is denoted ∩ S. By Theorem 2.3, T is a tree. Definition 7.1 A graph is planar if it can be drawn in the plane without its edges crossing. We begin with an example. At line 5, since key ('S') is equal to s4 ('S'), we return 4, the index of key in the sequence s. 5 → Quantifiers The logic in Sections 2 and 3 that deals with propositions is incapable of describing most of the statements in mathematics and computer science. These five binary trees are shown in Figure 8.12. In Exercises 42–47, decompose the function into simpler functions as in Example 1.44. Furthermore, every edge that touches Wor must be used. 30. The formalism in Definition 2.1 means: Start at vertex v0 ; go along edge e1 to v1 ; go along edge e2 to v2 ; and so on. Helmand No 3 Stars ? Carry out this procedure for n = 4 for each game tree of Exercises 23–25. (In Exercises 5– 8, x, y ∈ {1, 2, 3, 4, 5}.) 1. 46 43. How many are taking French and music but not business? A corollary is a theorem that follows easily from another theorem. 1356 2. Draw all nonisomorphic full binary trees having nine vertices. insertion sort(s, n) { if (n == 1) return insertion sort(s, n − 1) i=n−1 temp = sn while (i ≥ 1 ∧ si > temp) { si+1 = si i=i−1 } si+1 = temp } Let bn be the number of times the comparison si > temp in line 7 is made in the worst case. Input: Output: a and b (nonnegative integers, not both zero) s and t of Theorem 3.7 (stored in parameters s and t) and the greatest common divisor of a and b (which is returned) STgcdr(a, b, s, t) { // make a largest if (a < b) swap(a, b) if (b == 0) { s=1 t=0 // now a = sa + tb return a } q = a/b r = a mod b // a = bq + r g = STgcdr(b, r, s , t) // g = s b + t r // ∴ g = t a + (s − t q)b s = t t = s − t ∗ q return g } Computing an Inverse Modulo an Integer Suppose that we have two integers n > 0 and φ > 1 such that gcd(n, φ) = 1. For example, an infinite sequence v whose initial index is 0 can be denoted {vn }∞ n=0 . Prove the complement laws for sets. 1 2 8 24 2 14 26 6 18 4 7 12 28 6 5 22 30 16 3 10 4 20 8 9 14. Input: PT , the root of a binary tree Output: 9. Therefore, xy = −qz + w; that is, w is the remainder when xy is divided by z. False 21. Vendor Percent purchased Percent defective Acme DotCom Nuclear 55 1 10 3 35 3 Let A denote the event "the computer was purchased from Acme," let D denote the event "the computer was purchased from Dot- 48. 3 + 5 > 2 if and only if 1 + 3 = 4. The constants can be determined by substituting g(n) into the recurrence relation and equating coefficients on the two sides of the resulting equation. The union consists of all elements belonging to either X or Y (or both). If X is a finite set, what is X∗ ? f(v1) = w1 , f(v2) = w4 , f(v3) = w3 , f(v4) = w2 , f(v5) = w6 , f(v6) = w5 , f(v7) = w7 , f(v8) = w8 . Not each x satisfies P(x). When you assume the negation of the hypotheses, additional statements become available that might make some of the conditions of the definitions and theorems apply. 390 Demand Quantity q4 q2 q1 q3 Quantity Figure 1.7 A cobweb with increasing price swings. This representation may be expressed 1011012 = 1 · 25 + 0 · 24 + 1 · 23 + 1 · 22 + 0 · 21 + 1 · 20 . Write an algorithm whose input is a sequence s1 , . There are many versions of pseudocode. This exercise shows that this error is minimized by choosing x to be a median of the values x1 , . More precisely, if n = 5, any n × n deficient board can be tiled with trominoes if and only if 3 divides n2 − 1 (see Exercises 27 and 28, Section 5). Informally, we are "rounding down." Examples: 2.3 = 2, 5 = 5, −2.7 = −3. How many outcomes give the sum of 7 or the sum of 11? Suppose that k ≥ n + p and that we can make postage for m satisfying n ≤ m < k. Obtain a contradiction by showing that r ∈ X. (The dinner consisting of an appetizer whose first letter is X, a main course whose first letter is Y , and a beverage whose first letter is Z is denoted XYZ.) Notice that there are two appetizers, three main courses, and four beverages and 24 = 2 · 3 · 4. (1.9) Inequalities (1.8) and (1.9) combine to give cn = dn . In a problem that involves the greatest common divisor, the preceding equation may be helpful. How many strings contain the letter A, allowing repetitions? If it is not the case that (6 < 6 and 7 is not less than 10), then 6 < 6. (The given proposition is true because 3 < 5 is true.) The final operator on a proposition p that we discuss in this section is the negation of p. ∅ 4. We call vn−1 the parent of vn . In words, we are assuming that x < 1 and y < 1. 3 4 3 1 4 2 B G W G 2 2 R B 3 3 1 2 1 3 4 7. a1 + a2 + · · · + a2n , n = 1, 2, . This sequence is an infinite sequence. Thus if 40 coins are distributed among nine bags so that each bag contains at least one coin, at least two bags contain the same number of coins. (4.3) Suppose that someone claims that n(n + 1) for all n ≥ 1. Since τ(R1) ∪ D is transitive, τ(τ(R1) ∪ D) = τ(R1) ∪ D (Exercise 56). He was appointed to the Schlumberger Centennial Chair in Computer Science at the University of Texas at Austin in 1984 and retired as Professor Emeritus in 1999. You can use the preceding characterizations in various ways. Let G be a directed graph with vertices corresponding to all bit strings of length n − 1. Since k ≤ n for every k in X, n is the largest element in X. Two important types of sequences are increasing sequences and decreasing sequences, and their relatives: nonincreasing sequences and nondecreasing sequences. Prove the bound laws for sets. a b 18. To compute the chance or probability that an even number appears, we first count how many ways an even number can appear (three: 2, 4, 6) and how may ways an arbitrary number can appear (six: 1, 2, 3, 4, 5, 6); then, the probability is the quotient: 3/6 = 1/2. The first step is "select the main course" and the second step is "select an optional beverage." There are n1 = 3 ways to select the main course (hamburger, cheeseburger, fish filet) and n2 = 5 ways to select the optional beverage (tea, milk, cola, root beer, none). 36. 75. Discussion We first give a direct proof like the proof in Example 1.11. Two important operations on numerical sequences are adding and multiplying terms. P = {a, d, e} for Exercise 2, Section 1 3. In Exercises 10–18, T2 (x, y) is the propositional function "x is taller than or the same height as y." Write each proposition in Exercises 10–13 in words. s6 t 6 ; (2s − t)12 18. State the associative laws for sets. The intended meaning is: Not everything is sweetness and light. 334 Counting Methods and the Pigeonhole Principle Thus each equivalence class has n1 ! n2 ! · · · nt ! elements. For all n, if block n is marked, then block n + 1 is also marked. The digraph of this relation (see Figure 3.1) has a directed edge from 2 to 3, but there is no directed edge from 3 to 2. Thus ns mod φ = 1. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is D2 × D2 . Write an algorithm that, given a sequence a, finds the length of a longest increasing subsequence of a. Find all buyers who buy parts that are produced by the department for which Suzuki works. an = −4an−1 − 4an−2 ; a0 = 2, a1 = 4 7. The n-cube has 2n vertices labeled 0, 1, . Vertices b, c, and d represent intermediate pumping stations. Section 3 Review 3. We must show that an ≤ 3n + 2. Find a recurrence relation and an initial condition for the sequence A0 , A1 , . It is not a function (from X to Y). We construct an n-vertex binary tree whose left subtree has k vertices and whose right subtree has n − k − 1 vertices by a two-step process: Construct the left subtree, construct the right subtree. ¬(p ∨ q) ∧ (¬p ∨ r) In Exercises 49–54, represent the proposition symbolically by letting p : There is a hurricane. = n+1 n+1 n1 · · · nk n+1 n+1 C1 j=1 n n i=1 j=1 n(n + 1) −2 2 j2 2 6 n(n + 1)(2n + 1) 6 n2 (n + 1)2 n(n + 1)(2n + 1) − 6 2 n2 (n + 1)[2(2n + 1) − 3(n + 1)] 6 n2 (n + 1)[n − 1] n2 (n2 − 1) = = . If all the weights in G are distinct, distinct spanning trees of G have distinct weights. (f) To prove equation (2.2), we will replace sn−1 and sn−2 in the right side of equation (2.2) by the formulas of parts (d) and (e). Hint: In Exercises 5 and 6, replace → and ↔ with logically equivalent expressions that use or and and. Thus if n = 2k , the statement x = x + 1 is executed 1 + lg n times. Answer parts (a)–(c) of Exercise 8 for this situation. Each time we arrive at Wor on some edge, we must leave Wor on a different edge. What are series edges? Then, choose 23 additional items. The following pseudocode determines whether ∀x∀y P(x, y) is true or false: for i = 1 to n for j = 1 to n if (¬P(di , dj)) return false return true The for loops examine members of the domain of discourse. Next, insertion sort inserts s3 into the now-sorted subsequence s1 , s2 . i i Proof Let V be the set of vertices. If we let q be the quotient of a divided by b, we have a = bq + r. Define Ackermann's function. Find the coefficient of x2 y3 z4 in the expansion of (x + y + z)9 . Give an example of a graph that does not have an Euler cycle. R is reflexive and symmetric. Does the graph of Exercise 3 have a Hamiltonian path? Specifically, write C(n + 1, k) in terms of C(n, i) for appropriate i. Connect each vertex above the x-axis to its mirror image below the x-axis. Suppose that S is a collection of subsets of a set X and X = ∪ S. Suppose that we choose c in U and remove it from U at line 15. ak . One technique is to find a value of x ∈ X (again one value suffices!) that has the property that P(x, y) is false for every y ∈ Y . Choose one beverage. Similarly, if Egbert is secretary, there are 20 possibilities, and if Egbert is treasurer, there are 20 possibilities. Write an algorithm that uses the output of your algorithm in Exercise 35 to print each vertex and its parent. Exercises 50–54 refer to a group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music. For example, if we substitute "Barry Bonds" for "baseball player," statement (c) is Barry Bonds hit over .300 in 2003, which is true. We first verify that S(1) is true (Basis Step). , vj), (vi , vi−1 , . 69. (b) A maximal flow corresponds to a maximal matching. , ckj } of {c1 , . Not satisfied with the chances, the person computes the probability of two bombs on a plane to be 0.0000012 = 0.000000000001. 554 Trees For two rooted trees T1 and T2 to be isomorphic, there must be a one-to-one, onto function f from T1 to T2 that preserves the adjacency relation and that preserves the root. ∃x∀y P(x, y) 70. Universal quantifier Universally quantified statement Counterexample Existential quantifier Existentially quantified statement Generalized De Morgan's laws for logic: ¬(∀xP(x)) and ∃x¬P(x) have the same truth values. How many reflexive and symmetric relations are there on an n-element set? [Codd; Date; Kroenke; and Ullman] are recommended references on databases in general and the relational model in particular. We first prove that X ⊆ Z. At lines 15–22, we reverse the order of the elements s4 s5 s6 = 532. What is a Venn diagram? Definition 5.9 Let P be a propositional function with domain of discourse D. In the tree that results, add two edges to a vertex labeled fi + fj , and label the added vertices fi and fj . Decrypt 28. Notice that this function is one-to-one. For our next attempt at choosing two edges incident on vertex B, let us select the edges labeled 2 and 3, as shown in Figure 8.8. Since this choice includes one edge incident on R, we must choose one additional edge incident on R. B ∩ ∅ 11. Is it always true, according to your definition of radius, that 2r = d? Since p(n) = O(nk) and p(n) = (nk), p(n) = (nk). Consider block 3. Now c = bq2 = (aq1)q2 = a(q1 q2). Use the If S(n) is true, then S(n + 1) is true form of the Inductive Step to prove the statement in Exercise 1. , sj , where m = (i + j)/2. A list of the letters as they appear in a word is an example of a sequence. Suppose that n satisfies 2k−1 ≤ n < 2k . Since (5 + 45) mod 10 = 0, the check digit is 5. How is it denoted? At this point, the value of a1 might be equal to any one of the original values in the sequence. The graph must be connected. Suppose there is one-to-one, onto function f from the vertex set of G1 to the vertex set of G2 satisfying the following: Vertices v and w are adjacent in G1 if and only if the vertices f(v) and f(w) are adjacent in G2 . Suppose that if four teams play seven games, no pair of teams plays at least two times; or, equivalently, if four teams play seven games, each pair of teams plays at most one time. Let i be the largest index such that S uses an ai coin. It is often easiest to prove that a graph is not bipartite by arguing by contradiction. The value x = 2 makes (x > 1) → (x2 > x) true. sr , which represents the r-combination {s1 , . 88. Thus a0 = 2 and the inequality holds when n = 0. We add a 2-cent stamp to make n-cents postage. We note that T is connected since there is a simple path from any vertex to any other vertex. 200 80. We will illustrate the algorithm with two examples. ∃x (P(x) ∧ Q(x)) 43. 300 11. , vn , and let A2 be the adjacency matrix of G2 relative to the ordering f(v1), . Subsets of Y not containing a. We omit the Basis Step. 2 2 5 Surprisingly,√even though fn is an integer, the preceding formula involves the irrational number 5. Write a program to determine whether a graph is planar. Now tn = tn−1 tn−2 = 1·1 by the inductive assumption = 1. Well-Ordering Property: every nonempty set of nonnegative integers has a least element. Similarly the missing square cannot be a 3. There exists x such that x < y (x, y real numbers). Discussion We can devise a proof of Theorem 5.6 by looking carefully at the technique used in long division. Vertex b is labeled "a, 2" to indicate its value and the fact that it was labeled from a. There are C(980, 5) ways to select 5 good microprocessors since there are 1000 − 20 = 980 good microprocessors. Define a relation R on X as x R y if x and y are in the same state. If the numbers to add are bn bn−1 · · · b1 b0 and bn bn−1 · · · b1 b0 , at the iteration i > 0 the algorithm adds bi , bi , and the carry bit from the previous iteration. Example 2.4 Use proof by contrapositive to show that for all x ∈ R, if x2 is irrational, then x is irrational. Also, determine whether the cut is minimal. What is a universal set? , 97 and 70, 71, . We consider changing the truth table for p → q when p is false. Thus an actor's Bacon number is the length of a shortest path from the vertex corresponding to that actor to the vertex corresponding to Bacon. Because of the importance of counting, a variety of useful aids, some quite sophisticated, have been developed. We first find an upper bound for the number of comparisons in the worst case when the total number of vertices in the trees input to the procedure is n > 0 and neither tree is empty. 601 Network Models Example 2.5 In this discussion, if vertex v satisfies predecessor(v) = p val(v) = t, and we show v's label on the graph as (p, t). At this point, we have two solutions S and T of (2.9), given by Sn = 2n , Tn = 3n . First suppose that a = 1, b = 5, c = 3. How many outcomes are doubles? Give an alternative proof of the existence of q and r in Theorem 5.6 using the form of mathematical induction where the Inductive Step is "if S(n) is true, then S(n + 1) is true." Hint: First assume that n > 0. Following lines 3–7 (left/right/root) of Algorithm 6.5, we obtain the postorder listing CEDBIJHGFA. We let E1 denote the event "get doubles" and E2 denote the event "get a sum of 6." Since doubles can be obtained in six ways, P(E1) = 1 6 = . C(10 + 3 − 1, 10) n+1 19. In this chapter, we introduce set terminology and notation. Furthermore, either s or t is negative. 232 Algorithms To derive an omega notation for an expression f(n) directly, you must find a constant C2 and a simple expression g(n) such that |f(n)| ≥ C2 |g(n)| for all but finitely many n. Some of these tasks must be done before others. A two-person game is played in which a fair coin is tossed until either the sequence HT (heads, tails) or the sequence TT (tails, tails) appears. What is the value of S({a, c})? Multiplication Principle If an activity can be constructed in t successive steps and step 1 can be done in n1 ways, step 2 can then be done in n2 ways, . Then x2 + x − 6 = 0. To prove that p1 , p2 , . (There are too many possibilities to list them all.) Also verify the recurrence relation for n = 5. This sum is 102 ; thus we write 0 and carry 1. Analyze the following argument that, assuming independence, purportedly shows that the runner is almost, but not quite, certain to complete the marathon. Thus as we look systematically at some examples, we should try to see how the case for n persons relates to the cases for n−1 and n−2 persons. Is E1 one-to-one? Write a program that solves an arbitrary Instant Insanity puzzle. We assign this minimum value to the circle vertex. In a linear homogeneous recurrence relation with constant coefficients, each term is of the form cak . In this section we discuss two methods of solving recurrence relations: iteration and a special method that applies to linear homogeneous recurrence relations with constant coefficients. Terminate Kruskal's Algorithm after k iterations. Assume that the children are evaluated left to right. 3 3 Thus the multiples of 3 between 1 and 10,000 correspond to the values k = 1, 2, . Therefore, the n-cube is bipartite. If data item d(v) is stored in vertex v and data item d(w) is stored in vertex w, then if v is a left child (or right child) of w, some ordering relationship will be guaranteed to exist between d(v) and d(w). Two dice are rolled, one blue and one red. (For Exercise 42) 33 54. Network Models Richard Johnsonbaugh 591 11. However, if n = 1, 2m + 5n2 = 2m + 5, which is odd being the sum of an even integer and an odd integer. By Exercise 27, n + 2 < Cn . f ◦ g = {(1, x), (2, z), (3, x)} 1 2 3 f g X w x y z Z 41. If 4 > 6, then 9 > 12. , xi } E2 = {y1 , . Write an algorithm to list all of the ways the robot can walk n meters. All owners of Porsches are accountants. As hardware costs have declined, it has become feasible to build parallel computers with many processors that are capable of executing several instructions at a time. Recursively invoke this algorithm to move the n − kn disks at the top of peg 1 to peg 2. p ∧ ¬p ∴q ∴ ¬p 14. Show that if modified Instant Insanity has a solution, the version given in the text must also have a solution. 69 Sets and Logic 38. For the relations R1 and R2 of Exercise 36, Section 3, find ρ(Ri), σ(Ri), τ(Ri), and τ(σ(ρ(Ri))) for i = 1, 2. Notice that the relation R is reflexive, antisymmetric, and transitive. n+1 131 Proofs In particular, D < 1. Compute S3 and S4 . 2 + sin n = O(2 + cos n) (a) Show, by consulting the figure, that 44. Thus D2 = 1. Write the truth table of each proposition in Exercises 22–29. The remainder of the path must be a path of length n − 2 from v1 to v1 . Figure 1.2 14. Explain how Algorithm 3.12 computes an . Someone knowledgeable about basketball will take this into account. At the same time, ∀x∀y P(x, y) is false. If ai = aj + 6, two are six apart. As confirmation, we list the 15 dinners (N = no beverage): HT, HM, HC, HR, HN, CT, CM, CC, CR, CN, FT, FM, FC, FR, FN. 49F7 + C66 38. Corollary 6.11 If E1 and E2 are mutually exclusive events, P(E1 ∪ E2) = P(E1) + P(E2). v2 e1 v1 e2 e3 v3 v4 v5 v6 e4 G Figure 2.2 A graph that is not connected. Examples of upper bounds for X are 4, 6.9, 3π, 9072. Level Number of Vertices 0 1 2 3 4 5 1 2 4 8 16 32 Figure 7.9 Level compared with the maximum number of vertices in that level in a binary tree. (If there was a solution for n − am using fewer coins, we could add one am coin to it to obtain a solution for n using fewer coins than S, which is impossible.) By the inductive assumption, the greedy solution for n − am is optimal. Thus the string that follows 13467 must begin 135. From x ≤ d and d ≤ d2 , deduce x ≤ d2 . Explain how the formula is derived. r : I get rich. The cobweb in economics first appeared in [Ezekiel]. Let s0 = 0. We see that the path that visits the vertices a, b, c, d, e, in this order, has minimum length. (c) T is connected and acyclic. Your goal is to try to make the propositional function true, and Farley's goal is to try to make it false. , n} in increasing lexicographic order r comb(r, n) { s0 = −1 for i = 1 to r si = i println(s1 , . A graph with numbers on the edges (such as the graph of Figure 1.7) is called a weighted graph. More precisely, a collision occurs for a hash function H if H(x) = H(y), but x = y. , vj) are distinct simple paths from vi to vj , which contradicts the definition of a tree. The following pseudocode determines whether ∃x P(x) is true or false: for i = 1 to n if (P(di)) return true return false The for loop examines the members di in the domain of discourse one by one. The n × n board is tiled, and the inductive step is complete. We will arrange for (1, 2), (2, 3) ∈ R ◦ S, but (1, 3) ∈ / R ◦ S. In computer

science, we have this theorem: For some program P, the output of P is P itself. Figure 1.8 shows the same structure as a rooted tree. Since 6 divides both 7n − 1 and 6 · 7n , it divides their sum, which is 7n+1 − 1. Inclusive-or: To enter Utopia, you must show a driver’s license or a passport or both. 10 5 [Bain] invented an algorithm to draw the n-cube in the plane. (7) (5) (6) Finding a Solution Let’s first try to prove (1). Antisymmetric: Suppose that (x1 , x2)R(x1 , x2) and (x1 , x2)R(x1 , x2). For every positive integer n, n divides 77. Show that by a suitable ordering of the vertices, the adjacency matrix of a bipartite graph can be written 0 AT A , 0 where 0 is a matrix consisting only of 0’s and AT is the transpose of the matrix A. 5, a z 6, 4, 7. If b > c, then 4. If each person chose a cola randomly, what is the probability that no one chose Coke? The problem is that one of s or t must be negative in order for the sum sp + tq to be 1 (since p and q are greater than 1). What kind of labeling corresponds to a maximal matching? Given an ordered pair (A, B) satisfying A ⊆ B ⊆ X, we see that each element in X is in exactly one of A, B − A, or X − B. For all real numbers d, d1 , d2 , x, if d = min{d1 , d2 } and x ≤ d, then x ≤ d1 and x ≤ d2 . The statement ∃x∀y P(x, y) is false if, for every x ∈ X, there is at least one y ∈ Y such that P(x, y) is false. Give an algorithm to execute a postorder traversal. If a1 , a2 , ...) 1)) 1))) ... 1 2 ... 9. Model this system as a transport network. In this case, n − dn is in X. Define inverse function. Furthermore, among those that have the HIV virus, approximately 95 percent test positive on the ELISA test. Notice that 7 together with GS ∗ is the greedy solution GS to the n-problem. w1 C D 13. It is not the case that today is Monday or it is raining if and only if it is hot. Discussion By trial and error, we find that 2p − 1 is prime for p = 2, 3, 5, 7 but not p = 11 since 211 − 1 = 2048 − 1 = 2047 = 23 · 89. We will next find all nonisomorphic trees with five vertices in which the maximum vertex degree that occurs is 3, and so on. The subset-sum problem is: Given a set {c1 , . (The problem can be solved in fewer than 3 + 2lg n array comparisons in the worst case.) 12. Concerning the traveling salesperson problem, [Applegate] is accurately self-described as “the definitive book on the subject.” 493 Graph Theory In many cases, so-called branch-and-bound methods (see, e.g., [Tucker]) often give solutions to the traveling salesperson problem more efficiently than will exhaustive search. 23 Sets and Logic (a) The hypothesis is the clause following if ; thus an equivalent formulation is If Mary studies hard, then she will be a good student. Inductive Step We assume that 5n − 1 is divisible by 4. Input: 1. If we define f(x, y) = x + y, where x, y ∈ X, then f is a binary operator on X. Suppose that Ri is a partial order on Xi , i = 1, 2. The algorithm is not general; that is, it does not apply to a set of inputs. {1, 2, 3, 4, 5, 7, 10} 64. An existence proof of ∃x P(x) that exhibits an element a of the domain of discourse that makes P(a) true is called a constructive proof. , dn }. Exercises 39–44 refer to Algorithm 3.13. i=k+1 Now P(E) + P(E) = = k i=1 n P(xi) + n P(xi) i=k+1 P(xi) = 1. Proof For the prime p = 11, 2p − 1 is composite: 211 − 1 = 2048 − 1 = 2047 = 23 · 89. The possibilities are: A (correct), B (incorrect), C (incorrect); and A (incorrect), B (incorrect), C (incorrect). 52. Choose x = x in the domain of discourse. The original problem was to find the maximum sum in a rectangular submatrix of an n × n matrix of real numbers. p → q ¬q ∴ ¬q 13. Any number different from 1, 2, 3 would do.) So far, so good! In order for (2, 3) ∈ R ◦ S, we must have (2, b) ∈ S and (b, 3) ∈ R, for some b. Since α1 = The function f is onto X∗ if given any string γ ∈ X∗ , there exist (α, β) ∈ X∗ × X∗ such that f(α, β) = γ. , b. |x| = 0 = 0 · 0 = sgn(x)x. Functions, Sequences, and Relations To obtain a different example, we would have to start with a string whose first four bits are different from 0111, say 1011. Give a recursive algorithm to insert a word in a binary search tree. In this case, m = 4. Thus the problem is to count the number of ways of choosing k integers, with repetitions allowed, from the set {1, 2, . TABLE 1.1 ■ Paths in the graph of Figure 1.7 from a to e that pass through every vertex exactly one time, and their lengths. If An represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence {An }. In words, this last proposition can be stated as: There exists a rock fan who does not love U2. Therefore, the remainder 36844 mod 2427 is r = 36844 − 2427 · 15 = 439. The subtree whose root is the parent of w may be deleted (pruned). If the algorithm is correct, prove it; otherwise, give an example of a connected, weighted graph and vertices a and z for which it fails. If we add equations (1.1) and (1.2), we obtain m + n = dq1 + dq2 = d(q1 + q2). Problem-Solving Tips Depth-first search and breadth-first search are the basis of many graph algorithms. ¬(∃x P(x)) 19. Thus when we describe how the best-case or worst-case time grows as the size of the input increases, we not only seek the dominant term [e.g., 60n2 in the formula for t(n)], but we also may ignore constant coefficients. Derive a recurrence relation and an initial condition for the number of ways to parenthesize the product Exercises 19–21 refer to the sequence S where Sn denotes the number of n-bit strings that do not contain the pattern 00. prune the subtree with root H. Algorithm 1.1 has correctly found the largest of three numbers. In (2.7) or (2.8), i is called the index, m is called the lower limit, and n is called the upper limit. This is where the specifics of the game must be dealt with. Write recursive and nonrecursive programs to compute n!. We state the method of Example 1.21 as Theorem 1.22. For example, a0 = 1 since there is one binary tree having no vertices; a1 = 1 since there is one binary tree having one vertex; a2 = 2 since there are two binary trees having two vertices (see Figure 8.13); and a3 = 5 since there are five binary trees having three vertices (see Figure 8.12). Remember you’re trying to derive an inequality, not an equality, so you can replace terms in f(n) with other terms if the result is larger (see, e.g., Example 3.3). Therefore, the total number of r-permutations is n(n − 1) · · · (n − r + 1). Hand courtesy of Ben Schneider.] An experiment is a process that yields an outcome. 136 f = {(1, a), (2, b), (3, a)} is a function from X = {1, 2, 3} to Y = {a, b, c}. h(x) = (x2 + x) mod 17; cells and data as in Exercise 56 a function from X = {a, b, c} to X: (a) Write f ◦ f and f ◦ f ◦ f as sets of ordered pairs. , pn are equivalent. The domain of discourse is Z+ × Z+ . h1 h0 ? It is a linear homogeneous recurrence relation with nonconstant coefficients. (5.4) 45 Sets and Logic The proposition (5.3) is true if and only if Pi is true for every i = 1, . Example 2.12 Given that proposition p is false, proposition q is true, and proposition r is false, determine whether the proposition ¬p ∨ q ∧ r is true or false. 132. If n = 1 or if m = 1, there is no cycle and, in particular, there is no Hamiltonian cycle. Exercises 77–81 use the following definitions. Find a formula for the sequence c defined by yi zi . b1 = 1, b2 = 2, b3 = 3, b4 = 4, b5 = 5, b6 = 126 114. advance(i, j) while (i ≥ 1 ∧ j ≥ 1) { // search for a legal value s(i, j) = s(i, j) + 1 // not valid(i, j) returns true if the value s(i, j) // conflicts with the previously chosen and specified // values, and false otherwise. a b h c g d f e 9. Find b1 , b2 , and b3 . Describe the sets Z, Q, R, Z+ , Q+ , R+ , Z− , Q− , R− , Znonneg , Qnonneg , and Rnonneg , and give two examples of members of each set. Prove that the following are equivalent for sets A and B: (a) A ⊆ B (b) A ∩ B = ∅ (c) A ∪ B = B Section 3 9. It is important to recognize when to apply each principle. A universal set is a set that contains all of the sets under discussion. 12 11 4 7 532 2 7 5 6 13 8 3 8 3 14 10 9 9 2 6 8 4 7 3 3 10 5 4 3 9 6 2 4 4 3 1 6 5 1 10 2 6 2 3 4 2 5 2 5 2. 6 + 9 = 15. How is the intersection denoted? Thus the set on the left-hand side of equation (*) contains one set. Loosely speaking, ∀ and ∃ are interchanged. If x > 1, the hypothesis and conclusion are both true; hence the conditional proposition if x > 1, then x + 1 > 1 is true. Assume that the algorithm is correct for all k < n. wi . Definition 1.15 The complete bipartite graph on m and n vertices, denoted Km,n , is the simple graph whose vertex set is partitioned into sets V1 with m vertices and V2 with n vertices in which the edge set consists of all edges of the form (v1 , v2) with v1 ∈ V1 and v2 ∈ V2 . This suggests that we might prove the first statement by contradiction or by proving its contrapositive. , Xn is the set of all n-tuples (x1 , x2 , . Since line 2 depends on the order of the subsets generated in line 1, the determinism property is lacking here as well. Solve the recurrence relation of Exercise 67 in case n is a power of 2, assuming that b1 = 1. Then x = 2n for some n ∈ Z+ . 1 2 3 4 An informative way to picture a relation on a set is to draw its digraph. By the inductive assumption, the values computed are correct. A similar argument shows that if (x, y) and (y, z) happen to both be in 173 Functions, Sequences, and Relations S, then (x, z) ∈ R ∪ S. We could continue in this manner until we find an i for which si ≥ A. How many outcomes have either the blue die 3 or an even sum or both? ¬(∃x P(x)) 34. We can think of a propositional function as defining a class of propositions, one for each element in the domain of discourse. A function (see Section 1) is a special type of relation. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is D1 × D1 . Write a program to determine whether X = Y , given arrays representing X and Y . One technique for proving that ∀x P(x) is true is to let x denote an arbitrary element of the domain of discourse D. sj . 24 . 2n − 2 4. (e = 2.71828 . (In fact, it is false that for all m and n, if m and n are even integers then mn is a square. Example 6.19 Let R denote class premium, T denote class table wine, and S denote class swill. This latter statement is: There exists a person x such that for all y, x loves y. If T is a graph with n vertices, the following are equivalent (Theorem 2.3): (a) T is a tree. Now assume that the order in which funnyorder visits the nodes of a tree T having fewer than n nodes is revpost(T). True 1+ 46. (p ∨ q) → r ≡ ¬(p ∨ q) ∨ r ≡ ¬p¬q ∨ r ≡ (¬p ∨ r)(¬q ∨ r) 10. Theorem 3.5 If X is a set containing t elements, the number of unordered, k-element selections from X, repetitions allowed, is C(k + t − 1, t − 1) = C(k + t − 1, k). The car comes with a cupholder that heats or cools your drink. Then P is a propositional function with domain of discourse Z+ since for each n ∈ Z+ , P(n) is a proposition [i.e., for each n ∈ Z+ , P(n) is true or false but not both]. 484 Graph Theory Theorem 7.7 Kuratowski’s Theorem A graph G is planar if and only if G does not contain a subgraph homeomorphic to K5 or K3,3 . The modus ponens rule of inference now gives Q(Ralphie), which represents the proposition “Ralphie understands the trigonometric functions.” We conclude that the conclusion does follow from the hypotheses. Thus the set on the right-hand side of equation (*) also contains two distinct sets: {c} and {c, d}. How is the ceiling denoted? 486 Graph Theory by the inductive assumption, (7.3) holds for G . We let revpost(T) denote the reverse of post(T). Fill in the details of the following proof that there exist irrational numbers a and b such that ab is rational. If P ends at v2 , then v2 is in the same component as v. It then places one right tromino in the center. First, select an ordering, say abcdefgh, of the vertices of G. Exercise 7, Section 1 14. Basis Step (n = 4) 29. Finally, the prospective recipient computes the unique number s, 0 < s < φ, satisfying ns mod φ = 1. We then add the clauses ¬b and ¬d to the hypotheses to obtain 1. We count the number of n-bit strings not containing the pattern 000. , sn) // print the ith permutation We will show how Algorithm 4.14 generates the permutation that follows 163542. We can consider the assignment of names to people to be that of assigning pigeonholes to the pigeons. ¬p → ¬q, if |4| ≥ 3, then −3 ≥ 4 or 4 ≥ 3; true. 341 Counting Methods and the Pigeonhole Principle Algorithm 4.9 Generating Combinations This algorithm lists all r-combinations of {1, 2, . Conclusion: I’ll buy the compact disc and the compact disc player. The sequence g1 , g2 , . How many integer solutions of x1 + x2 + x3 + x4 = 17 satisfy x1 ≥ 0, x2 ≥ 1, x3 ≥ 2, x4 ≥ 3? We consider two cases: x ≥ y and x < y. Example 1.10 Let C = {1, 3} and A = {1, 2, 3, 4}. For all sets X and Y , either X is a subset of Y or Y is a subset of X. Therefore, there are exactly k edges g(e1), . 1 − (1/4) 8. logically equivalent to 93. No string that begins 134 and represents a 5-combination of X exceeds 13467. The population of Utopia increases 5 percent per year. 118 Proofs Suppose that parentheses are inserted in the product a1 a2 · · · an . sort is then called recursively on the first n−1 elements. 16. If at some point, you are forced to put a vertex into both V1 and V2 , the graph is not bipartite. Is nondecreasing? What is proof by cases? Sometimes, as for example in using a computer to analyze a graph, we need a more formal representation. an−1 . If a graph has a Hamiltonian cycle, must it have a Hamiltonian path? Given a tree T ∈ X1 , for each vertex v in T , we construct two trees in X2 as follows. Since the last element is a one, all of the zeros precede all of the ones for all n elements. Then (x, y) ∈ Rm and (y, z) ∈ Rn . Since |f(α)| ≥ 2 for all α ∈ X∗ , f(α) = λ for all α ∈ X∗ . In Exercises 13–17, draw the graph represented by each adjacency matrix. The 16 four-bit strings listed previously determine the 16 equivalence classes. Definition 2.3 states that the conjunction p ∧ q is true provided that p and q are both true; p ∧ q is false otherwise. What is a flow out of a vertex? Let p: The Skyscrapers win. Definition 3.11 If an algorithm requires t(n) units of time to terminate in the best case for an input of size n and t(n) = O(g(n)), we say that the best-case time required by the algorithm is of order at most g(n) or that the best-case time required by the algorithm is O(g(n)). Show that the sets {1, 2, . , pn using operators such as ¬ and ∨, a truth table will always supply all possible truth values of P for various truth values of the constituent propositions p1 , . The number of strings that contain AB = number of strings that contain BE = 4!. We begin with a = 273 and b = 110. No. Each edge is incident on at least one vertex. 8. Definition 1.1 Example 1.2 A graph (or undirected graph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge e ∈ E is associated with an unordered pair of vertices. Thus T is even. 91. In other words, for sufficiently large n, the probability that no one receives the correct order is essentially insensitive to the number of orders! Calculus tells us that ∞ xi . 1 2 3 4 [ ) 2 1 1) 4. Recursive algorithm Recursive function Divide-and-conquer technique Base cases: Situations where a recursive function does not invoke itself 26. If i = j, the entry is twice the number of loops incident on i. In this section we use decision trees to specify algorithms and to obtain lower bounds on the worst-case time for sorting as well as solving certain coin puzzles. In either case, we have proved that |x| < d. Relative to the vertex orderings a, b, c, d, e, f, g for G1 , and 1, 3, 5, 7, 2, 4, 6 for G2 , the adjacency matrices of G1 and G2 are equal. A where the vertices are arranged so that w1 , v, w2 crosses w3 , v, w4 at v as in the following figure. Comment on the following item from The New York Times: Big pickups also appeal because of the seemingly infinite ways they can be personalized; you need the math skills of Will Hunting to total the configurations. If we let P(x) be the propositional function “x glitters” and Q(x) be the propositional function “x is gold,” the first interpretation becomes ∀x(P(x) → ¬Q(x)), 46 (5.6) Sets and Logic and the second interpretation becomes ∃x(P(x) ∧ ¬Q(x)). The program must be able to accept as input an arbitrary number of cities and the distances between cities directly connected by roads and produce as output the shortest paths (routes) between each distinct pair of cities. State the disjunctive syllogism rule of inference. The name “inclusion-exclusion” in Theorem 1.12 results from including |X ∩ Y | twice when computing |X ∪ Y | as |X| + |Y | and then excluding it by subtracting |X ∩ Y | from |X| + |Y |. Inductive Step [Condition (4.8)] We assume that the inequality is true for n ≥ 1; that is, we assume that n! ≥ 2n−1 (4.10) is true. Now a ∈ Y1 if and only if s1 = 1 if and only if a ∈ Y2 . = Section 9 Review 6. Suppose that x is incident on vertices a and b. Assuming that the host chooses a door that hides a goat, what is the probability of winning the car for each strategy in Exercises 45–47? Implement Dijkstra’s shortest-path algorithm as a program. In all three, prior instances of the current case are assumed known. In part (b) we are given a major hint: Divide the routes into classes based on when the route first meets the top edge. Having explained the key idea of insertion sort, we now complete the explanation of the algorithm. Specifically, we assume that pj is true. (f) Determines whether v is a terminal vertex. (4.1) Proof We have already pointed out that if there is a complete matching in G, condition (4.1) holds. Finally, we introduce a supersource and supersink. Draw all nonisomorphic, simple graphs having exactly five vertices and two edges. To count the number of dinners, we use the Multiplication Principle. [The notation (x, y) means that we obtain x on the first die and y on the second die.] Therefore, the probability is 3/36 = 1/12. j=1 for i = 2, 4, 6, . The I’s and L’s can then be placed in only one way because the L’s must come first. The characters, used by the blind, consist of raised dots. Show that a directed graph G contains a directed Euler cycle if and only if the undirected graph obtained by ignoring the directions of the edges of G is connected and in(v) = out(v) for every vertex v in G. Therefore A ∪ U = U. Discussion For much the same reasons that a direct proof seemed unpromising in Examples 2.1 and 2.2, a direct proof in which we assume only that x2 is irrational seems to be too little to get us started. (3.1) Using the generalized De Morgan’s laws for logic and the fact that ¬(p → q) ≡ p ∧ ¬q, we find that (3.1) is equivalent to ∃x∃y[[(x, y) ∈ R] ∧ ¬[(y, x) ∈ R]] or, equivalently, ∃x∃y[[(x, y) ∈ R] ∧ [(y, x) ∈ R]]. In Exercises 90–94, state whether the given function f is a binary operator on the set X. Since there are Dn−1 such distributions, there are Dn−1 distributions in which q does not have p’s order. In Theorem 8.11 we showed that there are five nonisomorphic binary trees having three vertices. Prove that {rn } satisfies rn = 7rn−1 − 10rn−2 , zn = (2 + n)3n , 79. 338 17. f(x) = log2 (x2 + 2) 1 2x2 45. Prove that T is a tree if and only if T is connected but the removal of any edge (but no vertices) from T disconnects T. Find the coefficient of x3 yz4 in the expansion of (2x + y + z)8 . Select Chapter Self-Test Section 1 1. There are two equivalence classes. For more powerful methods, such as methods that make use of generating functions, consult [Brualdi]. Hint: Use the ideas suggested in the hint for Exercise 36, Section 4. (3.4) 287 Introduction to Number Theory [The first inequality in (3.4) holds because q > 0; q cannot equal 0, because a > b.] Inequalities (3.3) and (3.4) give a ≥ fn+3 b ≥ fn+2 . Let us temporarily insert an edge e from v to w. After initializing the given flow, we move to lines 4–7, where we set all labels to null. , sn sorted in nondecreasing order and a value x. The relation obtained by executing this join is shown in Table 6.3. TABLE 6.3 ■ PLAYER [ID Number = PID] ASSIGNMENT ID Number 58199 01180 26710 39826 Name Position Age Team p 1b p 2b 18 37 22 31 Jackalopes Mutts Mutts Blue Sox Battey Homer Score Singleton Most queries to a relational database require several operations to provide the answer. We say that α is lexicographically less than β and write α < β if either (a) p < q and si = ti for i = 1, . Modify Algorithm 4.9 so that line 5 5. If P ends at v1 , then v is in the same component as v1 in G . We then repeatedly apply (3.1) until we derive a contradiction. Does the graph of Exercise 4 have a Hamiltonian path? Mesh model for parallel computation 50. We cannot have two of (8.3) or two of (8.4) identical; thus some number in (8.3) is equal to some number in (8.4). Let R = {(1, 2)}, S = {(2, 1)}. In this section we discuss the binary number system, which represents integers using bits, and the hexadecimal number system, which represents integers using 16 symbols. If I go to the store, then I will get a soda. Theorem 2.21 If G is a graph with m edges and vertices {v1 , v2 , . G1 G2 Figure 6.3 Nonisomorphic graphs. And a couple of other towns in Massachusetts, and on to Portland and Bangor and straight home! FROM DEATH OF A SALESMAN Although the first paper in graph theory goes back to 1736 (see Example 2.16) and several important results in graph theory were obtained in the nineteenth century, it is only since the 1920s that there has been a sustained, widespread, intense interest in graph theory. The next (not yet selected) item can be paired in 2n − 3 ways, and so on. ∃x∀y ¬P(x, y) 92. Represent the assertion “someone does not know anyone” symbolically. Prove that the board contains a nontrivial rectangle (not 1 × k or k × 1) whose four corner squares are all black or all red. Since R1 ⊆ τ(R1), R1 ∪ D ⊆ τ(R1) ∪ D. Why is it sometimes necessary to relax the requirements of an algorithm as stated in Section 1? Write a program that tests whether a proposed isomorphism is an isomorphism. The remainder when x is divided by y 6. Ordered lists need not be restricted to two elements. Consider the construction of a binary tree with n vertices, n > 0. Thus statement (c) is a propositional function. If S ∗ were not optimal, S could not be optimal. Is q decreasing? † The first-class postage rate for retail letters was different: 41 cents for the first ounce or fraction thereof and 17 cents for each additional ounce or fraction thereof up to 3.5 ounces. Thus in the original graph, v and w have odd degree and all other vertices have even degree. These two statements seem quite different. , an and moves the numbers to random positions. Put an edge between states v and w if v and w share a border. We are not counting the individual items available by constructing them using a step-by-step process. Next we turn to the worst-case analysis of binary search. if (n == 0) 3. There are three ways to assign Egbert an office. The black lines show a maximal matching (i.e., the maximum number of applicants have jobs). 261 Introduction to Number Theory Algorithm 1.8 Testing Whether an Integer Is Prime This algorithm determines whether the integer n > 1 is prime. We cannot include edge (b, h) or we will complete a cycle. For example, at the root when we compare C1 with C2 , if the left side is heavier than the right side, we know that either C1 is the heavy coin or C2 is the light coin. C(n, 0) = C(n, n) = 1, for all n ≥ 0; and C(n + 1, k) = C(n, k − 1) + C(n, k), for all 1 ≤ k ≤ n Section 7 1. The capacity of edge (a, b), Cab , is 3 and the capacity of edge (b, c), Cbc , is 2. False. We let Pn denote a particular problem of size n. Each function in Exercises 32–37 is one-to-one on the specified domain X. Then each edge (namely none!) is incident on one vertex in V1 and one vertex in V2 . Since 8 students are taking calculus, psychology, and computer science, we write 8 in the region representing CALC ∩ PSYCH ∩ COMPSCI. What kind of proof did you give? The element tm is equal to sm + 1. Thus binary insertion sort is not optimal if n = 6. Again x ∈ (A ∪ B) − (A ∩ B) = A B. Now P can be seated to the right of someone in n ways. There is also an exclusive-or (see Exercise 66) that defines p exor q to be true if p or q, but not both, is true, and false otherwise. We must show that x ∈ A ∩ B. r−1 Section 4 1. The Inclusion-Exclusion Principle generalizes the Addition Principle by giving a formula to compute the number of elements in a union without requiring the sets to be pairwise disjoint. An argument like that in the preceding paragraph shows that b = d. In this case, the algorithm returns i: if (j > m) return i If the for loop runs to completion, a match was never found; so the algorithm returns 0. Domain of discourse Sets and Logic 55. Then n2 +1 = (2k−1)2 +1 = (4k2 −4k+1)+1 = 2(2k2 −2k+1). We prove only (A∪B)∪C = A∪(B∪C). Can this road system be designed so that the expressways do not cross? A = {1, 2}, B = {3, 2, 1} 34. What is an indirect proof? Trisha, Roosevelt, and Jos´e write programs that schedule tasks for manufacturing dog toys. 41. x = 1 y=2 z=3 a=x+y b=y+z c =x+z c =c+1 x=a+b+c 53. Sentences (a)–(d) are propositions, whereas sentences (e) and (f) are not propositions. A longstanding, open conjecture is that for every positive integer n, un is defined and equal to 1. If the members of X are listed as in (1.1), we simply look to see whether or not x appears in the listing. In case x ∈ / X, CX−Y (x) = 0 = 0 · [1 − CY (x)] = CX (x)[1 − CY (x)]. Also, find the truth value of each conditional proposition, its converse, and its contrapositive. In order for a recurrence relation such as (1.3) to define a sequence, a “start-up” value or values, such as (1.2), must be given. If h is the height of the tree, then h comparisons are required in the worst case. Finally, substituting the first formula for 84 into the previous equation yields 12 = 3 · 396 − 14 · (480 − 396) = 17 · 396 − 14 · 480. But each of these edges can be converted to a path of length 2 from c to c: (c, b, c), (c, d, c), (c, e, c). 59 Sets and Logic 67. Theorem 1.22 Let m and n be integers, m > 1, n > 1, with prime factorizations m = pa11 pa22 · · · pakk and n = pb11 pb22 · · · pbkk . List the members of the event “doubles occur” (i.e., the numbers are the same on both dice). In Exercises 6–10, write the given argument in words and determine whether each argument is valid. (3.7) It then sets a = 4 and b = 1. p → r r→q For each argument in Exercises 18–20, tell which rule of inference is used. ∃x∃y T2 (x, y) 6. Suppose that x ∈ A ∪ B. (Conclusion) The gap (· · ·) represents the part of the proof to be completed that leads from the hypotheses to the conclusion. Now suppose that x ∈ (A−B)∪(B−A). 1 = (337 − 258)49 − 258 · 15 = 337 · 49 − 258 · 64. A domino is a rectangle divided into two squares with each square numbered one of 0, 1, . ¬c ∨ d 4. Let U be a universal set and let A, B, and C be subsets of U. Discussion The outline of the proof is d = min{d1 , d2 } and x ≤ d (Hypotheses) ··· x ≤ d1 and x ≤ d2 (Conclusion) To help understand what is being asserted, let us look at a specific example. p ∨ q ¬p ∴q We observe that whenever the hypotheses p and q are true, the conclusion p ∧ q is also true; therefore, the argument is valid. Prove that the function f(x) = 1 x2 from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y . How many round-trips are there of the form Oz–Mid Earth– Fantasy Island–Mid Earth–Oz in which on the return trip we do not reverse the original route from Oz to Fantasy Island? We then invoke this algorithm with i = 3, j = 4 to search for key in s3 = ‘F ’, s4 = ‘S’. What is a parallel computer? Give an example of a function different from those presented in the text that is onto but not one-to-one, and prove that your function has the required properties. Executing power = power ∗ b sets power to 2. x1 x6 y1 x2 = y 4 = z 1 x7 E1 x4 y6 x5 = y 5 = z 2 x3 y3 y2 E2 Figure 6.1 Events E1 and E2 . [In Figure 1.1, we marked the points (1, 0) and (k, k − 1), k = 3, with diamonds, and we isolated the (k − 1) × (k − 1) subgrid.] Thus there are Ck−1 routes from (0, 0) to (k, k) that first meet the diagonal at (k, k). Prove that the function f from Z+ × Z+ to Z+ defined by f(m, n) = 2m 3n is one-to-one but not onto. 67. Model this system as a network. If true, prove it; otherwise, provide a counterexample. Full binary tree; height = 4; nine terminal vertices 12. You can write up the proof of (1) by beginning a new paragraph with a sentence that indicates that you are about to prove “if p then q.” You would then follow with a proof of (2) by beginning a new paragraph with a sentence that indicates that you are about to prove “if q then p.” Another common technique is to write [p → q.] Proof of p → q goes here. Use proof by cases to prove that x + y + |x − y| max{x, y} = 2 for all real numbers x and y. The symbol ∃ means “there exists.” Thus the statement there exists x, P(x) may be written ∃x P(x). Write a program that constructs a Gray code. Select an ordering of the vertices. At line 1, i is set to 8. (The loop incident at R counts as two edges and so cannot be chosen.) If we select the edge labeled 1 incident on R and G, we would need a loop at W labeled 4. The preceding observations explain how Theorem 5.14 generalizes De Morgan’s laws for logic (Example 3.11). Example 3.3 In general, a graph will have several spanning trees. , sj Recurrence Relations are solved recursively. In the argument p1 , p2 , . False 40. Inductive Step Assume that the theorem is true for n ≥ 1. A A B B C 25. It is also a fact (although we will not prove it in this book) that, except for the order of the prime factors, the prime factors are unique. † These sections can be omitted without loss of continuity. Show that S is a partition of X. Therefore, if the height of a binary search tree is small, searching the tree will always be very fast (see Exercise 25). Example 1.16 We find the greatest common divisor of 30 and 105 by looking at their prime factorizations 30 = 2 · 3 · 5 105 = 3 · 5 · 7. Thus the level of a vertex v is the length of the simple path from the root to v. You heard the “Flying Pigs” rock concert, and you have sore eardrums. Six edges; eight vertices 23. 46. bn = bn/2 + b(n+1)/2 + 1, b1 = 0 30. We have just observed that the function computes 1 as the value of 0!. b 2 a e 1 a, 2 0 2 1 c 3 a, 1 d z e, 4 d, 2 1 2 e Figure 4.11 The conclusion of Dijkstra’s shortest-path algorithm. Let R be the relation defined on the set of eight-bit strings by s1 R s2 provided that s1 and s2 have the same number of zeros. Proof [Second proof] We use a theorem that gives laws of sets and the fact that Y − X = Y ∩ X, which follows immediately from the definition of set difference. Let’s write out carefully what it is we have to count. Since most of the statements in mathematics and computer science use variables, we must extend the system of logic to include such statements. Suppose 374 C(4, 2)[C(26, 5) − 2C(13, 5)]. Implement Algorithm 2.16, exponentiation by repeated squaring, as a program. The proof will often use mathematical induction and the previous equation (see Example 4.7). An edge connects an applicant to a job for which the applicant is qualified. Eventually we obtain a simple cycle from v to v. , (n, n + 1)}. In Example 1.6, to determine whether a positive integer n > 1 was prime, we checked the potential divisors 2, 3, . We have just observed that the function computes 2 as the value of 2!. Since “110” does occur in t at index 2, the algorithm returns the value 2. A third form of an expression can be obtained by applying preorder traversal to a binary tree representation of an expression. 416 si , . If we set Un = br1n + dr2n , then U is a solution of (2.13). Find the probability of obtaining a bridge hand with 5–4–2–2 distribution, that is, five cards in one suit, four cards in another suit, and two cards in each of the other two suits. The algorithm can be described by a decision tree, and since the worst-case time is 2 or less, the height of the decision tree is 2 or less. The pair z, n = 713, 29 is made publicly available. Find a formula for rn−1 . The selection sort algorithm sorts the sequence s1 , . (Notice that the best-case, averagecase, and worst-case times are all the same for this algorithm.) We immediately obtain the initial condition b1 = 0. Since a is an upper bound for X, x ≤ a. Find the sum In Exercises 10 –12, find the number of terms in the expansion of each expression. f1 = 1, f2 = 1, f3 = 2, f4 = 3 . A function f from X to Y is said to be onto Y if the range of f is Y . However, P also serves as a path from a to b in G; thus G is connected. Given a set of features F , one computes the probability of a class given F for each class and places the item into the most probable class; that is, the class C chosen is the one for which P(C | F) is greatest. We conclude that the worst-case time of Algorithm 8.13 is O(n). For which values of n are the events “there are children of both sexes” and “there is at most one girl” independent? ∃x∀y P(x, y) Assume that ∀x∀y P(x, y) is false and that the domain of discourse is nonempty. Some statements not of the form (3.2) may be rephrased as conditional propositions, as the next example illustrates. Since v4 and v6 are adjacent and v4 is in V1 , v6 is in V2 . Problem Simplified Problem 5! 4! 3! 2! 1! 0! 5 · 4! 4 · 3! 3 · 2! 2 · 1! 1 · 0! None TABLE 4.2 ■ Combining subproblems of the factorial problem. Define a relation R on X by xRy if f(x) = f(y). The proof is by strong induction on n. Therefore, Dividing by n gives s1 + s2 + · · · + sn > A, n which contradicts the hypothesis. For n = 1, 2, 3, it is possible to draw the n-cube in the plane without having any of its edges cross: 4 1 3 1 2 1 W G 3 2 30. Exercises Which of the graphs in Exercises 1–4 are trees? Once Dolph and Francisco have been assigned, there are four ways to fill the remaining office. Define what it means for two graphs to be isomorphic. Solve the recurrence relation of Exercise 9. Focus camera. How many distinct Braille characters are possible? If we take a = 3x, b = −2y, and n = 4 in Theorem 7.1, we obtain Example 7.3 (3x − 2y)4 = (a + b)4 = C(4, 0)a4 b0 + C(4, 1)a3 b1 + C(4, 2)a2 b2 + C(4, 3)a1 b3 + C(4, 4)a0 b4 = C(4, 0)(3x)4 (−2y)0 + C(4, 1)(3x)3 (−2y)1 + C(4, 2)(3x)2 (−2y)2 + C(4, 3)(3x)1 (−2y)3 + C(4, 4)(3x)0 (−2y)4 = 34 x4 + 4 · 33 x3 (−2y) + 6 · 32 x2 (−2)2 y2 + 4(3x)(−2)3 y3 + (−2)4 y4 = 81x4 − 216x3 y + 216x2 y2 − 96xy3 + 16y4 . A = {1, 2, 3}, B = {1, 2} 38. , use iteration. If we think of the vertices in a weighted graph as cities and the edge weights as distances, the traveling salesperson problem is to find a shortest route in which the salesperson can visit each city one time, starting and ending at the same city. What is a simple cycle? Let g = {(1, a), (2, c), (3, c)} be a function from X = {1, 2, 3} to Y = {a, b, c, d}. 3: (1, 3), (2, 2), (3, 1), where (b, r) means the blue die shows b and the red die shows r. E1 and E2 7. At this point, U = {b, d}. How many ways can we select three books each from a different subject from a set of six distinct history books, nine distinct classics books, seven distinct law books, and four distinct education books? 22 + 42 + · · · + (2n)2 + [2(n + 1)]2 = 3 2(n + 1)(n + 2)[2(n + 1) + 1] + [2(n + 1)]2 = 3 15. The difference of X and Y is the set of elements that are in X but not in Y. We show that R ◦ S need not be transitive. Prove that Algorithm 2.3 is correct. (3.12) Substituting (3.12) into (3.11), we obtain rn−1 = tn−3 [−qn−2 rn−3 + rn−4] + sn−3 rn−3 = [−tn−3 qn−2 + sn−3]rn−3 + tn−3 rn−4 . Let X be the set of all four-bit strings (e.g., 0011, 0101, 1000). Then (a) vn−1 is the parent of vn . A relation R on a set X is symmetric if for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R. 3 3 3 1 2 4 4 5 2 5 3 6 6 8 5 11 6 8 9 7 2 4 12 Trees 5. We first note that since two numbers are involved, we will need two variables, say x and y. You should verify the preceding formula, find the six routes that first meet the top edge at the third position from the left, and see why the number of such routes is equal to the number of routes from the lower-left corner to the upper-right corner of a 2 × 2 grid. Theorem 8.14 The worst-case time of Algorithm 8.13 is (n), where n is the total number of vertices in the two trees. (a) In how many ways can this be done? Now E1 , . If α and β are two strings, the string consisting of α followed by β, written αβ, is called the concatenation of α and β. If p is in t, m comparisons must be performed to verify that p is, in fact, in t. Section 8 29. Show that if d1 | m and d2 | n, then d1 d2 | mn. v1 v2 w1 w2 a b h c g d c b e v5 v4 G1 f w3 v3 w4 w5 v1 v2 v3 w1 v4 v5 w3 w2 v6 G1 h j i 27. (Any vertex in G is either in V or in W .) (a) A flow in the matching network gives a matching in G. h(x) = x2 mod 11; cells and data as in Exercise 55 50. gcd(a, b) = gcd(a − b, b). Step 1: We find the worst play in the world—a sure fire flop. Example 5.5 Let R1 be the relation from X = {1, 2, 3} to Y = {a, b} defined by R1 = {(1, a), (2, b), (3, a), (3, b)}, and let R2 be the relation from Y to Z = {x, y, z} defined by R2 = {(a, x), (a, y), (b, y), (b, z)}. Let E be the sum of the lengths of the simple paths from the root to the terminal vertices. Its arrow diagram is xn = (7xn−1 + 5) mod 11 with seed 3 gives a sequence that begins 3, 4, 0, 5, . To do so, you can assume all preceding cases (not just the immediately preceding case as in Section 4). Given any term, add 3 to get the next term. Using this result, one can show (see Exercise 25) that if n is odd, say n = 2k + 1, when the two binary trees shown in Figure 8.15 are input to Algorithm 8.13, the number of comparisons is equal to 3n + 1. We omit the arrow diagram of f . If the pair is isomorphic, specify an isomorphism. Has n vertices of degree k e G1 2 f 2 12 f 1 e d 1 i b 6 G2 b l 3 4 G2 a 4 G1 11. We now prove (4.1). The argument is invalid. 1 10. The recurrence relation (2.6) is of order 1 and (2.7) is of order 2. List the elements of R. 1, 1, 2, 4, 16, 128, 4096, . Write an algorithm that stores n distinct words in a binary search tree T of minimal height. ∃x∀y T1 (x, y) 4. If we buy orange juice the first day, we have n − 1 dollars left, which may be spent in Rn−1 ways. Once the third letter has been selected, the fourth letter can be selected in two ways. Given g = {(1, b), (2, c), (3, a)}, a function from X = {1, 2, 3} to Y = {a, b, c, d}, and f = {(a, x), (b, x), (c, z), (d, w)}, a function from Y to Z = {w, x, y, z}, write f ◦ g as a set of ordered pairs and draw the arrow diagram of f ◦ g. Therefore, logb n ≤ C loga n for all n ≥ 1, where C = logb a. What is the capacity of a cut? , ej , vj , ej+1 , vj+1 , . Consider the final multiplication, which looks like (a1 · · · at)(at+1 · · · an), for some t, 1 ≤ t < n. τ(R1 ∩ R2) = τ(R1) ∩ τ(R2) 61. By Exercise 34, the 3 × 2 subboards have tilings. The next theorem shows that in this case, nr n furnishes the other basic solution. Therefore A is not a subset of B. An upper bound for X is a real number a having the property that x ≤ a for every x ∈ X. Factoring x2 − 4x + 4, we find that (x − 2)2 = 1. a x2 x1 e b x3 x5 x4 d c G1 A y1 C y2 y4 D y5 y3 B E G2 Figure 6.1 Isomorphic graphs. Define descendant in a rooted tree. Thus y = z. Then a = qg + r, 0 < r < g. Find the months with Friday the 13th in 2040. 8 37. Suppose that s1 = 1, s2 = 6, s3 = 3, s4 = 5, s5 = 4, s6 = 2 and that we are at line 6. , 449, 450 for potential divisors of 451, we find that 11 divides 451 (451 = 11 · 41); thus, 451 is composite. Depth-first search 4. CLEAR A D 25. Trace Algorithm 2.1 for the input t = “111011” and p = “110”. T1 and T2 as in Exercise 3 9. What is proof by contradiction? If a graph contains a cycle that includes all the edges, the cycle is an Euler cycle. Module 81 has no numerical error. Use mathematical induction to show that for all n ≥ 1, fn ≤ 2n−1 . An ordering of x1 , . Combine the evaluation function in Exercise 27 with a twolevel search of the game tree to obtain a game-playing algorithm for Mu Torere. (b) Find all the solutions to the Instant Insanity puzzle of Figure 8.5. 8. (An alphanumeric character is one of A to Z or 0 to 9.) How many valid FORTRAN identifiers are there? How many consist of distinct digits? (¬r ∨ q) → ¬q 18. Let f be a characteristic function in X. Thus f is not onto Y . a 2 5 g 8 G2 G1 10. Does the graph of Exercise 5 have a Hamiltonian path? D G E H E F B F B H I J Figure 6.6 Reverse postorder traversal. A right child of vertex v is a child designated as “right.” 22.

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